A thermodynamic uncertainty relation for a system with memory

Ivan Di Terlizzi[*,†]

Marco Baiesi[*,†]

∗ Dipartimento di Fisica e Astronomia "Galileo Galilei", Università di Padova, Via Marzolo 8, 35131, Padova, Italy

† INFN, Sezione di Padova, Via Marzolo 8, 35131, Padova, Italy

Abstract. We introduce an example of thermodynamic uncertainty relation (TUR) for systems modeled by a generalised Langevin dynamics with memory, determining the motion of a micro-bead driven in a complex fluid. Contrary to TURs typically discussed in the previous years, our observables and the entropy production rate are one-time variables. The bound to the signal-to-noise ratio of such state-dependent observables can only in some cases be mapped to the entropy production rate. For example, this is true for steady states and for a subclass of Markovian systems. In fact, the presence of memory in the system complicates the thermodynamic interpretation of the uncertainty relation.

Keywords: nonequilibrium inequalities, memory effects, diffusion, fluctuations, entropy production

1. Introduction

The performances of non-deterministic systems are intrinsically limited by several laws. For instance, the *quantum speed limit* determines the minimum time needed to transform a quantum state to another [1]. Recently is was noted that similar speed limits exist also in the evolution of stochastic classical systems [2–5]. Other classical bounds limit the statistics of currents and other observables, whose squared average must be lower that their variance times some cost function [6]. The *thermodynamic uncertainty relation* (TUR) [7, 8] is the primary example of such nonequilibrium inequalities and includes the entropy production as a cost function [9–25] (dissipation may also limit the speed of a process [26]). The TUR and its generalisations [27–33] are inequalities usually discussed and proven for discrete and continuous diffusive Markov systems. Fewer results are available for non-Markovian systems [34–36], namely for systems with some form of memory.

On the practical side, TURs may help in thermodynamic inference [37], for instance in evaluating the entropy production rate from data [14, 38–41]. Theoretically, while the proof of the TUR in steady states is provided by the machinery of large deviation theory [11,15], an approach by Dechant and Sasa (DS) [18,23] adopts information theory as the main theoretical tool. Moreover, a unifying view [42] may explain the mechanism behind uncertainty relations.

The DS approach leads to quite general results for stochastic systems, finite times statistics, and regimes outside steady states. For instance, DS showed that various forms of TUR hold for diffusion processes and anticipated that a TUR holds also for Langevin dynamics with inertia. Following this approach, Van Vu and Hasegawa [32] indeed provided an explicit nonequilibrium inequality for inertial stochastic dynamics when reversible currents are present. Their formula shows that the mean entropy production cannot by itself constitute the cost function in inertial systems. In equilibrium, where on average dissipation is absent while currents are eventually present thanks to inertia, there is a form of dynamical activity that naturally enters in the upper bound, as in a *kinetic uncertainty relation* [31] or similar inequalities including time-symmetric nondissipative observables [27–30, 32].

A linear response formula was used by DS to compare the probability of observing an event for two different processes [23]. Usually, in nonequilibrium statistical physics, the comparison has been done between path weights of two kinds of dynamics (see for instance the derivation of linear response formulas in [43–45]). However, one may also consider the instantaneous probability of observables as position and velocity [19, 23]. We are going to follow this path.

In this work we provide nonequilibrium inequalities for a simple system with memory. This is achieved by developing the DS approach for a one-dimensional Langevin system subject to a friction dynamics with memory and to a time dependent force. We will then proceed to consider a multidimensional generalisation of this scenario. A practical realisation of this dynamics is a colloidal bead dragged by optical tweezers in a complex medium [46,47], say a viscoelastic fluid, or driven by a space-independent time-modulated force. Analytical solutions show that the average and variance of observables for this non-Markovian system also obey a form of generalised TUR, especially in the unidimensional case. In this inequality there appear instantaneous quantities: the instantaneous rate of entropy production is eventually present, while original TURs include the accumulated entropy production and may reduce to forms including its rate only if a steady state is established. Similarly, the observable R entering in our formula is state dependent and its variance is meant as the instantaneous variance in a statistical ensemble at the same time. Thus, the formulas in this work are not for integrated currents, as in usual TURs, but for state-dependent quantities. The importance of such instantaneous quantities was recently highlighted in a novel version of TUR for Markovian systems [24] (see also previous examples [19]).

For the simplest one-dimensional Langevin dynamics without memory (that is, with friction depending on the instantaneous velocity and with white noise) and with harmonic trap, both in steady states (where the trap has been moving with a steady velocity for a long time) and in transients from an initial equilibrium, but only for velocity-independent observables $R(x_t)$ in the latter case, we find that the entropy production rate is the only component of the cost function in the TUR. However, it comes with a prefactor depending on the trap strength and the fluid viscosity. The same holds by replacing Markovianity with the long time limit in systems with memory. The multidimensional generalisation, however, shows that such thermodynamic interpretation is not always possible. For the case of unbound diffusion we also find a rich phenomenology in which, under some specific conditions, there emerges entropy production as the upper bound of the nonequilibrium inequality.

Our calculations are based on the modified Laplace transforms that we recently introduced [48]. In order to deal with steady states, in this modification the initial time is shifted earlier than time zero, at which we start the statistical analysis. By sending the initial time to minus infinity, we may well describe systems where the memory effects are perfectly developed.

In the following section we introduce the model and some of its features. The main formulas are discussed in Section 3 and their applications can be found in Section 4. After our conclusions in Section 5, some technical details are discussed in a set of Appendices.

2. Model

We study a Langevin dynamics with coloured noise and friction with memory, by focusing first on the unidimensional case. We would like to characterise uncertainty relations for the motion of a colloidal particle in a complex viscoelastic fluid, subject to a harmonic potential due to the action of optical tweezers, and eventually to a time-modulated space-independent external force $f(t)$ that could represent a uniform electric field acting on a charged particle. A simpler Langevin equation with white noise and

Markovian dynamics would not be suitable for modelling memory effects by the complex fluid. By introducing a memory kernel $\Gamma(t)$, we thus consider the generalised Langevin equation (GLE)

$$m\ddot{x}(t) = -\int_{t_m}^{t} \Gamma(t-t')\dot{x}(t')\mathrm{d}t' - \kappa\left[x(t) - \lambda(t)\right] + f(t) + \eta(t)\,, \tag{1}$$

where κ is the spring constant associated to a harmonic trap with time dependent minimum $\lambda(t)$ and $t_m \leq 0$ is the time to which the effects of memory extend. Moreover, m is the mass of the colloidal bead and $\eta(t)$ is a coloured Gaussian noise such that $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta\left(s\right)\rangle = k_B T\Gamma\left(|t-s|\right)$ obeying the fluctuation-dissipation theorem [49]. For future convenience, we collect the space-independent terms in

$$F(t) \equiv f(t) + \kappa\lambda(t)\,. \tag{2}$$

A trajectory thus starts from an initial condition (x_{t_m}, v_{t_m}) and evolves according to (1). The initial time $t_m \leq 0$ is finite. However, the limit $t_m \to -\infty$ will be useful for describing steady regimes when, for example, the harmonic trap is moving at constant velocity and no other external force is present.

In this paper we will only focus on processes for which the joint probability density function (PDF) of position and velocity at time t is Gaussian. For linear systems, as in our case, this can be obtained either by starting already from a Gaussian PDF $P(x_{t_m}, v_{t_m}, t_m)$, or by starting from an arbitrary distribution and wait long enough till it becomes a Gaussian. The latter scenario occurs if either $t_m \to -\infty$ or $t \to \infty$. Under these hypothesis, the PDF $P(x_t, v_t, t|t_m)$, were t_m indicates the dependence on some initial conditions at time t_m as those discussed above, is a bivariate Gaussian

$$P(x_t, v_t, t|t_m) = \frac{1}{\sqrt{(2\pi)^2|\mathcal{S}_{t_m,t}|}} \exp\left[-\frac{1}{2}(\mathbf{x}_t - \langle\mathbf{x}\rangle_{t_m,t})^T \mathcal{S}^{-1}_{t_m,t}(\mathbf{x}_t - \langle\mathbf{x}\rangle_{t_m,t})\right]\,, \tag{3}$$

with $\mathbf{x}_t = (x_t, v_t)$, $\langle\mathbf{x}\rangle_{t_m,t} = (\langle x\rangle_{t_m,t}, \langle v\rangle_{t_m,t})$ and $\mathcal{S}_{t_m,t}$ the covariance matrix

$$\mathcal{S}_{t_m,t} = \begin{pmatrix} \langle\Delta^2 x\rangle_{t_m,t} & \mathrm{Cov}_{t_m}(x_t, v_t) \\ \mathrm{Cov}_{t_m}(x_t, v_t) & \langle\Delta^2 v\rangle_{t_m,t} \end{pmatrix} \tag{4}$$

whose components are the variances, defined as $\langle\Delta^2 r\rangle_{t,t_m} = \langle r^2\rangle_{t_m,t} - \langle r\rangle^2_{t_m,t}$, of position and velocity along with their covariance $\mathrm{Cov}_{t_m}(x_t, v_t) = \langle xv\rangle_{t_m,t} - \langle x\rangle_{t_m,t}\langle v\rangle_{t_m,t}$. The distribution is of course completely characterised by the above mentioned quantities, which can be calculated starting from (1) by applying a *modified* Laplace transform, defined as

$$\hat{g}_{t_m}(k) = \mathcal{L}_k^{t_m}[g(t)] \equiv \int_{t_m}^{+\infty} \mathrm{e}^{-kt} g(t)\mathrm{d}t\,. \tag{5}$$

Its details are discussed in [48]. The sub/superscript t_m is useful for reminding us that this transform is different from the standard Laplace transform. For a causal function

$g(t < 0) = 0$, there is no difference between the usual Laplace transform and the modified one. However, this is not the case in general.

An important quantity that appears while solving the GLE is the "position susceptibility" $\chi_x(t)$, defined via its modified Laplace transform

$$\hat{\chi}_x(k) = [mk^2 + k\hat{\Gamma}(k) + \kappa]^{-1} \,. \tag{6}$$

In the following we will also use its integral $\chi(t)$ and its derivative $\chi_v(t)$ (velocity susceptibility),

$$\chi(t) \equiv \int_0^t \chi_x(t')\mathrm{d}t' \,, \tag{7}$$

$$\chi_v(t) \equiv \partial_t \chi_x(t) \,. \tag{8}$$

With these definitions and following the procedure in [48], it can be shown that the average position and velocity are equal to

$$\langle x \rangle_{t_m,t} = \langle x_{t_m} \rangle (1 - \kappa\chi(t - t_m)) + m\langle v_{t_m} \rangle \chi_x(t - t_m) + \int_{t_m}^t \chi_x(t - t')F(t')\mathrm{d}t' \,, \tag{9}$$

$$\begin{aligned} \langle v \rangle_{t_m,t} =& -\kappa\langle x_{t_m} \rangle \chi_x(t - t_m) + m\langle v_{t_m} \rangle \chi_v(t - t_m) + \\ &+ \chi_x(0)F(t) + \int_{t_m}^t \chi_v(t - t')F(t')\mathrm{d}t' \,, \end{aligned} \tag{10}$$

where $\langle x_{t_m} \rangle$ and $\langle v_{t_m} \rangle$ are respectively the average position and velocity at the initial time t_m. This also explains terming $\chi_x(t)$ and $\chi_v(t)$ susceptibilities. We are using the fact that the presence of the the external force $f(t)$ does not change the results obtained in [48]. Moreover, note that $\chi_x(0) \neq 0$ only for overdamped dynamics, see Appendix A for more details. For the components of the covariance matrix one finds that

$$\begin{aligned} \langle \Delta^2 x \rangle_{t_m,t} =& k_B T \Big[2\chi(t - t_m) - m\chi_x^2(t - t_m) - \kappa\chi^2(t - t_m) \Big] + \\ &+ \langle \Delta^2 x_{t_m} \rangle (1 - \kappa\chi(t - t_m))^2 + m^2 \langle \Delta^2 v_{t_m} \rangle \chi_x^2(t - t_m) + \\ &+ 2m\mathrm{Cov}(x_{t_m}, v_{t_m}) \chi_x(t - t_m)(1 - \kappa\chi(t - t_m)) \,, \end{aligned} \tag{11}$$

$$\begin{aligned} \langle \Delta^2 v \rangle_{t_m,t} =& k_B T \Big[1/m - m\chi_v^2(t - t_m) - \kappa\chi_x^2(t - t_m) \Big] + \kappa^2 \langle \Delta^2 x_{t_m} \rangle \chi_x^2(t - t_m) + \\ &+ m^2 \langle \Delta^2 v_{t_m} \rangle \chi_v^2(t - t_m) - 2\kappa m \mathrm{Cov}(x_{t_m}, v_{t_m}) \chi_v(t - t_m) \chi_x(t - t_m) \,, \end{aligned} \tag{12}$$

$$\begin{aligned} \mathrm{Cov}_{t_m}(x_t, v_t) =& k_B T \Big[\chi_x(t - t_m) - m\chi_v(t - t_m)\chi_x(t - t_m) - \kappa\chi_x(t - t_m)\chi(t - t_m) \Big] + \\ &- \kappa \langle \Delta^2 x_{t_m} \rangle \chi_x(t - t_m)(1 - \kappa\chi(t - t_m)) + \\ &+ m^2 \langle \Delta^2 v_{t_m} \rangle \chi_x(t - t_m)\chi_v(t - t_m) + \\ &+ m\mathrm{Cov}(x_{t_m}, v_{t_m}) \big(\chi_v(t - t_m)(1 - \kappa\chi(t - t_m)) - \kappa\chi_x^2(t - t_m) \big) \,, \end{aligned} \tag{13}$$

with initial variances $\langle \Delta^2 x_{t_m} \rangle$, $\langle \Delta^2 v_{t_m} \rangle$ and covariance $\mathrm{Cov}(x_{t_m}, v_{t_m})$.

