Interoccurrence Times in the Bak-Tang-Wiesenfeld Sandpile Model: A Comparison with the Observed Statistics of Solar Flares

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A sequence of bursts observed in an intermittent time series may be caused by a single avalanche, even though these bursts appear as distinct events when noise and/or instrument resolution impose a detection threshold. In the Bak-Tang-Wiesenfeld sandpile, the statistics of quiet times between bursts switches from Poissonian to scale invariant on raising the threshold for detecting instantaneous activity, since each zero-threshold avalanche breaks into a hierarchy of correlated bursts. Calibrating the model with the time resolution of GOES data, qualitative agreement with the interoccurrence time statistics of solar flares at different intensity thresholds is found.

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A simple picture of intermittency in turbulent flows places rare, hot regions that dissipate energy inside a cold laminar sea. As the hot regions evolve, they maintain a clustered structure, and the dissipation, occurring at small scales, remains correlated at large ones. For instance, examining the sky at night, one sees stars, galaxies, and clusters of galaxies against a dark background [1,2]. Many striking examples of turbulent intermittency occur in astrophysical, space, or laboratory plasmas, such as flares in the solar corona [3,4], magnetic substorms [5–7], bursty bulk flows [8], auroral emissions [9–11], turbulence in the solar wind [12–14], or bursts observed in plasma experiments [15–17]. Important parameters include the Reynolds number(s), $R = Vl_0/\nu$ (or $R_m = Vl_0/\eta$, etc.), where ν is the viscosity, η is the magnetic diffusivity, and V is the velocity difference over the integral scale l_0. For increasing R, characteristic widths of the dissipating regions decrease, while their intensity increases. Turbulence becomes an "on/off" phenomena as the Reynolds number(s) become large [2,18].

The controversial hypothesis that turbulent intermittency is a manifestation of self-organized criticality (SOC) has been discussed by Bak and others [2,4,5,19–26]. In this scenario, intermittent energy dissipation is a stick-slip or threshold process. Each slip can trigger further slips—either through short or long range interactions. Eventually, a regime materializes where sparse, sporadic avalanches of intense energy dissipation interrupt laminar regions of space-time which rest near static equilibrium with little dissipation despite the continuous, global input of energy.

Up to now, SOC has mostly been studied with vividly plain models, such as the original Bak-Tang-Wiesenfeld (BTW) sandpile, that ignore many features of hydrodynamic or plasma turbulence. Rather, they generate patterns of rapid energy dissipation in space and time, which is the hallmark of intermittency. On the other hand, most studies of turbulence [18] examine structure functions and multiscaling phenomena [27,28], and/or nonlinear instabilities and coherent structures, etc. [5,29,30], all of which are associated with intermittency but do not directly characterize the bursts of energy dissipation in space and time. As a result, some comparisons that have been made are superficial and should be made more definitive. Further, one argument used so far to distinguish SOC from turbulence is misleading and erroneous.

While BTW and other SOC models exhibit a broad distribution of avalanche sizes and durations, which are comparable to, e.g., solar flare data, a marked difference has been noted regarding the time intervals between bursts. For instance, Boffetta *et al.* [31] found that the distribution of times between flares exhibits power-law statistics, while intervals between subsequent avalanches in BTW are (approximately) Poissonian. Further, they and other groups [12,15–17,32] found that shell or reduced magnetohydrodynamics (MHD) models gave a better description of the waiting time statistics, and some [15,16,31,32] used this to distinguish SOC from turbulence, or to question the applicability of the SOC paradigm for magnetically confined plasmas in thermonuclear research [15].

We make three key points. First, intermittent bursts can never be detected, nor distinguished from the background, at arbitrarily low thresholds. Even the studies above comparing reduced MHD or shell models with flares or bursts in man-made plasmas have to use a threshold for defining bursts. Such a threshold is realistic because the emission associated with, e.g., flares, decays slowly after a local peak, allowing overlaps with subsequent peaks. Although a threshold is unavoidably connected to the precise definition of events, robust features may be observed with rescaled distributions measured at different thresholds [33,34]. For such time series, event durations and quiet times between them are measured on a single clock with equal precision. Thus we propose that the sequence of bursts arises from a single avalanche observed at finite detection threshold. The bursts within a single avalanche in a SOC system can be correlated in space and time, being part of the same, critical process.