3. General result

In this paper we will use the DS approach [23] (see below) to obtain new stochastic inequalities involving average and variance of generic observables $R(x_t, v_t)$ depending on position and velocity. This is done by performing a perturbation dependent on a small parameter α such that $P\left(x_t, v_t, t|t_m\right) \to P^\alpha\left(x_t, v_t, t|t_m\right)$ and $\langle R\rangle_{t_m,t} = \int \mathrm{d}x R\left(x_t, v_t\right) P\left(x_t, v_t, t|t_m\right) \to \langle R\rangle^\alpha_{t_m,t} = \int \mathrm{d}x R\left(x_t, v_t\right) P^\alpha\left(x_t, v_t, t|t_m\right)$. For $\alpha \approx 0$, to leading order it holds that

$$\frac{\left(\left\langle R\right\rangle^\alpha_{t_m,t} - \left\langle R\right\rangle_{t_m,t}\right)^2}{\left\langle\Delta^2 R\right\rangle_{t_m,t}} \leq 2\mathbb{K}^\alpha_t(t|t_m)\,, \tag{14}$$

where there appears the Kullback-Leibler (KL) divergence between P and P^α,

$$\mathbb{K}^\alpha_t(t|t_m) \equiv \int \mathrm{d}x_t \mathrm{d}v_t P^\alpha\left(x_t, v_t, t|t_m\right) \ln\left[\frac{P^\alpha\left(x_t, v_t, t|t_m\right)}{P\left(x_t, v_t, t|t_m\right)}\right]\,. \tag{15}$$

As in a previous study [31], we now choose an α-dependent perturbation that maps to an ensemble at a time rescaled by $1+\alpha$, namely

$$P^\alpha(x_t, v_t, t|t_m) = P\left(x_t, v_t, (1+\alpha)\,t|t_m\right) \implies \langle R\rangle^\alpha_{t_m,t} = \langle R\rangle_{t_m,(1+\alpha)t} \approx \langle R\rangle_{t_m,t} + \alpha t\langle\dot{R}\rangle_{t_m,t} \tag{16}$$

so that equation (14) becomes (neglecting orders higher than α^2)

$$\frac{\left(\alpha t\left\langle\dot{R}\right\rangle_{t_m,t}\right)^2}{\left\langle\Delta^2 R\right\rangle_{t_m,t}} \leq 2\mathbb{K}^\alpha_t(t|t_m)\,. \tag{17}$$

Since the KL divergence (15) is at its minimum when $\alpha = 0$, for small α one can further approximate it as

$$\mathbb{K}^\alpha_t(t|t_m) = \frac{\alpha^2 t^2}{2}\mathcal{I}(t|t_m)\,, \tag{18}$$

where $\mathcal{I}(t|t_m)$ is the Fisher information, i.e., the concavity of the KL divergence evaluated at its minimum. This allows us to rewrite (17) as

$$\frac{\left\langle\dot{R}\right\rangle^2_{t_m,t}}{\left\langle\Delta^2 R\right\rangle_{t_m,t}} \leq \mathcal{I}(t|t_m)\,, \tag{19}$$

that is a form of the generalised Cramer-Rao bound, see [19] for an extended discussion. Moreover, in this article, the authors Ito and Dechant find the expression for Fisher information associated to the perturbation shown above, that is

$$\mathcal{I}\left(t|t_m\right) = \left(\langle\dot{x}\rangle_{t_m,t}, \langle\dot{v}\rangle_{t_m,t}\right)^T \mathcal{S}^{-1}_{t_m,t}\left(\langle\dot{x}\rangle_{t_m,t}, \langle\dot{v}\rangle_{t_m,t}\right) + \frac{1}{2}\mathrm{tr}\left(\mathcal{S}^{-1}_{t_m,t}\dot{\mathcal{S}}_{t_m,t}\mathcal{S}^{-1}_{t_m,t}\dot{\mathcal{S}}_{t_m,t}\right)\,, \tag{20}$$

where, as usual, the dot superscript stays for time derivative. In the next subsections we will calculate explicit expressions for the Fisher information starting from different initial conditions and eventually show their connection with the entropy production rate. To do so, we will use expressions from [48] for the entropy production rate of the system ($\langle\sigma_{\rm sys}\rangle_{t_m,t}$), of the environment ($\langle\sigma_{\rm med}\rangle_{t,t_m}$), and the total one ($\langle\sigma_{\rm tot}\rangle_{t_m,t}$), all valid for systems described by a Gaussian PDF

$$\langle\sigma_{\rm sys}\rangle_{t_m,t} = \frac{\partial_t|\mathcal{S}_{t_m,t}|}{2|\mathcal{S}_{t_m,t}|}\,, \tag{21}$$

$$\langle\sigma_{\rm med}\rangle_{t_m,t} = \beta\widehat{\gamma}(t-t_m)\langle v\rangle_{t_m,t}\langle v_{\rm ret}\rangle_{t_m,t} - \frac{\beta\kappa}{2}\partial_t\langle\Delta^2 x\rangle_{t_m,t} - \frac{\beta m}{2}\partial_t\langle\Delta^2 v\rangle_{t_m,t}\,, \tag{22}$$

$$\langle\sigma_{\rm tot}\rangle_{t_m,t} = \langle\sigma_{\rm sys}\rangle_{t_m,t} + \langle\sigma_{\rm med}\rangle_{t_m,t}\,, \tag{23}$$

where $|\mathcal{S}_{t_m,t}|$ is the determinant of the covariance matrix and

$$\widehat{\gamma}(t) \equiv \int_0^t \Gamma(t')\mathrm{d}t'\,, \tag{24}$$

$$\widehat{\gamma} \equiv \lim_{t\to\infty}\widehat{\gamma}(t) = \int_0^\infty \Gamma(t')\mathrm{d}t' < \infty\,, \tag{25}$$

$$\langle v_{\rm ret}\rangle_{t_m,t} \equiv \frac{1}{\widehat{\gamma}(t-t_m)}\int_0^{t-t_m}\langle v\rangle_{t_m,t-t'}\Gamma(t')\mathrm{d}t' \tag{26}$$

are respectively the effective time dependent friction coefficient, its long time limit and the so called *retarded velocity* (see [48]).

To sum up, the inequality defined by (19) and (20) is an instantaneous nonequilibrium uncertainty relation for processes with a Gaussian distribution and following a GLE with memory. Of course, this formula works also for Markov dynamics, see again [19]. By instantaneous we mean that both the observable R and the cost function on the right hand side are quantities that depend only on the (PDF of the) position and velocity at time t. Moreover, the Fisher information can be related to entropy production rates in some cases discussed in the following section.

3.1. Particle confined by a harmonic trap

For an active harmonic trap, we focus on two interesting regimes where the covariance matrix is constant and diagonal: the case $t_m \to -\infty$ and a dynamics starting from thermodynamic equilibrium. For this cases we find that

$$\mathcal{S}_t^{eq} = \mathcal{S}_{-\infty,t} = \begin{pmatrix} \frac{k_BT}{\kappa} & 0 \\ 0 & \frac{k_BT}{m}\end{pmatrix}\,. \tag{27}$$

This is obtained by using equations (11), (12) and (13) along with the limits of the susceptibilities calculated in Appendix A. Indeed, since κ is not modulated and since the force $f(t)$ is space-independent, we have that the covariance matrix remains constant.

In the first case, this *steady state* for the covariance matrix is achieved by starting from $t_m \to -\infty$. To justify terming steady state such regime, we anticipate that we will illustrate it for a particle that is being dragged since a long time by a trap moving at constant velocity. However, the results below hold also for a more complex scenario with general $\lambda(t)$ and $f(t)$.

Using again the limits of the susceptibilities, in particular using that that $\lim_{t\to\infty}\chi_v(t)=0$, $\lim_{t\to\infty}\chi_x(t)=0$ and $\lim_{t\to\infty}\chi(t)=1/\kappa$, from (9) and (10) we obtain that

$$\langle x\rangle_t^{-\infty}=\int_{-\infty}^{t}\chi_x(t-t')F(t')\mathrm{d}t'\,, \tag{28}$$

$$\langle v\rangle_t^{-\infty}=\chi_x(0)F(t)+\int_{-\infty}^{t}\chi_v(t-t')F(t')\mathrm{d}t'\,. \tag{29}$$

where $\chi_x(0)\neq 0$ only for overdamped dynamics. Moreover, the notation $\langle\ldots\rangle_t^{-\infty}$ denotes an average obtained for $t_m\to-\infty$. The asymptotic decay of the position susceptibility $\lim_{t\to\infty}\chi_x(t)=0$ is expected in a constrained system (this will not be the case for $\kappa=0$).

In the second case, equipartition as in (27) holds because we start at $t_m=0$ from an equilibrium distribution under the potential $\frac{\kappa}{2}x^2$. This implies that $\langle v_0\rangle=\langle x_0\rangle=0$, $\langle\Delta^2 v_0\rangle=k_BT/m$, $\langle\Delta^2 x_0\rangle=k_BT/\kappa$ and $\mathrm{Cov}(x_{t_0},v_{t_0})=0$, so that the covariance matrix remains constant for all $t\geq 0$ (this can be checked by plugging the parameters just listed into equations (11), (12) and (13)). Moreover, it can be readily seen that

$$\langle x\rangle_t^{eq}=\int_{0}^{t}\chi_x(t-t')F(t')\mathrm{d}t'\,, \tag{30}$$

$$\langle v\rangle_t^{eq}=\chi_x(0)F(t)+\int_{0}^{t}\chi_v(t-t')F(t')\mathrm{d}t'\,. \tag{31}$$

For these two cases of confined particle, the estimates for the average of position and velocity and for the covariance matrix along with (19) and (20) lead to the uncertainty relation

$$\mathrm{g}_{R,t_m}^{\mathrm{trap}}(t)\leq\mathcal{C}_{t_m}^{\mathrm{trap}}(t) \tag{32}$$

with

$$\begin{aligned}\mathrm{g}_{R,t_m}^{\mathrm{trap}}(t)&\equiv\frac{\left\langle\dot{R}\right\rangle_{t_m,t}^2}{\left\langle\Delta^2 R\right\rangle_{t_m,t}}\\ \mathcal{C}_{x,v,t_m}^{\mathrm{trap}}(t)&\equiv\frac{\kappa\langle\dot{x}\rangle_{t_m,t}^2+m\langle\dot{v}\rangle_{t_m,t}^2}{k_BT}\,,\end{aligned} \tag{33}$$

where t_m is either 0 or $-\infty$, $\mathcal{C}_{t_m}^{\mathrm{trap}}(t)$ is the cost function and $\mathrm{g}_{R,t_m}^{\mathrm{trap}}(t)$ is essentially a (squared) signal-to-noise ratio (SNR), a quantity that encodes the precision associated to the observable $R(x_t,v_t)$. Instead, if we consider the position PDF $P(x_t,t|t_m)$, obtained from the marginalisation of $P(x_t,v_t,t|t_m)$ with respect to v_t, we obtain another expression for the cost function $\mathcal{C}_{t_m}^{\mathrm{trap}}(t)$ from the Fisher information, i.e.

$$\mathcal{C}_{x,t_m}^{\mathrm{trap}}(t)\equiv\frac{\kappa\langle\dot{x}\rangle_{t_m,t}^2}{k_BT}\,, \tag{34}$$

which is also valid for overdamped dynamics, where $m = 0$. In this case, the bound is only valid for observables that depend solely on x_t, i.e. $R(x_t)$.