To test this idea, we study a BTW sandpile [19] in which the time unit is that of a parallel update step, and consider both (infinitely) slow driving (A) and running sandpile [25,35,36] (B) conditions. In the time series of the activity, $n(t)$, bursts are defined as consecutive intervals during which $n(t) > n_c$, where the detection threshold for events is $n_c \geq 0$ (see Fig. 1). A data analysis technique analogous to that developed by Baiesi *et al.* [33] to examine interoccurrence statistics of flares allows a direct comparison and shows that BTW shares several features with the intermittent statistics of flare emission, which is assumed to be simply related to the released magnetic energy.

The BTW sandpile consists of a $L \times L$ lattice with a discrete number z_i of sand grains occupying each site i. We study two versions: (A) in the slow driving limit, a grain is added to the pile at a randomly chosen location as soon as the previous ($n_c = 0$) avalanche ends. The durations (t_d) and quiet times (t_q) then refer to intervals between local peaks within each avalanche and statistics are obtained over many avalanches. In the second case (B), one grain of sand is dropped every ΔT update steps at a randomly chosen site. In both cases, at each update step, t, all sites that exceed a threshold for stability, $z_i > z_c = 3$, topple in parallel by distributing one grain of sand to each of their four nearest neighbors or, for boundary sites, over the edge of the lattice. Taken as the instantaneous dissipation signal,

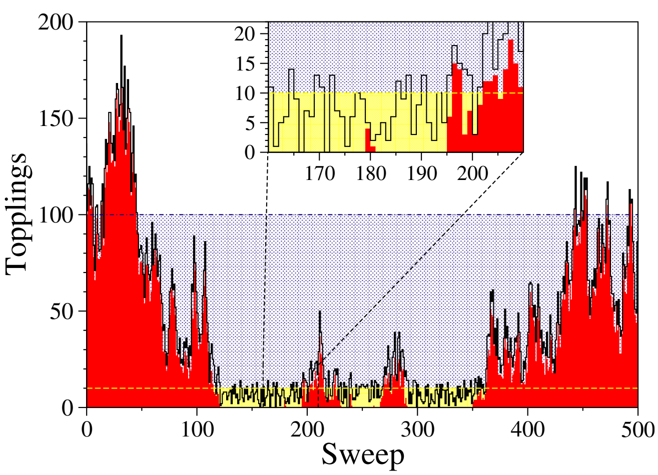

FIG. 1 (color online). Time series of the instantaneous number of topplings, $n(t)$, in the two dimensional BTW with $L = 2048$ and $\Delta T = 100$. Dark (red online) areas and the black line represent the actual signal and the signal with some additive noise, respectively. Shaded regions indicate a detection thresholds $n_c = 10$ (yellow, dashed line) and $n_c = 100$ (blue, dot-dashed line); any signal below n_c is considered undetectable. Intervals between a rise of the signal above n_c and its next fall below constitute measured durations t_d of events, followed by a "quiet time" t_q until the next rise. The "waiting time" between two consecutive rises is $t_w = t_d + t_q$.

$n(t)$ is the number of unstable sites toppling at each parallel update step. In model B, if $\Delta T > \langle t_d \rangle_L$, the sequence of topplings is also interrupted by instances where the activity completely stops ($n(t) = 0$) [36]. The quantity $\langle t_d \rangle_L$ is the average duration of avalanches on a lattice of scale L in the stationary state of model A. For both A and B, consecutive stopping points separated by intervals where $n(t) > 0$ delimit $n_c = 0$ avalanches. The time series of Model B has similar character to the flare data studied in Ref. [33], with a broad distribution of events that exceed each threshold n_c, albeit with a finite-size cutoff curtailing the power-law tail observed in Fig. 2(a) of Ref. [33].