We have previously shown in [48] that, if the memory kernel is integrable as in (25), for large observation times the entropy production rate of the system becomes

$$\lim_{t\to\infty}\langle\sigma_{\rm tot}\rangle_{t_m,t} = \lim_{t\to\infty}\langle\sigma_{\rm med}\rangle_{t_m,t} = \frac{\widehat{\gamma}}{k_B T}\left(\lim_{t\to\infty}\langle\dot{x}\rangle_{t_m,t}\right)^2 . \tag{35}$$

In this limit equation (32) can be rewritten as

$$\lim_{t\to\infty} {\rm g}^{\rm trap}_{R,t_m}(t) \le \frac{\kappa}{\widehat{\gamma}}\langle\sigma_{\rm tot}\rangle_{t_m,t} + \frac{m\langle\dot{v}\rangle^2_{t_m,t}}{k_B T} . \tag{36}$$

This means that for long times, instantaneous observables $R(x_t, v_t)$ have a SNR ${\rm g}^{\rm trap}_{R,t_m}(t)$ bounded from above by the mean total entropy production rate times a ratio of the trap strength by the low-frequency damping coefficient, plus a positive term dependent on the average acceleration of the particle. Instead, for observables dependent solely on the position x_t or overdamped systems, the last term on the r.h.s. is not present and the bound has a purely entropic interpretation for large observation times. Again, as noted in [32], the dependence on odd variables, such as v_t, of the observables considered in the SNR ratio generates a new term in the cost function in addition to entropy production. However, for non-oscillating external forces such as $F(t) \propto t^n$ or more specifically $\lambda(t) = vt$, it holds that $\langle\dot{v}\rangle_t/\langle\dot{x}\rangle_t \to 0$ as $t\to\infty$ and hence, in this limit, the entropy production rate becomes the dominant component of the cost function in (36).

As an example, we consider a particularly interesting regime that can be achieved by choosing $\lambda(t) = vt$ and by sending $t_m \to -\infty$. Again in [48], we show that this can be considered as a steady state for which

$$\langle x\rangle^{ss}_t = vt - \frac{\widehat{\gamma}v}{\kappa}, \qquad \langle v\rangle^{ss}_t = \langle v_{\rm ret}\rangle^{ss}_t = v, \qquad \langle\sigma_{\rm tot}\rangle^{ss}_t = \widehat{\gamma}v^2 t . \tag{37}$$

Hence, in this case, we may express the cost function in terms of the instantaneous mean entropy production rate times $\kappa/\hat{\gamma}$ for all times and all observables $R(x_t, v_t)$, i.e.

$${\rm g}^{ss,{\rm trap}}_R(t) \le \frac{\kappa}{\widehat{\gamma}}\langle\sigma_{\rm tot}\rangle^{ss}_t . \tag{38}$$

Finally, in the Markovian (mk) case, inequality (36) is valid for every t_m and t. Indeed in this case

$$\Gamma^{\rm mk}(t) = 2\gamma_0\delta(t), \qquad \widehat{\gamma}^{\rm mk}(t) = \int_0^t \Gamma^{\rm mk}(t){\rm d}t' = \gamma_0, \qquad \langle v_{\rm ret}\rangle^{\rm mk}_{t_m,t} = \langle v\rangle^{\rm mk}_{t_m,t} \tag{39}$$

and from equation (23) we get

$$\frac{\kappa}{\widehat{\gamma}}\langle\sigma\rangle^{\rm mk}_{t_m,t} = \frac{\kappa}{k_B T}(\langle\dot{x}\rangle^{\rm mk}_{t_m,t})^2 , \tag{40}$$

$$\mathcal{C}^{\rm mk,trap}_{x,v,t_m}(t) = \frac{\kappa}{\gamma_0}\langle\sigma\rangle^{\rm mk}_{t_m,t} + \frac{m}{k_B T}(\langle\dot{v}\rangle^{\rm mk}_{t_m,t})^2 , \tag{41}$$

$$\mathcal{C}^{\mathrm{mk,trap}}_{x,t_m}(t) = \frac{\kappa}{\gamma_0}\langle\sigma\rangle^{\mathrm{mk}}_{t_m,t}\,. \tag{42}$$

In the last line we noted that, for Markovian dynamics and observables that depend only on x_t as well as for overdamped dynamics, the cost function is always proportional to the entropy production rate.

3.2. Particle not confined

When no confinement is present ($\kappa = 0$), the only way to drive the system out of equilibrium is through the space-independent force $f(t)$. We again analyse two situations.

First, we consider an initial distribution that can be factorised as $P(x_0, v_0) = \delta(x_0 - \widetilde{x}_0)P^{eq}(v_0)$ (a Dirac delta is a limit of a Gaussian), with

$$P^{eq}(v_0) = \sqrt{\frac{m}{2\pi k_B T}}\exp\left[-\frac{mv_0^2}{2k_B T}\right] \tag{43}$$

and meaning that $\langle x_0\rangle = 0$, $\langle v_0\rangle = 0$, $\langle\Delta^2 x_0\rangle = 0$, $\langle\Delta^2 v_0\rangle = k_B T/m$ and $\mathrm{Cov}(x_{t_0}, v_{t_0}) = 0$. With this initial conditions we find that

$$\langle x\rangle^{dd}_{x_0,t} = \widetilde{x}_0 + \int_0^t \chi_x(t-t')f(t')\mathrm{d}t'\,, \quad \langle v\rangle^{dd}_t = \chi_x(0)f(t) + \int_0^t \chi_v(t-t')f(t')\mathrm{d}t'\,, \tag{44}$$

$$\mathcal{S}_t = \begin{pmatrix} 2k_B T\chi(t) & k_B T\chi_x(t) \\ k_B T\chi_x(t) & k_B T/m \end{pmatrix}\,. \tag{45}$$

Experimentally, this can obtained by selecting any occurrence where $x(t) = \widetilde{x}_0$ and use it as an initial point for the future dynamics. Note that this kind of initial distribution could have been also used for the trapped case but we simply chose not to consider it.

Otherwise, we can prepare the system in an initial equilibrium distribution with an optical trap, say with stiffness κ', and switch it off when the external force is turned on. This would correspond to $\langle v_0\rangle = \langle x_0\rangle = 0$, $\langle\Delta^2 v_0\rangle = k_B T/m$, $\langle\Delta^2 x_0\rangle = k_B T/\kappa'$ and $\mathrm{Cov}(x_{t_0}, v_{t_0}) = 0$, so that

$$\langle x\rangle^{\kappa'}_{x_0,t} = \int_0^t \chi_x(t-t')f(t')\mathrm{d}t'\,, \qquad \langle v\rangle^{\kappa'}_t = \chi_x(0)f(t) + \int_0^t \chi_v(t-t')f(t')\mathrm{d}t'\,, \tag{46}$$

$$\mathcal{S}_t = \begin{pmatrix} k_B T(2\chi(t) + 1/\kappa') & k_B T\chi_x(t) \\ k_B T\chi_x(t) & k_B T/m \end{pmatrix}\,. \tag{47}$$

In both cases, using the limits of the susceptibilities for the diffusive scenario, one sees that the long time limit of the covariance matrix reads

$$\lim_{t\to\infty}\mathcal{S}_t = \begin{pmatrix} 2k_B Tt/\gamma & k_B T/\gamma \\ k_B T/\gamma & k_B T/m \end{pmatrix}\,. \tag{48}$$

In all the cases shown above, the averages of position and velocity along with the covariance matrix fully characterise the joint PDF of position and velocity, as its Gaussian character is preserved by construction.

Moreover, for both scenarios we can effectively write the covariance matrix, its time derivative and its inverse as

$$\mathcal{S}_t = k_B T \begin{pmatrix} 2\chi(t) + \langle \Delta^2 x_0 \rangle / k_B T & \chi_x(t) \\ \chi_x(t) & 1/m \end{pmatrix}, \tag{49}$$

$$\dot{\mathcal{S}}_t = k_B T \begin{pmatrix} 2\chi_x(t) & \chi_v(t) \\ \chi_v(t) & 0 \end{pmatrix}, \tag{50}$$

$$\mathcal{S}_t^{-1} = \frac{1}{k_B T |\widetilde{\mathcal{S}}_{t_m,t}|} \begin{pmatrix} 1 & -m\chi_x(t) \\ -m\chi_x(t) & 2m\chi(t) + m\langle \Delta^2 x_0 \rangle / k_B T \end{pmatrix}, \tag{51}$$

where

$$|\widetilde{\mathcal{S}}_t| = \frac{m}{(k_B T)^2} |\mathcal{S}_t| = 2\chi(t) + \langle \Delta^2 x_0 \rangle / k_B T - m\chi_x^2(t) \tag{52}$$

and $\langle \Delta^2 x_0 \rangle = 0$ or $\langle \Delta^2 x_0 \rangle = k_B T / \kappa'$ depending on the case considered.

By using equations (49), (50) and (51) along with the generalised Cramér-Rao bound (19) and the expression of the Fisher matrix for Gaussian PDFs (20) we get

$$\mathrm{g}_R^{\mathrm{diff}}(t) \leq \mathcal{C}^{\mathrm{diff}}(t) \tag{53}$$

with

$$\begin{aligned} \mathrm{g}_R^{\mathrm{diff}}(t) &\equiv \frac{\langle \dot{R} \rangle_t^2}{\langle \Delta^2 R \rangle_t / t} \\ \frac{\mathcal{C}_{x,v}^{\mathrm{diff}}(t)}{t} &\equiv \frac{1}{k_B T |\widetilde{\mathcal{S}}_t|} \left(\langle \dot{x} \rangle_t^2 + \frac{m \langle \Delta^2 x \rangle_t}{k_B T} \langle \dot{v} \rangle_t^2 - 2m\chi_x(t) \langle \dot{x} \rangle_t \langle \dot{v} \rangle_t \right) + \Phi(t) \end{aligned} \tag{54}$$

$$\Phi(t) \equiv \frac{1}{2} \mathrm{tr} \left(\mathcal{S}_{t_m,t}^{-1} \dot{\mathcal{S}}_{t_m,t} \mathcal{S}_{t_m,t}^{-1} \dot{\mathcal{S}}_{t_m,t} \right) = \frac{2\chi_x^2(t)}{|\widetilde{\mathcal{S}}_t|^2} \Big(1 - m\chi_v(t) \Big(1 - m\chi_v(t)/2 \Big) \Big) \xrightarrow{t\to\infty} \frac{\chi_x(t)}{2t\chi(t)} \tag{55}$$

where we used that $\langle \Delta^2 x \rangle_t = 2 k_B T \chi(t) + \langle \Delta^2 x_0 \rangle$ and that for free diffusion $\lim_{t\to\infty} \chi(t) = t/\widehat{\gamma}$, $\lim_{t\to\infty} \chi_x(t) = 1/\widehat{\gamma}$ and $\lim_{t\to\infty} \chi_v(t) = 0$. For the case of observables depending only on x_t, as well as for overdamped dynamics, it can be shown that the cost function reduces to

$$\frac{\mathcal{C}_x^{\mathrm{diff}}(t)}{t} \equiv \frac{\langle \dot{x} \rangle_t^2}{\langle \Delta^2 x \rangle_t} + \frac{1}{2} \left(\frac{\partial_t \langle \Delta^2 x \rangle_t}{\langle \Delta^2 x \rangle_t} \right)^2 = \frac{\langle \dot{x} \rangle_t^2}{\langle \Delta^2 x \rangle_t} + 2 \left(\frac{\chi_x(t)}{2\chi(t) + \langle \Delta^2 x_0 \rangle} \right)^2. \tag{56}$$

Note that the SNR $\mathrm{g}_R^{\mathrm{diff}}(t)$ and the cost function $\mathcal{C}^{\mathrm{diff}}(t)$ for this diffusive case are defined differently from (33). Now there is an additional factor t to let these quantities converge to a constant value at large observation times.

Finally, we consider the entropy production rate of the system

$$\langle\sigma_{\mathrm{sys}}\rangle_{t_m,t} = \frac{\partial_t|\mathcal{S}_{t_m,t}|}{2|\mathcal{S}_{t_m,t}|} = \frac{\chi_x(t)(1-m\chi_v(t))}{2\chi(t)+\langle\Delta^2 x_0\rangle/k_BT - m\chi_x^2(t)} \xrightarrow{t\to\infty} \frac{\chi_x(t)}{2\chi(t)} \,. \tag{57}$$

Except for a factor $1/t$, we recover the same limit as for (55) as well as for the second term on the right hand side of (56). Moreover, if again we consider external forces that have a polynomial asymptotic behaviour $f(t)\propto t^n$, the term involving the squared derivative of the position in $\mathcal{C}^{\mathrm{diff}}_{x,v}(t)$ becomes dominant. As a consequence, using the expressions for the entropy production rate of the environment (22) as well as its long time limit (35), we get

$$\lim_{t\to\infty}\mathcal{C}^{\mathrm{diff}}(t) = \frac{\widehat{\gamma}\langle\dot{x}\rangle_t^2}{2k_BT} + \frac{\chi_x(t)}{2\chi(t)} = \frac{1}{2}(\langle\sigma_{\mathrm{tot}}\rangle_t + \langle\sigma_{\mathrm{sys}}\rangle_t) \,. \tag{58}$$

However, in general the term proportional to the squared acceleration may be not negligible. This is the case for example for an oscillating external force as $f(t) = A\sin(\omega t)$. Nevertheless, for overdamped dynamics ($m=0$) and for observables only dependent on x_t, the limit (58) holds for all $f(t)$. Hence, again the presence of the odd variable v_t in the PDF modifies the interpretation of the cost function in the large time limit.

Finally, it worth considering the two paradigmatic cases where

(a) $f(t)=0$, corresponding to $\langle\dot{x}\rangle_t = 0$ and $\langle\dot{v}\rangle_t = 0$ so that

$$\mathcal{C}^{\mathrm{diff}}(t) = t\Phi(t) \xrightarrow{t\to\infty} \langle\sigma_{\mathrm{sys}}\rangle_t = \langle\sigma_{\mathrm{tot}}\rangle_t \,, \tag{59}$$

(b) $f(t)=f$, corresponding to $\langle\dot{x}\rangle_t = f\chi_x(t)$ and $\langle\dot{v}\rangle_t = f\chi_v(t)$ and implying that

$$\mathcal{C}^{\mathrm{diff}}(t) \xrightarrow{t\to\infty} \frac{\widehat{\gamma}\langle\dot{x}\rangle_t^2}{2k_BT} = \frac{1}{2}\langle\sigma_{\mathrm{med}}\rangle_t = \frac{1}{2}\langle\sigma_{\mathrm{tot}}\rangle_t \,, \tag{60}$$

where we used that $\langle\sigma_{\mathrm{sys}}\rangle_t \propto t^{-1}$ as $t\to\infty$

We thus find different behaviours for the cost function in the large time limit depending on the external forces considered. While in all cases we recover the entropy production rate, there is a prefactor $1/2$ only for forced diffusion.

To sum up, for (i) underdamped dynamics, observables of the form $R(x_t,v_t)$ and external force with a polynomial large time limit $f(t)\propto t^n$, (ii) for underdamped dynamics and observables $R(x_t)$ and (iii) for overdamped dynamics, we recover an entropic interpretation of the cost function for the large time limit and the inequality reads

$$\lim_{t\to\infty}\mathrm{g}_R^{\mathrm{diff}}(t) \le \frac{1}{2}(\langle\sigma_{\mathrm{tot}}\rangle_t + \langle\sigma_{\mathrm{sys}}\rangle_t) \,. \tag{61}$$

We close this section by noting that the bound above becomes valid for all times in the special case of Markovian dynamics in the overdamped limit and starting from an initial distribution that is a Dirac delta for the position and an equilibrium distribution for the initial velocitiy.

3.3. Multidimensional case

In this section we explore a multidimensional generalisation of our results in which uncoupled degrees of freedom may be subject to different temperatures. The starting point is of course the multidimensional GLE for the n-dimensional vector $\bar{x}_t$ with components x^i_t defined as

$$m\ddot{x}^i(t) = -\int_{t_m}^{t} \Gamma^i(t-t')\dot{x}^i(t')\mathrm{d}t' - \partial_{x^i} U(\bar{x}_t) + f^i(t) + \eta^i(t)\,, \tag{62}$$

$$\langle \eta^i(t')\eta^j(t'')\rangle = k_B T^i \delta_{ij} \Gamma^i(|t'-t''|)\,. \tag{63}$$

We choose the external potential to have the following form

$$U(\bar{x}_t, t) = \sum_i \frac{\kappa^i}{2}(x^i(t) - \lambda^i(t)) \quad \Longrightarrow \quad \partial_{x^i} U(\bar{x}_t, t) = \kappa^i(x^i(t) - \lambda^i(t))\,. \tag{64}$$

With this choice of the confining potential the multidimensional GLE becomes

$$m\ddot{x}^i(t) = -\int_{t_m}^{t} \Gamma^i(t-t')\dot{x}^i(t')\mathrm{d}t' - \kappa^i x^i(t) + F^i(t) + \eta^i(t) \tag{65}$$

where we set $F^i(t) = \kappa^i\lambda^i(t) + f^i(t)$. By defining $\hat{\chi}_x(k) = \left[mk^2 + k\hat{\Gamma}^i(k) + \kappa^i\right]^{-1}$, one can solve (65) by noting that all n equations are uncoupled and can hence be solved independently, namely

$$x^i(t) = x^i_{t_m}\left(1 - \kappa^i\chi^i(t-t_m)\right) + m v^i_{t_m}\chi^i_x(t-t_m) + \int_{t_m}^{t} \mathrm{d}t'\chi^i_x(t-t')\left[F^i(t') + \eta^i(t')\right]\,. \tag{66}$$

The averages of position and velocity can also be readily obtained

$$\langle x^i\rangle_{t_m,t} = \langle x^i_{t_m}\rangle\left(1 - \kappa^i\chi^i(t-t_m)\right) + m\langle v^i_{t_m}\rangle\chi^i_x(t-t_m) + \int_{t_m}^{t}\mathrm{d}t'\chi^i_x(t-t')F^i(t')\,, \tag{67}$$

$$\langle v^i\rangle_{t_m,t} = -\kappa^i\langle x^i_{t_m}\rangle\chi^i_x(t-t_m) + m\langle v^i_{t_m}\rangle\chi^i_v(t-t_m) + \chi^i_x(0)F^i(t) + \int_{t_m}^{t}\mathrm{d}t'\chi^i_v(t-t')F^i(t')\,, \tag{68}$$

while the components of the $2n \times 2n$ multidimensional covariance matrix

$$\mathcal{S}_{t_m,t} = \begin{pmatrix} \mathrm{Cov}_{t_m}(x^i_t, x^j_t) & \mathrm{Cov}_{t_m}(x^i_t, v^j_t) \\ \mathrm{Cov}^T_{t_m}(x^i_t, v^j_t) & \mathrm{Cov}_{t_m}(v^i_t, v^j_t) \end{pmatrix} \tag{69}$$

are explicitly calculated in Appendix B.

In this section we will again focus on initial conditions such that the joint PDF $P(\bar{q}_t, t|t_m)$ is Gaussian, where we have set $\bar{q}_t = (\bar{x}_t, \bar{v}_t)$. In other words we are going to consider PDFs of the form

$$P(\bar{q}_t, t|t_m) = \frac{1}{\sqrt{(2\pi)^{2n}|\mathcal{S}_{t_m,t}|}} \exp\left[-\frac{1}{2}(\bar{q}_t - \langle\bar{q}\rangle_{t_m,t})^T \mathcal{S}^{-1}_{t_m,t}(\bar{q}_t - \langle\bar{q}\rangle_{t_m,t})\right]\,, \tag{70}$$

so that, for a given observable $\bar{R}(\bar{q}_t)$ with average

$$\langle \bar{R} \rangle_{t,t_m} = \int P(\bar{q}_t, t|t_m)\bar{R}(\bar{q}_t)\mathrm{d}\bar{q}_t \,, \tag{71}$$

the inequality deriving from the generalised Cramér-Rao bound reads

$$\langle \dot{\bar{R}} \rangle^T_{t,t_m} \mathcal{S}^{-1}_{\bar{R},t_m,t}(t) \langle \dot{\bar{R}} \rangle_{t,t_m} \leq \langle \dot{\bar{q}} \rangle^T_{t,t_m} \mathcal{S}^{-1}_{t,t_m} \langle \dot{\bar{q}} \rangle_{t,t_m} + \frac{1}{2}\mathrm{tr}\left(\mathcal{S}^{-1}_{t_m,t}\dot{\mathcal{S}}_{t_m,t}\mathcal{S}^{-1}_{t_m,t}\dot{\mathcal{S}}_{t_m,t}\right) . \tag{72}$$

where $\mathcal{S}^{ij}_{\bar{R},t_m,t}(t) = \langle R^i R^j \rangle_{t_m,t} - \langle R^i \rangle_{t_m,t} \langle R^j \rangle_{t_m,t}$, see again [19] for more details.

In the following we analyse this bound for the same cases of trapped dynamics and free diffusion discussed in the previous subsections.

3.3.1. Multidimensional confined particle. Here we consider the dynamics of a Brownian particle generated by a parabolic confining potential that is being dragged with a driving protocol $\bar{\lambda}(t)$, i.e.

$$U(\bar{x}_t, t) = \sum_i \frac{\kappa^i}{2}(x^i(t) - \lambda^i(t)) \tag{73}$$

with $\kappa_i \neq 0$ for every i. Moreover, remembering that for this scenario it holds that

$$\lim_{t\to\infty} \chi^i_v(t) = 0 \qquad \lim_{t\to\infty} \chi^i_x(t) = 0 \qquad \lim_{t\to\infty} \chi^i(t) = 1/\kappa^i \tag{74}$$

we immediately see from (B.6), (B.7) and (B.8) that, in the long time limit as well as for $t_m \to -\infty$, the covariance matrix becomes

$$\lim_{t-t_m\to\infty} \mathcal{S}_{t,t_m} = \lim_{t-t_m\to\infty} \begin{pmatrix} \mathrm{Cov}_{t_m}(x^i_t, x^j_t) & \mathrm{Cov}_{t_m}(x^i_t, v^j_t) \\ \mathrm{Cov}^T_{t_m}(x^i_t, v^j_t) & \mathrm{Cov}_{t_m}(v^i_t, v^j_t) \end{pmatrix} = \begin{pmatrix} k_B T^i \delta_{ij}/\kappa^i & 0 \\ 0 & k_B T^i \delta_{ij}/m \end{pmatrix} , \tag{75}$$

which is a constant matrix and also corresponds to the covariance matrix of a system that is in equilibrium. In fact, using this covariance matrix as initial condition at time t_m, we see that $\mathcal{S}_{t,t_m}$ remains constant at all times. We conclude that for the dynamics starting from an initial equilibrium condition and for a stationary state, i.e. $t_m \to -\infty$, the covariance matrix is always constant and diagonal. Moreover, from (67) and (68) we see that average positions and velocities can be in both cases effectively written as

$$\langle x^i \rangle_{t_m,t} = \int_{t_m}^t \mathrm{d}t' \chi^i_x(t-t')F^i(t') \tag{76}$$

$$\langle v^i \rangle_{t_m,t} = \chi^i_x(0)F^i(t) + \int_{t_m}^t \mathrm{d}t' \chi^i_v(t-t')F^i(t') \tag{77}$$

where $t_m = 0$ for equilibrium initial conditions or $t_m = -\infty$. Hence, for these initial setups, the inequality (72) becomes

$$\mathrm{g}^{\mathrm{trap}}_R(t) \leq \mathcal{C}^{\mathrm{trap}}(t) \,, \tag{78}$$

with

$$
\mathrm{g}_{\bar{R}}^{\mathrm{trap}}(t) = \langle \dot{\bar{R}} \rangle_{t,t_m}^T \mathcal{S}_{\bar{R},t_m}^{-1}(t) \langle \dot{\bar{R}} \rangle_{t,t_m} \,, \tag{79}
$$

$$
\begin{aligned}
\mathcal{C}_{x,v,t_m}^{\mathrm{trap}}(t) &\equiv \sum_i \frac{\kappa_i \langle \dot{x}^i \rangle_{t,t_m}^2 + m \langle \dot{v}^i \rangle_{t,t_m}^2}{k_B T_i} \,, \\
\mathcal{C}_{x,t_m}^{\mathrm{trap}}(t) &\equiv \sum_i \frac{\kappa_i \langle \dot{x}^i \rangle_{t,t_m}^2}{k_B T_i} \,,
\end{aligned} \tag{80}
$$

where the range of validity of the two cost functions was discussed in the previous subsections.