Consider an observer who measures the global activity sequence with a finite error, so that the time series she records is $n_{\text{obs}}(t) = n(t) + \eta(t)$. For instance, let $\eta(t)$ be an independent random number uniformly distributed between 0 and 15. The effect of this noise is shown in Fig. 1 for model B. One way for the observer to separate the signal from the noise is to increase her threshold for detecting events, and coarse-grain her unit of measurement. In observing natural phenomena these detection thresholds are an unavoidable part of the measurement. For simplicity, we study the original time series, without noise, but with a finite threshold $n_c > 0$ (e.g., $n_c = 100$ as in Fig. 1) to distinguish bursts, considering all instances with $n(t) \leq n_c$ to have no activity [37,38]. Simulation results shown in Figs. 2–4 are for BTW without noise.

As Fig. 2 shows, on increasing the threshold n_c from zero, the distribution of quiet times t_q for model B switches from an (approximately) exponential distribution to a power law,

$$P_{\text{quiet}}(t_q) \sim t_q^{-\gamma_q} \quad \text{with } \gamma_q^{\text{BTW}} = 1.67 \pm 0.05. \tag{1}$$

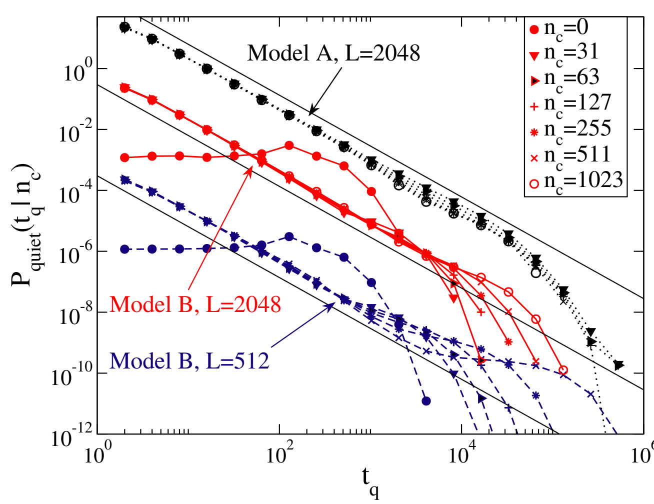

FIG. 2 (color online). Distribution of quiet times t_q in BTW for different thresholds n_c for model A and model B with $\Delta T = 100$ (groups of curves are offset vertically). In model B the distribution changes from approximately exponential at $n_c = 0$ to a power law for increasing n_c. The straight reference lines decay as $t_d^{-5/3}$, suggesting $\gamma_q^{\text{BTW}} = 1.67(5)$ in Eq. (1).

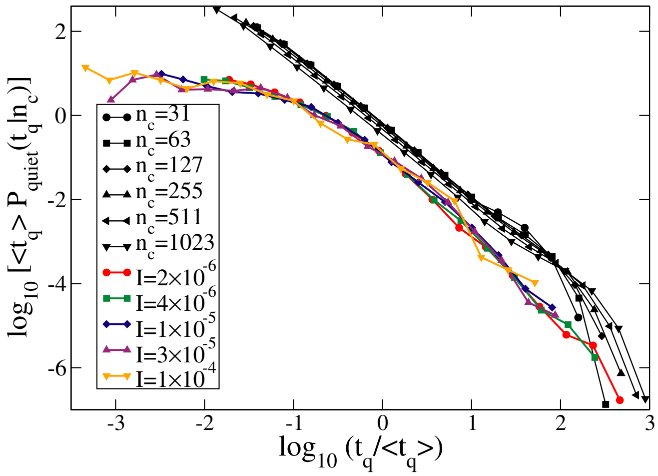

FIG. 3 (color online). Rescaled distribution of quiet times in BTW with $L = 2048$, $\Delta T = 100$ for different thresholds n_c and for the solar flare data corresponding to "all" at different thresholds I in Fig. 3(b) of Ref. [33]. The quiet times t_q have been divided by the average quiet time $\langle t_q \rangle$ at each threshold, and the distributions rescaled to preserve normalization. BTW and flare data are similar in the intermediate regime. The former are shifted up by one unit on the log scale.