Moving forward, we consider the total entropy production rate, which coincides with the entropy production rate of the environment because of the constancy of the covariance matrix, along with its large time limit

$$
\lim_{t\to\infty} \langle \sigma_{\mathrm{tot}} \rangle_{t_m,t} = \lim_{t\to\infty} \langle \sigma_{\mathrm{med}} \rangle_{t_m,t} = \sum_i \frac{\widehat{\gamma}_i}{k_B T_i} \left(\lim_{t\to\infty} \langle v \rangle_{t_m,t} \right)^2 \,. \tag{81}
$$

We see that, differently from the unidimensional case, it is not possible to find a simple proportionality relation between the first terms on the right hand side of (80) and the long time limit of the entropy production rate. We conclude that, for this multidimensional case, the cost functions in general have no entropic interpretation for large observation times. The recovery of a cost function proportional to the entropy production rate is only possible in specific cases: for example, for a dragging vector $\bar{\lambda}(t)$ with one nonzero component, or when all ratios $\kappa_i/\hat{\gamma}_i$ are the same.

3.3.2. Multidimensional particle not confined. We conclude this section by considering the case of diffusion without confinement. The GLE associated to this scenario is as follows:

$$
m\ddot{x}^i(t) = -\int_{t_m}^t \mathrm{d}t' \Gamma^i(t-t')\dot{x}^i(t') + f^i(t) + \eta^i(t) \,. \tag{82}
$$

In a similar way as for the unidimensional case, we chose

$$
\mathcal{S}_0 = \begin{pmatrix} \mathrm{Cov}(x_0^i, x_0^j) & \mathrm{Cov}(x_t^0, v_t^0) \\ \mathrm{Cov}^T(x_0^i, v_0^j) & \mathrm{Cov}_{t_m}(v_0^i, v_0^j) \end{pmatrix} = \begin{pmatrix} \langle \Delta^2 x_0^i \rangle \delta_{ij} \,. & 0 \\ 0 & k_B T \delta_{ij}/m \end{pmatrix} \tag{83}
$$

as the covariance matrix characterising the initial state at $t=0$, with $\langle \Delta^2 x_0^i \rangle = k_B T/\kappa'^i$. This corresponds to an initial state where the particle has reached thermal equilibrium in a parabolic trap as in (73), with parameters $\{\kappa'^i\}$. The potential is then switched off at time $t=0$ when $f(t)$ is turmed on. As a consequence we have that $\langle x_0^i \rangle = 0$ and $\langle v_0^i \rangle = 0$. Moreover, in the limit of $\kappa'^i \to \infty$, the covariance matrix (83) is the same as the one of a joint PDF that is factorised as $P(\bar{x}_0, \bar{v}_0) = \prod_i \delta(x_0^i - \widetilde{x}_0^i) P^{eq}(\bar{v}_0)$. This would rather correspond to $\langle x_0^i \rangle = \widetilde{x}_0^i$ and $\langle v_0^i \rangle = 0$. Both scenarios can hence be hence effectively described by the covariance matrix

$$\mathcal{S}_t = \begin{pmatrix} \big(\langle \Delta^2 x_0\rangle + 2k_B T^i \chi^i(t)\big)\delta_{ij} & k_B T^i \chi_x^i(t)\delta_{ij} \\ k_B T^i \chi_x^i(t)\delta_{ij} & k_B T^i \delta_{ij}/m \end{pmatrix} \tag{84}$$

as well as by

$$\langle x^i\rangle_t = \widetilde{x}_0^i + \int_{t_m}^t \mathrm{d}t' \chi_x^i(t-t') f^i(t')\,, \tag{85}$$

$$\langle v^i\rangle_{t_m,t} = \chi_x^i(0) F^i(t) + \int_{t_m}^t \mathrm{d}t' \chi_v^i(t-t') f^i(t') \tag{86}$$

where $\widetilde{x}_0^i \neq 0$ only for $\{\kappa^i \to \infty\}$. With all these informations we are able to evaluate the bound (72)

$$\mathrm{g}_R^{\mathrm{diff}}(t) \le \mathcal{C}^{\mathrm{diff}}(t)\,, \tag{87}$$

with

$$\mathrm{g}_{\bar{R}}^{\mathrm{diff}}(t) = t \langle \dot{\bar{R}}\rangle_{t,t_m}^T \mathcal{S}_{\bar{R},t_m}^{-1}(t) \langle \dot{\bar{R}}\rangle_{t,t_m} \tag{88}$$

$$\frac{\mathcal{C}_{x,v,t_m}^{\mathrm{diff}}(t)}{t} \equiv \sum_i \frac{\langle \dot{x}^i\rangle_t^2 + m\langle \Delta^2 x^i\rangle_t \langle \dot{v}^i\rangle_t^2/k_B T - 2m\chi_x^i(t)\langle \dot{x}^i\rangle_t \langle \dot{v}^i\rangle_t}{k_B T^i |\widetilde{\mathcal{S}}_t^i|} + \sum_i \Phi^i(t) \tag{89}$$

$$\Phi^i(t) \equiv \frac{2\big(\chi_x^i(t)\big)^2}{|\widetilde{\mathcal{S}}_t^i|^2}\Big(1 - m\chi_v^i(t)\Big(1 - m\chi_v^i(t)/2\Big)\Big) \xrightarrow{t\to\infty} \frac{1}{2t}\sum_i \frac{\chi_x^i(t)}{\chi^i(t)} \tag{90}$$

$$|\widetilde{\mathcal{S}}_t^i| = \langle \Delta^2 x^i\rangle_t / k_B T^i - m(\chi_x^i(t))^2\,, \tag{91}$$

where again we used that $\langle \Delta^2 x^i\rangle_t = 2k_B T^i \chi^i(t) + \langle \Delta^2 x_0^i\rangle$ along with the limits of the susceptibilities.

Hence, similarly to the the confined case, there is no straightforward connection between the cost function (89) and the total entropy production (see (81)). However, if we consider free diffusion with $\bar{f}(t) = \bar{0}$, that means $\langle x^i\rangle_t = 0$ and $\langle v^i\rangle_t = 0$, and note that

$$\langle \sigma_{\mathrm{sys}}\rangle_{t_m,t} = \frac{\partial_t |\mathcal{S}_{t_m,t}|}{2|\mathcal{S}_{t_m,t}|} = \sum_i \frac{\partial_t |\widetilde{\mathcal{S}}_{t_m,t}^i|}{2|\widetilde{\mathcal{S}}_{t_m,t}^i|} = \sum_i \frac{\chi_x^i(t)(1 - m\chi_v^i(t))}{\langle \Delta^2 x^i\rangle_t / k_B T - m(\chi_x^i(t))^2} \xrightarrow{t\to\infty} \sum_i \frac{\chi_x^i(t)}{2\chi^i(t)} \tag{92}$$

it is clear that, for this special case,

$$\lim_{t\to\infty} \mathcal{C}_{x,v,t_m}^{\mathrm{diff}}(t) = \langle \sigma_{\mathrm{sys}}\rangle_{t_m,t} = \langle \sigma_{\mathrm{tot}}\rangle_{t_m,t} \tag{93}$$

Finally, it is worth noting that, as usual, for observables that depend solely on $\bar{x}_t$, i.e. $\bar{R}(\bar{x}_t)$ as well as for overdamped dynamics, the cost function has a different form

$$\mathcal{C}_{x,t_m}^{\mathrm{diff}}(t) = \sum_i \left(\frac{\langle \dot{x}^i\rangle_t^2}{\langle \Delta^2 x^i\rangle_t} + 2\left(\frac{\chi_x^i(t)}{2\chi^i(t) + \langle \Delta^2 x_0^i\rangle}\right)^2 \right)\,. \tag{94}$$

Moreover, for overdamped free diffusion and for $\langle\Delta^2 x_0^i\rangle \to 0$, i.e. $\kappa'^i \to \infty$ for every i, we get that for all times

$$\mathcal{C}^{\mathrm{diff}}_{x,t_m}(t) = \langle\sigma_{\mathrm{tot}}\rangle_{t_m,t} \tag{95}$$

4. Applications

We discuss some regimes for which it is possible to derive explicit analytical expressions for the SNRs and for the cost function or the entropy production rates. Moreover, we will focus on unidimensional underdamped dynamics and observables that depend only on spatial variables. This will allow us to observe the typical oscillations associated to the underdamped scenario and will enable us to recover an entropic interpretation of the cost functions in the large time limit.

4.1. Exponential memory kernel with confinement

For our example, we focus on a simple memory kernel with one exponential component, with GLE

$$m\ddot{x}(t) = -\gamma_0\dot{x}(t) - \frac{\gamma}{\tau}\int_{t_m}^{t} \mathrm{e}^{-(t-t')/\tau}\dot{x}(t')\mathrm{d}t' - \kappa\left[x(t) - vt\right] + \eta(t)\,, \tag{96}$$

where we set $\lambda(t) = vt$ and $f(t) = 0$. As already hinted in the previous section, with this linear dragging protocol a steady state for $t_m \to -\infty$.

Here we analyse the the bound (32) for two different observables, i.e. $R_1(x) = \mathrm{sign}(x)$ and $R_2(x) = x^2$, starting from equilibrium or from a stationary state. In the latter case, the bound becomes a full-fledged entropic bound (36).

As a first standard example for viscoelastic fluids, we analyse the case of an exponential memory kernel,

$$\Gamma^{\exp}(t) = 2\gamma_0\delta(t) + \sum_{i=1}\frac{\gamma_i}{\tau_i}\mathrm{e}^{-t/\tau_i}\,, \qquad \hat{\Gamma}^{\exp}(k) = \gamma_0 + \sum_{i=1}\frac{\gamma_i}{1+k\tau_i}\,, \tag{97}$$

with

$$\widehat{\gamma} = \int_0^\infty \Gamma(t')\mathrm{d}t' = \sum_{i=0}\gamma_i\,, \qquad 0 \le \widehat{\gamma} < \infty\,. \tag{98}$$

This is an important example, as a finite sum of suitably sized exponential terms can approximate, up to a finite time scale, every memory kernel even if $\widehat{\gamma}$ does not converge, see [50] for details.

For a memory kernel that is purely exponential, i.e. when $\gamma_0 = 0$, we note that the SNRs as well as the bounds exhibit strong oscillations when starting from an equilibrium distribution, depending of course on the values of the parameters (Figure 1). When $\gamma_0 \neq 0$ instead, these oscillations are smothered (Figure 2). No significant difference is seen instead if we start from a stationary state (thus we show only the case $\gamma_0 = 0$ in Figure 3), in fact if $t_m \to -\infty$ the memory effects are lost and the dynamics only

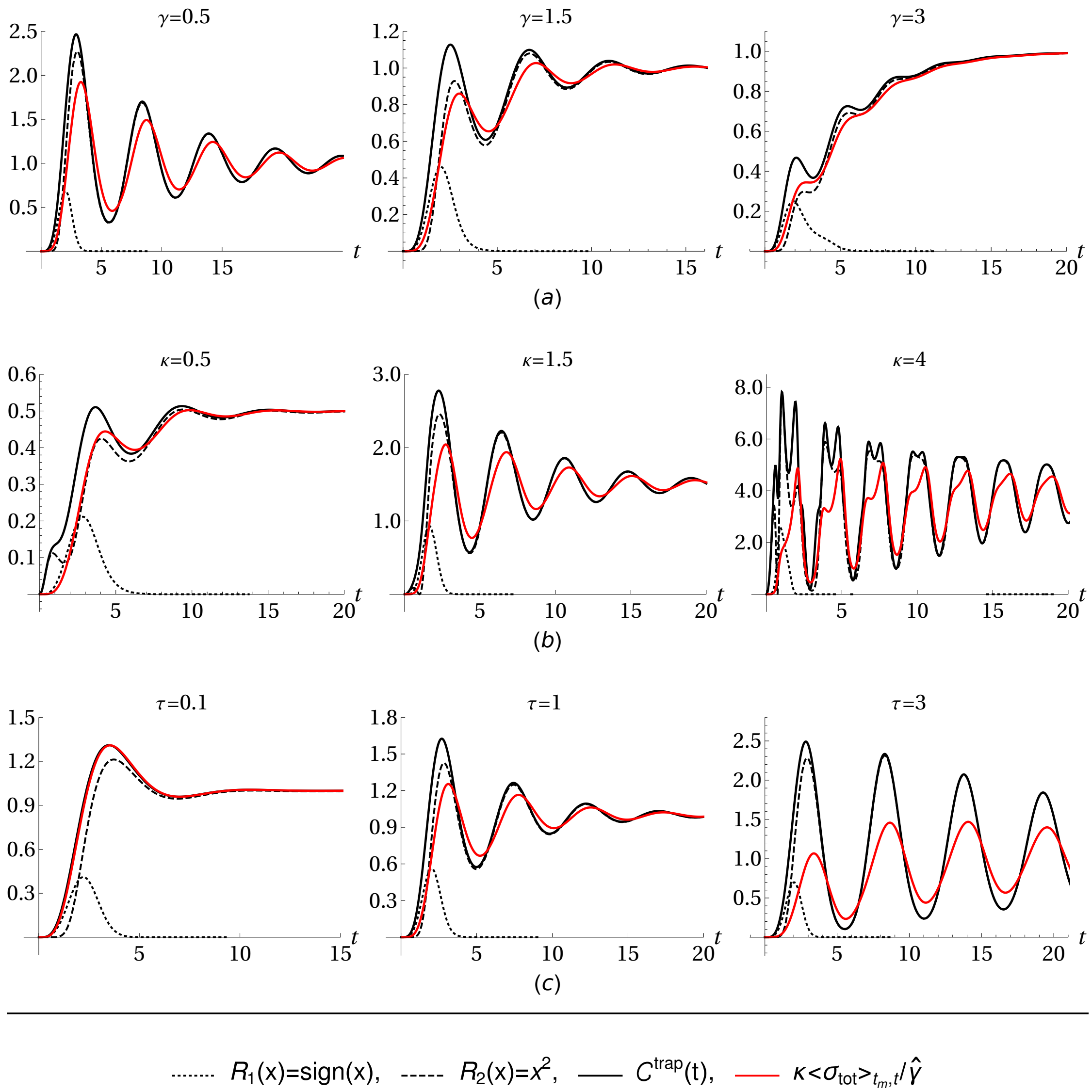

Figure 1. Upper bound (34) (dense black line) of the nonequilibrium inequality (32), and SNR ratios $g_R^{\mathrm{trap}} = \langle \dot{R} \rangle_t^2 / \langle \Delta^2 R \rangle_t$ for two observables (dashed lines, see legend), for a GLE with pure exponential kernel ($\gamma_0 = 0$) and starting from an equilibrium distribution (parameters $m = 1$, $v = 1$, $\kappa = 1$, $\gamma = 1$ and $\tau = 1$). We also show the term $\kappa \langle \sigma_{\mathrm{tot}} \rangle_t / \widehat{\gamma}$ (red line), which becomes the entropic bound for long times but in general is not a bound for the SNRs. In (a) we vary γ (quantities exhibit oscillations that become less pronounced as γ grows), in (b) we vary κ (oscillations become stronger and more persistent in time as κ becomes larger and the limit of the cost functions as well as the entropic bound grow linearly as the value of trap stiffness becomes larger), and in (c) we vary τ (note that for $\tau = 0.1$, i.e. at quasi-Markovianity, red and black continuous lines nearly coincide: this reflects the fact that for Markovian dynamics the cost function becomes proportional to the rate of entropy production).

depends on the limit of the time dependent friction coefficient $\widehat{\gamma}$, see equation (37). In other words it is not possible anymore to distinguish the effects of the exponential part

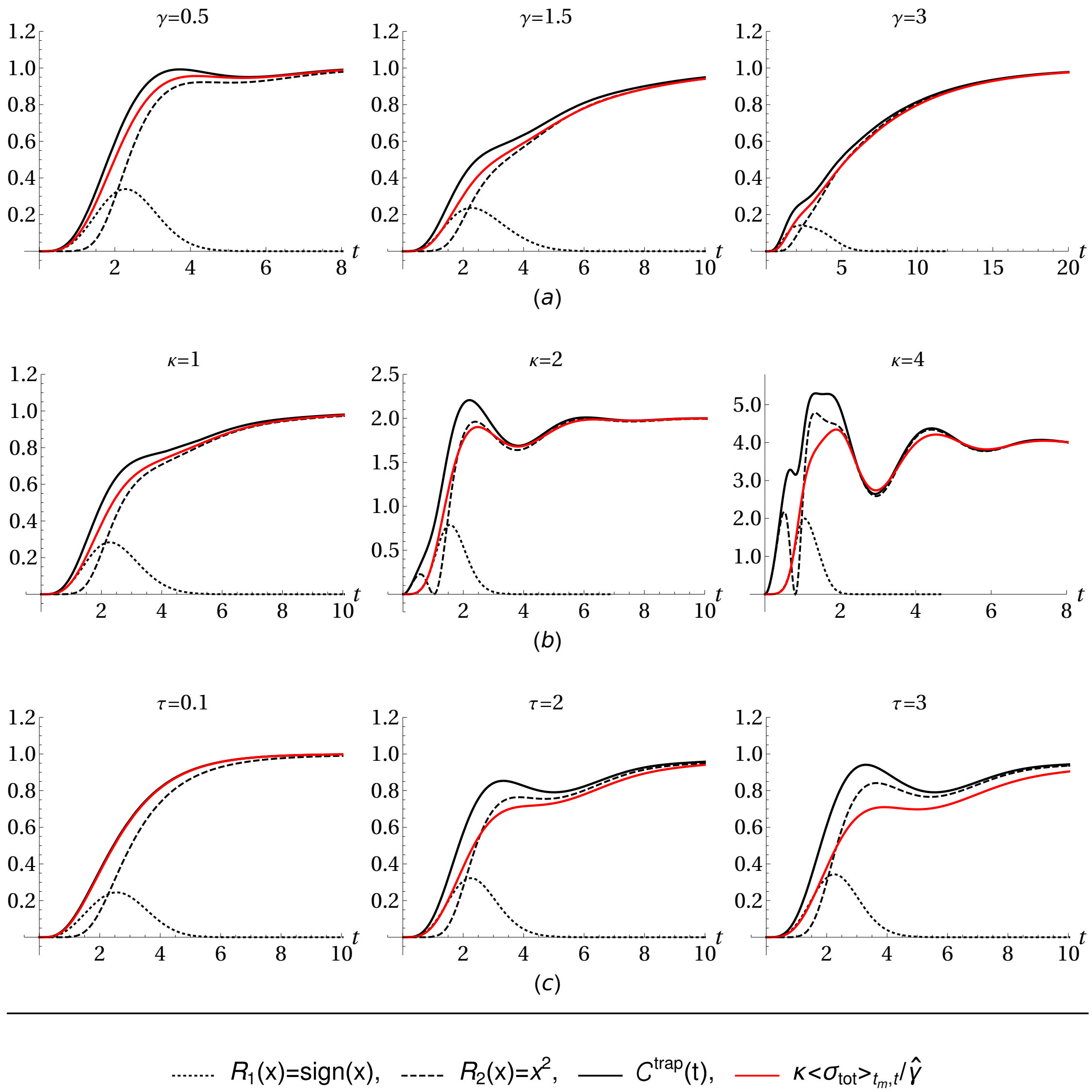

Figure 2. Similar to Figure 1 but for a complete exponential memory kernel with Markovian contribution $\gamma_0 = 0.5$ that smothers the oscillations.

of the memory kernel from the Markovian one.

4.2. Exponential memory kernel without confinement

We analyse diffusion dynamics ($\kappa = 0$) of the bead subject to an external force $f(t) = f$ that is constant both in space and time. The variance of the position grows in time, hence there exists no stationary distribution. Also the average position grows linearly in time due to f.

As in the previous subsection, for simplicity the associated GLE contains a single

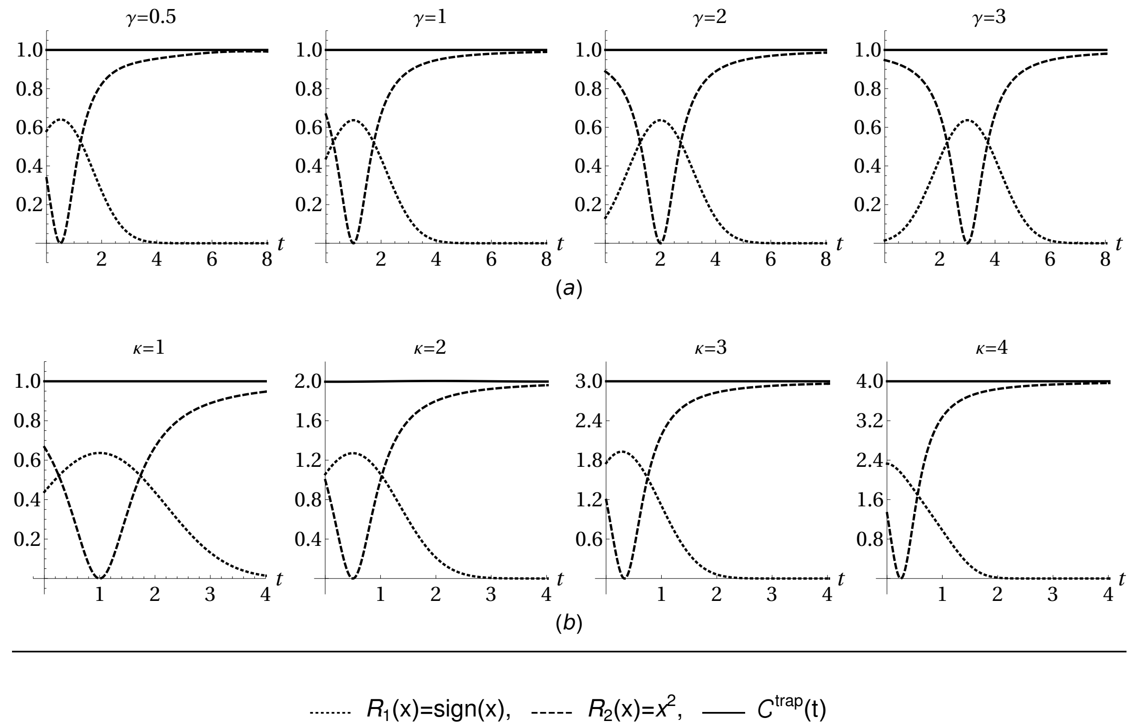

Figure 3. As in Figure 1 but for a trap dragging the particle with constant velocity (the "steady state") and passing with its minimum $\lambda(t) = vt$ at $\lambda(0) = 0$. In this case the cost function $\mathcal{C}^{\rm trap,ss}(t) = \frac{\kappa}{\gamma}\langle\sigma_{\rm tot}\rangle_t^{\rm ss} = \kappa v^2$ matches the entropic bound, proportional to the constant entropy production rate (see equation (37)). Variations of memory characteristic time are not considered as their effects are not present as $t_m \to -\infty$. (a) A larger value of γ corresponds to a shift of the minimum and maximum of the two SNRs towards larger observation times. The cost function remains unaffected by variations of γ. (b) As the trap stiffness grows, so does the cost function proportionally, while the minimum and maximum of the SNRs move towards smaller observation times.

exponential with characteristic time τ,

$$m\ddot{x}(t) = -\gamma_0\dot{x}(t) - \frac{\gamma}{\tau}\int_0^t \mathrm{e}^{-(t-t')/\tau}\dot{x}(t')\mathrm{d}t' + f + \eta(t)\,. \tag{99}$$

We thus discuss the bound (53) for the dynamics generated by the above equation, again noting the entropic nature of the bound in the large time limit. We will consider three observables: $R_1(x) = \mathrm{sign}(x)$, $R_2(x) = x^2$ and $R_3(x) = x$. The latter observable has a non-saturating SNR for this unbound diffusion dynamics.