For model A one observes an even cleaner and broader scaling regime, with the same γ_q^{BTW} [39]. The Abelian property of BTW assures that this behavior in model B cannot be due to overlapping avalanches. Thus, the natural introduction of detection thresholds leads to the discovery of a hierarchical sequence of correlated bursts (or subavalanches) within a large avalanche.

A similar analysis by Baiesi *et al.* [33] for flares, detected as intervals during which the emission intensity in the GOES time series exceeds I, used a particularly simple scaling ansatz for the quiet time distribution,

$$P_{\mathrm{quiet}}(t_q|I) = \frac{1}{\langle t_q \rangle_I} f_{\mathrm{flare}}\left(\frac{t_q}{\langle t_q \rangle_I}\right). \qquad (2)$$

Here f_{flare} is a scaling function and the average quiet time for a given threshold, $\langle t_q \rangle_I$, provides a factor that collapses distributions measured at different I. While BTW does not obey this particular scaling ansatz, for comparison we apply Eq. (2) to BTW with different values of n_c and compare the dependence of two dimensionless quantities $\langle t_q \rangle P_{\mathrm{quiet}}$ vs $t_q/\langle t_q \rangle$ at different thresholds for flares and BTW. As shown in Fig. 3, we find that although the scaling functions $f(x)$ are clearly different for $x \ll 1$, they are similar for intermediate arguments.

Event durations, t_d, observed at different thresholds in BTW and for flares occurring during solar minimum, are shown in Fig. 4 [40]. In order to compare with BTW, we set the time resolution of the GOES data equal to the time resolution of BTW. One minute is set equal to one sweep. Using this straightforward calibration, the statistics of event durations are similar as well. In both cases, power-law behavior is observed, $P_{\mathrm{dur}}(t_d) \sim t_d^{-\gamma_{\mathrm{dur}}}$. For flares at solar minimum $\gamma_{\mathrm{dur}} = 2.0 \pm 0.1$ [33], while this exponent for the $n_c \gg 1$ bursts in BTW is smaller.

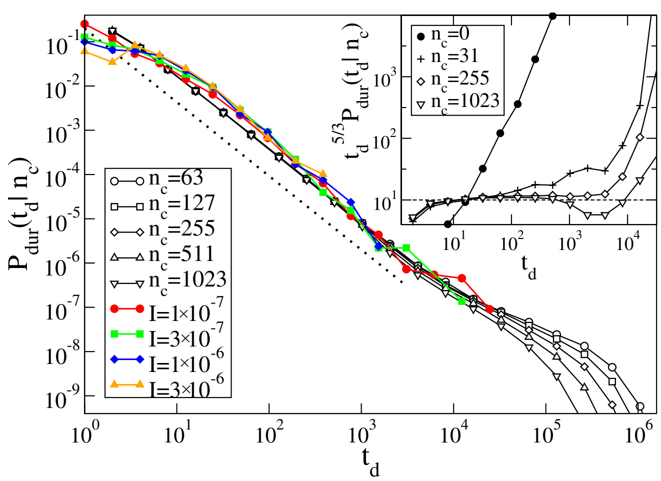

FIG. 4 (color online). The distribution of event durations for BTW model B with $L = 2048$, $\Delta T = 100$ and for flare data corresponding to "minimum" at different thresholds I in Fig. 4 of Ref. [33]. One parallel update step in BTW has been calibrated to the time resolution of the GOES data, or 1 min. Although the critical exponents are different, the overall distributions are roughly comparable. The inset shows that, for increasing n_c, the behavior changes from that in Refs. [41,42] towards a plateau indicating an asymptotic exponent $\approx 5/3$ for BTW at large thresholds [n_c: 0 (●), 15 (+), 255 (◇), and 1023 (▽)]. Similar behavior is found for model A (not shown).