Figure 4 and Figure 5 show the case of an initial distribution that is a Dirac delta for the starting position and an equilibrium distribution for the initial velocity, which implies $\langle\Delta^2 x_0\rangle = 0$. For small times this causes a divergence of the cost function $\mathcal{C}_x^{\rm diff}(t)$ due to its term $t(\partial_t\langle\Delta^2 x\rangle_t/\langle\Delta^2 x\rangle_t)^2$ in (56). While in this regime the bound becomes loose for $R_1(x)$ and $R_3(x)$ it is immediately saturated for $R_2(x)$.

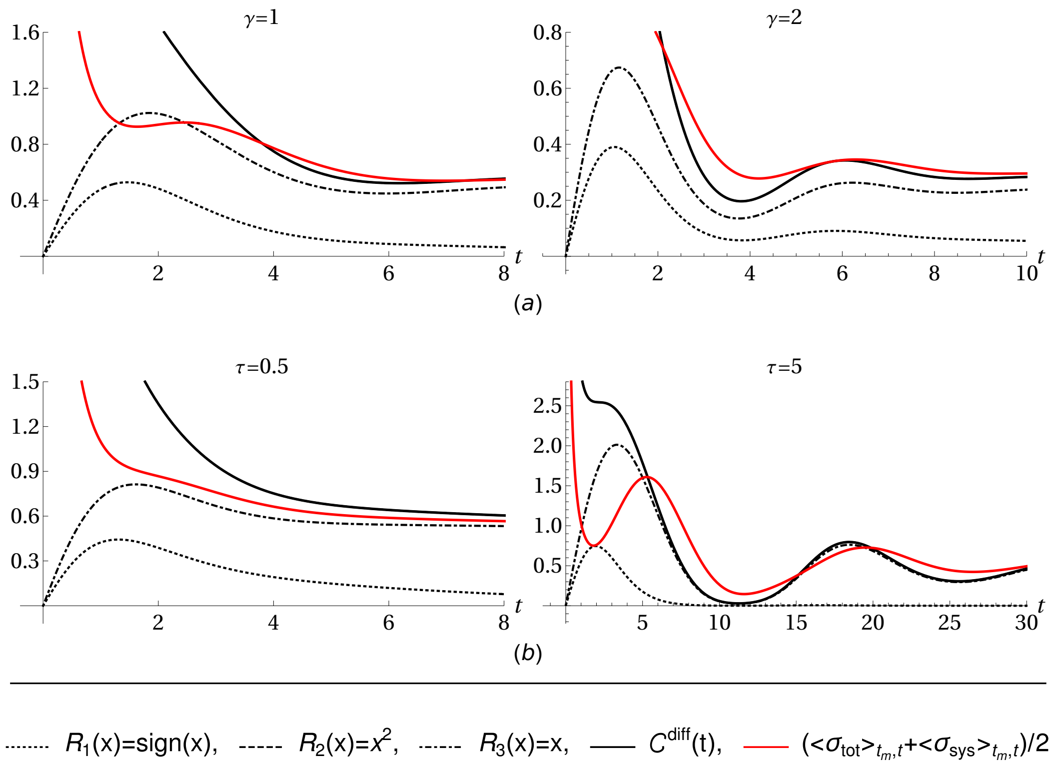

Figure 4. For free diffusion ($\kappa = 0$) under a constant force $f = 1$ from an initial distribution $P(x_{t_0}, v_{t_0}, t_0) = \delta(x - x_0)P^{eq}(v_0)$: SNRs $g_R^{\rm diff} = t\langle\dot{R}\rangle_t^2/\langle\Delta^2 R\rangle_t$ (dashed lines, see legend), cost function (dense black line) and entropic bound (red line) for pure exponential memory kernel ($\gamma_0 = 0$). As in previous figures, $m = 1$, $\gamma = 1$ and $\tau = 1$. In row (a) we vary γ. Differently from the bounded case (see Figure 1) oscillations become stronger in amplitude as γ increases while again the limit to which the cost functions and the entropy production rates does not change with γ. In (b) instead we vary τ. As before, the long time limit of the cost function is approached also by the corresponding entropic bound, while oscillations increase as the memory characteristic time gets larger. The bound is very quickly saturated for the observable $R_2(x) = x^2$, hence its SNR is not visible in the panels.

However, if the dynamics starts from an equilibrium condition in an optical trap of stiffness κ' (implying $\langle\Delta^2 x_0\rangle = k_B T/\kappa'$) no divergences occur and the bound becomes tighter for all observables. This can be all seen in Figure 6, for $\gamma_0 = 0$. The case $\gamma_0 > 0$ yields similar plots with less oscillations.

To summarise, the entropic bound is violated for finite times and is only valid asymptotically. It is perhaps surprising that the observable which goes closer to saturate the inequality is $R_2 = x^2$ instead of $R_3 = x$, which fully saturates the bound for trapped dynamics and, of course, for observables that are velocity-independent.

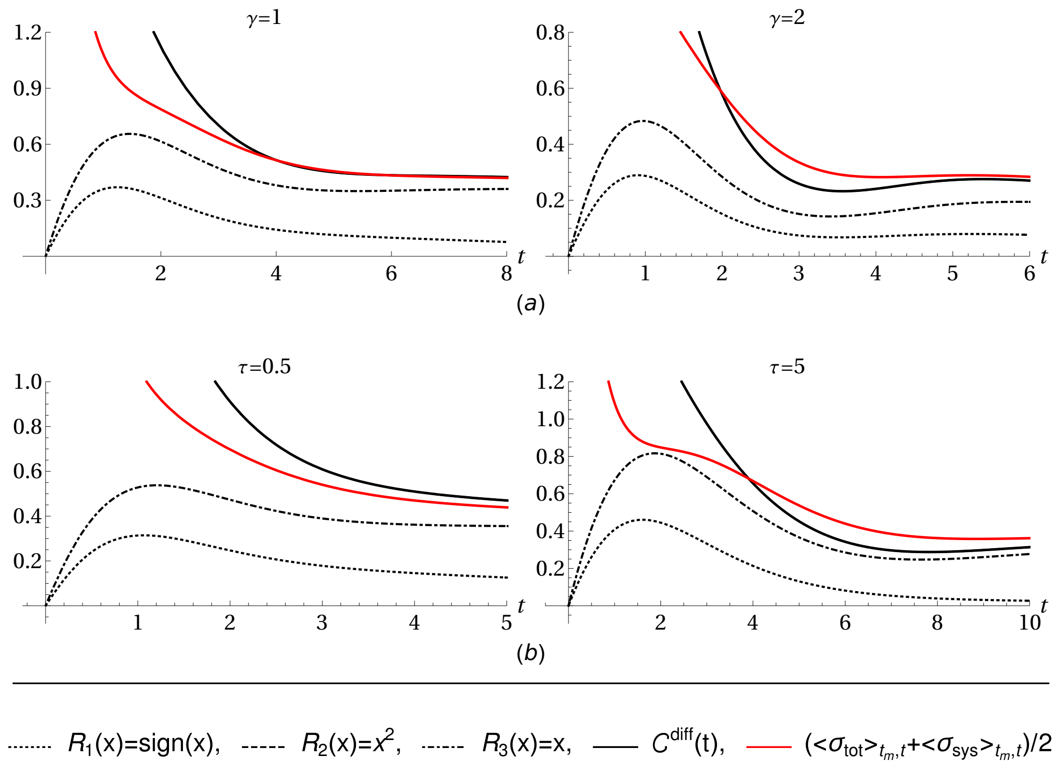

Figure 5. As in Figure 4 but for complete exponential memory kernel ($\gamma_0 = 0.5$, damping the oscillations) with initial distribution $P(x_{t_0}, v_{t_0}, t_0) = \delta(x - x_0)P^{eq}(v_0)$.

5. Conclusions

Considering a system as optical tweezers dragging a microbead in a complex fluid, we have derived nonequilibrium inequalities for Langevin equations with memory kernel, for the cases in which the position evolves distributed as a Gaussian. These inequalities cover either diffusion bounded by a harmonic trap or unbounded diffusion driven by a homogeneous time-dependent field.

The inequality (32) quantifies how the signal-to-noise ratio of observables is bounded by a cost function for a system in a moving harmonic trap. By focusing on instantaneous quantities, it is in line with a recent TUR for Markovian dynamics [24] and embodies a previous Markovian version [19]. An approach based on instantaneous quantities is a viable option for dealing with non-Markovian systems, which have more complicated path weights than those of Markovian systems.

The cost function in the inequality (32) can become proportional to the instantaneous entropy production rate in some limits. For a particle confined by a harmonic trap, (32) becomes (36) in the limit of large observation times. For the specific case of observables that are velocity-independent we recover a purely entropic interpretation of the cost function. Moreover, under the same hypothesis as above, the limit becomes an equality for all times for Markovian dynamics while for steady states

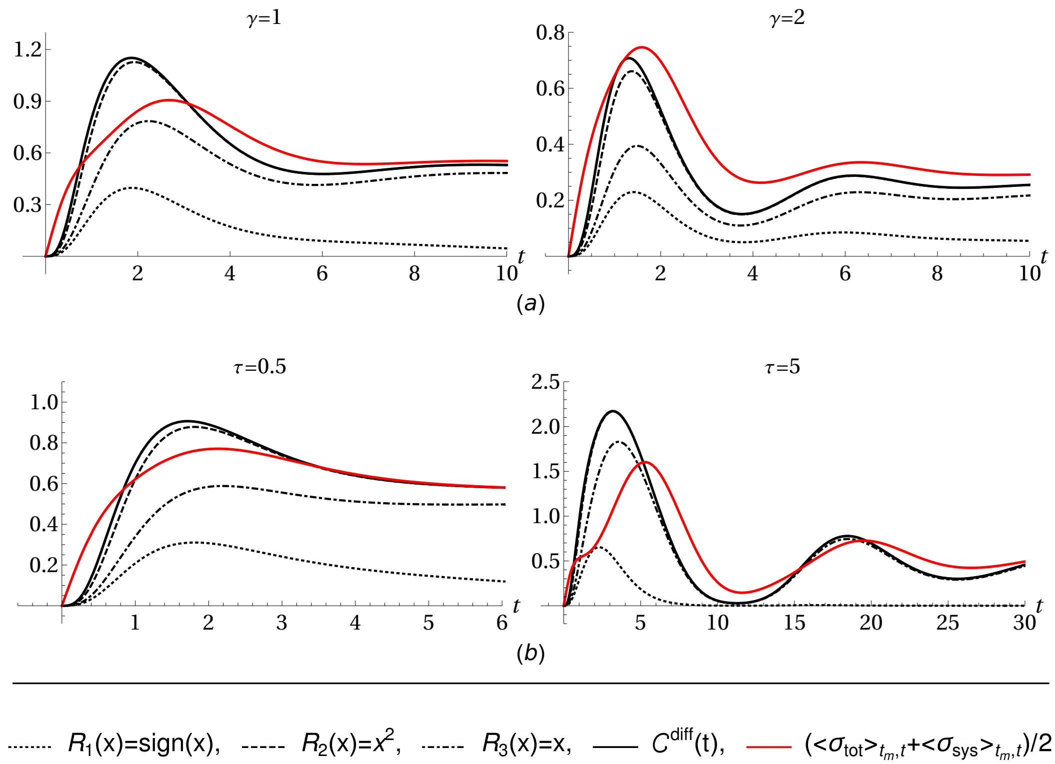

Figure 6. SNRs $g_R^{\rm diff} = t\langle \dot{R}\rangle_t^2/\langle \Delta^2 R\rangle_t$ (dashed lines, see legend), cost function (dense black line) and entropic bound (red line) for pure exponential memory kernel ($\gamma_0 = 0$) and for initial distribution $P(x_{t_0}, v_{t_0}, t_0) = P^{\kappa',eq}(v_0)P^{eq}(v_0)$ with $\langle \Delta^2 x_0\rangle = k_B T/\kappa'$. In row (a) we chose $m = 1$, $f = 1$ and $\tau = 1$. A finite initial variance of the position avoids a divergence of the cost functions. Differently from the bounded case (see Figure 1) oscillations become stronger in amplitude as γ increases while again the long time limit of cost function and entropy production rate does not change with γ. For (b) instead we have $\gamma = 1$, $m = 1$ and $f = 1$. As before, the long time limit of the cost function and rates is the same for both values of τ while oscillations increase as the memory characteristic time grows larger. Again the bound is very quickly saturated for $R_2(x)$.

generated by a linear dragging protocol $\lambda(t)$ started at $t_m \to -\infty$, it also holds for generic observables $R(x_t, v_t)$. In the multidimensional case, all these considerations are not valid in general and it is not possible to express the inequality (78) in terms of the entropy production.