The inset of Fig. 4 shows how the apparent power-law behavior for burst durations in BTW changes with increasing threshold n_c. At sufficiently large n_c a plateau appears in the function $t_d^{5/3} P_{\mathrm{dur}}(t_d)$. This suggests a critical exponent at high thresholds, $\gamma_{\mathrm{dur}}^{\mathrm{BTW}} = 1.67 \pm 0.05$, similar to γ_q^{BTW}. The figure also demonstrates that the scaling behavior of burst durations, for the system sizes that were studied, differs at large n_c from that at $n_c = 0$. In the latter case, Lübeck and Usadel [41] determined a critical exponent $\gamma_{\mathrm{dur}}^{\mathrm{BTW}}(n_c = 0) \simeq 1.48$, while Stella and De Menech [42] found multiscaling.

If the sequence of quiet times $(t_q)_i$ are uncorrelated, the cumulative variable $y_l(j) = \sum_{i=j}^{i=j+l} (t_q)_i$ exhibits diffusive behavior, with an average variance scaling as $\sigma = (\langle y^2 \rangle - \langle y \rangle^2)^{1/2} \sim l^H$, with $H = 1/2$ [43]. Our measurements of this quantity confirm that quiet time intervals for BTW at different thresholds are uncorrelated, in all cases giving $H = 1/2$. However, at solar minimum, we get $H \simeq 0.62$ (see also Ref. [44]). Therefore, BTW does not reproduce correlations *between* quiet times for flares.

In fact, a variety of SOC models exhibit power laws in times between events. Previous analyses [45–47] using a detection threshold considered a completely different limit than that discussed here, namely, that of an infinite time scale separation between the driving rate and the durations of events. Consequently, a threshold was imposed on a variable related to the total size or duration of each $n_c = 0$ avalanche, rather than its instantaneous dissipation. However, as explained earlier our limit of overlapping time scales for durations and quiet times may be the correct one to describe intermittency in turbulence. Indeed, cellular automata models of laboratory plasmas are running sandpiles; see, e.g., Newman *et al.* [48]. Within this scheme, some works [25,49], also using a threshold, have found power-law quiet times for the Hwa-Kardar [35] running sandpile driven at a sufficiently high rate. This was claimed to be due to interactions between overlapping avalanches. In contrast, here we show that the BTW model, considering one avalanche at a time, generates a power-law distribution of quiet times when a finite detection threshold is used. Its Abelian property assures that this correlation remains the same when the model is driven at a finite rate, in agreement with the results shown here. It is unlikely, though, that the Abelian property of BTW is essential to getting a scale free distribution of quiet times, although this remains to be clarified by studying other models.

In conclusion, we have demonstrated that SOC remains a viable alternative for the explanation of intermittent dissipation in turbulent phenomena by comparing results from numerical simulations of a BTW sandpile with solar flare statistics. By including an inevitable detection threshold in the analysis of BTW, as well as allowing an overlap of time scales between burst durations and quiet times, qualitative correspondence is obtained for the power-law statistics of interoccurrence times for solar flares. Studies of more physically realistic SOC models including detection thresholds for short time dissipation in the whole system may improve quantitative agreement.

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[1] L. M. Ozernoi and A. D. Chernin, Astron. Zh. **45**, 1137 (1968) [Sov. Astron. **12**, 901 (1969)].
[2] P. Bak and M. Paczuski, Physica (Amsterdam) **348A**, 277 (2005).
[3] M. Aschwanden *et al.*, Astrophys. J. **535**, 1047 (2000).
[4] E. Lu and R. Hamilton, Astrophys. J. **380**, L89 (1991).
[5] T. Chang, Phys. Plasmas **6**, 4137 (1999).
[6] A. J. Klimas *et al.*, J. Geophys. Res. **105**, 18 765 (2000).
[7] G. Consolini, Fractals **10**, 275 (2002).
[8] V. Angelopoulos, T. Mukai, and S. Kokubun, Phys. Plasmas **6**, 4161 (1999).
[9] B. V. Kozelov, V. M. Uritsky, and A. J. Klimas, Geophys. Res. Lett. **31**, L20804 (2004).
[10] V. M. Uritsky *et al.*, J. Geophys. Res. **107**, 1426 (2002).
[11] A. T. Y. Lui *et al.*, Geophys. Res. Lett. **27**, 911 (2000).
[12] M. P. Freeman, N. W. Watkins, and D. J. Riley, Phys. Rev. E **62**, 8794 (2000).
[13] M. P. Leubner and Z. Voros, Nonlin. Process. Geophys. **12**, 171 (2005).
[14] E. Marsch and C.-Y. Tu, Nonlin. Process. Geophys. **4**, 101 (1997).
[15] E. Spada *et al.*, Phys. Rev. Lett. **86**, 3032 (2001).
[16] V. Antoni *et al.*, Phys. Rev. Lett. **87**, 045001 (2001).
[17] L. Marrelli, L. Frassinetti, P. Martin, D. Craig, and J. S. Sarff, Phys. Plasmas **12**, 030701 (2005).
[18] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).
[19] P. Bak, C. Tang, and K. Weisenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[20] P. Bak, K. Chen, and C. Tang, Phys. Lett. A **147**, 297 (1990).
[21] M. De Menech and A. L. Stella, Physica (Amsterdam) **309A**, 289 (2002).
[22] A. Bershadskii and K. R. Sreenivasan, Eur. Phys. J. B **35**, 513 (2003).
[23] M. Paczuski and D. Hughes, Physica (Amsterdam) **342A**, 158 (2004).
[24] P. Charbonneau *et al.*, Sol. Phys. **203**, 321 (2001).
[25] R. Sánchez, D. E. Newman, and B. A. Carreras, Phys. Rev. Lett. **88**, 068302 (2002).
[26] M. Wheatland, Astrophys. J. **536**, L109 (2000).
[27] Z. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994).
[28] B. Dubrulle, Phys. Rev. Lett. **73**, 959 (1994).
[29] O. Métais and M. Lesieur, *Turbulence and Coherent Structures* (Springer, New York, 1991).
[30] A. A. Schekochihin, S. C. Cowley, J. L. Maron, and J. C. McWilliams, Phys. Rev. Lett. **92**, 064501 (2004).
[31] G. Boffetta *et al.*, Phys. Rev. Lett. **83**, 4662 (1999).
[32] V. Carbone *et al.*, Europhys. Lett. **58**, 349 (2002).
[33] M. Baiesi, M. Paczuski, and A. L. Stella, cond-mat/0411342.
[34] Á. Corral, Phys. Rev. E **68**, 035102(R) (2003).
[35] T. Hwa and M. Kardar, Phys. Rev. A **45**, 7002 (1992).
[36] Á. Corral and M. Paczuski, Phys. Rev. Lett. **83**, 572 (1999).
[37] For sufficiently large n_c and at large times, the event statistics with and without noise are the same.
[38] Quiet times occurring between $n_c = 0$ avalanches in model A happen at the slow time scale of the external drive and therefore are disregarded and not shown in Fig. 2.
[39] Error bars quoted here and elsewhere in the text represent our estimate of statistical and systematic errors associated with finite-size effects.
[40] For flares, the asymptotic distribution of burst durations is independent of threshold, but depends on the phase of the solar cycle [33].
[41] S. Lübeck and K. D. Usadel, Phys. Rev. E **55**, 4095 (1997).
[42] A. L. Stella and M. D. Menech, Physica (Amsterdam) **295A**, 101 (2001).
[43] M. De Menech and A. L. Stella, Phys. Rev. E **62**, R4528 (2000).
[44] P. Grigolini, D. Leddon, and N. Scafetta, Phys. Rev. E **65**, 046203 (2002).
[45] J. P. Norman, P. Charbonneau, S. W. McIntosh, and H.-L. Liu, Astrophys. J. **557**, 891 (2001).
[46] M. Hedges and G. Takacs, physics/0505015.
[47] M. Baiesi and C. Maes, cond-mat/0505274.
[48] D. E. Newman, B. A. Carreras, P. H. Diamond, and T. S. Hahm, Phys. Plasmas **3**, 1858 (1996).
[49] R. Woodward *et al.*, Phys. Rev. Lett. **93**, 249801 (2004).