For particles not constrained by optical tweezers, but eventually subject to a global force $f(t)$, the inequality reduces to (53). This may also become an instantaneous TUR with cost function proportional to some entropy production rates, see (61). For instance, for integrable memory kernels, at long times the effects of the memory are lost and essentially the system behaves as a Markovian one. Moreover, for the multidimensional case, we see that an entropic interpretation of the bound can be reached only in some specific cases such as for null external forces, see (95).

The fact that a nonequilibrium inequality contains a cost function not directly related only to the entropy production is not surprising. As discussed in the introduction, many previous examples show that other nondissipative aspects may be constraining the signal-to-noise ratio of observables, in conjunction with or in alternative to the entropy production.

However, for observables depending only on the current position, even in the underdamped case, the cost function has always a thermodynamic interpretation for long times. This result is similar to what has been seen in [25] where, for some "test" irreversible currents, the authors noted that the original TUR is still valid in the long time limit for underdamped dynamics, regardless of the expected bound for inertial systems found in [32]. Hence, it seems that the explicit appearance of odd variables, such as v_t or its odd powers, in the observables spoils the entropic interpretation of the cost functions in the long time limit, both for SNRs involving integrated currents [25] and for SNRs as those we have considered in this paper, i.e. concerning one point observables.

Our calculations were performed by remaining within the domain of Gaussian statistics. It seems interesting to check if and how one could generalise these results by preserving the scheme of observations that are local in time.

Acknowledgments

We thank Andreas Dechant, Gianmaria Falasco and Christian Maes for useful discussions.

Appendix A. Limits of susceptibilities

In this section we discuss the limits of the position susceptibility, a very useful quantity that appears throughout the whole article and defined as

$$\hat{\chi}_x(k) = [mk^2 + k\hat{\Gamma}(k) + \kappa]^{-1} \,, \tag{A.1}$$

along with the limit of its integral and derivative

$$\chi(t) \equiv \int_0^t \chi_x(t')\mathrm{d}t' \,, \qquad\qquad \chi_v(t) \equiv \partial_t \chi_x(t) \,, \tag{A.2}$$

both in the underdamped and overdamped case. To this end, we use the Tauberian theorem for Laplace transforms, which states that, for a given function $g(t)$,

$$\lim_{t\to 0} g(t) = \mathcal{L}_t^{-1}\left[\lim_{k\to\infty} \hat{g}(k)\right] \,, \qquad\qquad \lim_{t\to\infty} g(t) = \mathcal{L}_t^{-1}\left[\lim_{k\to 0} \hat{g}(k)\right] \,. \tag{A.3}$$

Furthermore, we focus on the memory kernel used in section 4, i.e.

$$\Gamma^{\exp}(t) = 2\gamma_0\delta(t) + \sum_i \frac{\gamma_i}{\tau_i}\mathrm{e}^{-t/\tau_i} \,. \tag{A.4}$$

The Laplace transform of this the memory kernel is

$$\hat{\Gamma}^{\exp}(k) = \gamma_0 + \sum_i \frac{\gamma_i}{1 + k\tau_i}\,. \tag{A.5}$$

We first consider the long time limit of the susceptibilities, distinguishing between the bounded ($\kappa \geq 0$) and free ($\kappa = 0$) case. We have

$$\begin{aligned}
\lim_{t\to\infty} \chi_x(t) &= \mathcal{L}_t^{-1}\left[\lim_{k\to 0} \frac{1}{mk^2 + k\hat{\Gamma}(k) + \kappa}\right] \approx \mathcal{L}_t^{-1}\left[\frac{1}{\kappa}\right] = \frac{2\delta(t)}{\kappa} \overset{t\to\infty}{=} 0\,,\\
\lim_{t\to\infty} \chi(t) &= \mathcal{L}_t^{-1}\left[\lim_{k\to 0} \frac{1}{k(mk^2 + k\hat{\Gamma}(k) + \kappa)}\right] \approx \mathcal{L}_t^{-1}\left[\frac{1}{k\kappa}\right] = \frac{\theta(t)}{\kappa} \overset{t\to\infty}{=} 1/\kappa\,,\\
\lim_{t\to\infty} \chi_v(t) &= 0\,,
\end{aligned} \tag{A.6}$$

where we used that $\chi(t) = \int_0^t \chi_x(t')\mathrm{d}t'$. The free case ($\kappa = 0$) instead is drastically different, in fact we get that

$$\begin{aligned}
\lim_{t\to\infty} \chi_x(t) &= \mathcal{L}_t^{-1}\left[\lim_{k\to 0} \frac{1}{mk^2 + k\hat{\Gamma}(k)}\right] \approx \mathcal{L}_t^{-1}\left[\frac{1}{k\hat{\Gamma}(0)}\right] = 1/\widehat{\gamma}\,,\\
\lim_{t\to\infty} \chi(t) &= \mathcal{L}_t^{-1}\left[\lim_{k\to 0} \frac{1}{k(mk^2 + k\hat{\Gamma}(k))}\right] \approx \mathcal{L}_t^{-1}\left[\frac{1}{k^2\hat{\Gamma}(0)}\right] = t/\widehat{\gamma}\,,\\
\lim_{t\to\infty} \chi_v(t) &= 0\,,
\end{aligned} \tag{A.7}$$

where we noted that $\hat{\Gamma}(0) = \widehat{\gamma} = \int_0^\infty \Gamma(t')\mathrm{d}t'$. Moreover, note that all this limits do not depend on m and hence they are valid also in the overdamped limit. Things become different in the limit of $t \to 0$.

Appendix A.1. Underdamped

Applying the Tauberian theorem to the underdamped position susceptibility, we get

$$\lim_{t\to 0} \chi_x(t) = \mathcal{L}_t^{-1}\left[\lim_{k\to\infty} [mk^2 + k\hat{\Gamma}(k) + \kappa]^{-1}\right] \approx \mathcal{L}_t^{-1}\left[\frac{1}{mk^2}\right] = \frac{t}{m} \overset{t\to 0}{=} 0\,, \tag{A.8}$$

where we used that in (A.5) it holds that

$$\lim_{k\to\infty} \frac{mk^2}{k\hat{\Gamma}(k)} \gg 1\,. \tag{A.9}$$

As for its integral and derivative of course we have that

$$\begin{aligned}
\lim_{t\to 0} \chi(t) &= \lim_{t\to 0} \int_0^t \chi_x(t')\mathrm{d}t' \approx \frac{t^2}{2m} \overset{t\to 0}{=} 0\,,\\
\lim_{t\to 0} \chi_v(t) &= \lim_{t\to 0} \partial_t \chi_x(t) = 1/m\,.
\end{aligned} \tag{A.10}$$

We see that this result does not depend on the form of the kernel. In fact, inertial effects dominate the particle behaviour in the small time limit.

Appendix A.2. Overdamped

Overdamped dynamics is obtained by taking $m = 0$ so that the Laplace transform of the position susceptibility becomes $\hat{\chi}_x^{ov}(k) = [k\hat{\Gamma}(k) + \kappa]^{-1}$. We have that

$$\lim_{t\to 0} \chi_x^{ov,\exp}(t) = \mathcal{L}_t^{-1}\left[\lim_{k\to\infty}[k\hat{\Gamma}^{\exp}(k) + \kappa]^{-1}\right] \approx \mathcal{L}_t^{-1}\left[\frac{1}{k\gamma_0}\right] = \frac{1}{\gamma_0}\,. \tag{A.11}$$

As it can be seen from first line of the last equation, in the overdamped limit it is important that $\Gamma^{\exp}(t)$ has a piece proportional to the Dirac delta, as pointed out in [51]. Finally, for the integral of the susceptibility we get

$$\lim_{t\to 0} \chi^{ov,\exp}(t) = \mathcal{L}_t^{-1}\left[\lim_{k\to\infty} \frac{1}{k(k\hat{\Gamma}^{\exp}(k) + \kappa)}\right] \approx \mathcal{L}_t^{-1}\left[\frac{1}{k^2\gamma_0}\right] = \frac{t}{\gamma_0}\,. \tag{A.12}$$

Note that in the small time limit the trapping plays no role as all the susceptibilities do not depend on κ.

Appendix B. Multidimensional covariance matrix

We show how to calculate the components of the $2n \times 2n$ covariance matrix associated to the multidimensional GLE (65). In order to do so, we use the expressions for the position and velocity which read

$$x^i(t) = x^i_{t_m}\left(1 - \kappa^i\chi^i(t - t_m)\right) + mv^i_{t_m}\chi^i_x(t - t_m) + \int_{t_m}^{t} \mathrm{d}t'\chi^i_x(t - t')\left[F^i(t') + \eta^i(t')\right]\,, \tag{B.1}$$

$$v^i(t) = -\kappa^i x^i_{t_m}\chi^i_x(t - t_m) + mv^i_{t_m}\chi^i_v(t - t_m) + \int_{t_m}^{t} \mathrm{d}t'\chi^i_v(t - t')\left[F^i(t') + \eta^i(t')\right]\,. \tag{B.2}$$

The velocity, if not averaged, is not defined in the overdamped limit (there should be an additional term of the form $\chi^i_x(0)(F^t + \eta^i(t))$ which is singular for overdamped dynamics and equal to zero in the underdamped case). Nevertheless, for $m = 0$, the covariance matrix has only $n \times n$ components and the cross correlation between position and velocity are not needed to define such matrix. Hence the expression above perfectly works for our scopes. In a similar way as done in [48] we define

$$\phi^i(t) = \int_{t_m}^{t} \chi^i_x(t - t')\eta^i(t')\mathrm{d}t' \tag{B.3}$$

and, by doing so, one can show that

$$\begin{aligned}\mathcal{C}_{ij}(t', t'') =& \langle \phi^i(t')\phi^j(t'')\rangle = \int_{t_m}^{t'} \mathrm{d}s' \int_{t_m}^{t''} \mathrm{d}s''\chi^i_x(t' - s')\chi^j_x(t'' - s'')\langle \eta^i(s')\eta^j(s'')\rangle \\ =& k_B T_i \delta_{ij}\Big[\chi^i(t' - t_m) + \chi^i(t'' - t_m) - \theta(t' - t'')\chi(t' - t'') - \theta(t'' - t')\chi^i(t'' - t') + \\ & - \kappa^i\chi^i(t' - t_m)\chi^i(t'' - t_m) - m\chi^i_x(t' - t_m)\chi^i_x(t'' - t_m)\Big]\,.\end{aligned} \tag{B.4}$$

Using this result, one can finally calculate the four matrices composing the the multidimensional covariance matrix

$$\mathcal{S}_{t,t_m} = \begin{pmatrix} \mathrm{Cov}_{t_m}(x_t^i, x_t^j) & \mathrm{Cov}_{t_m}(x_t^i, v_t^j) \\ \mathrm{Cov}^T_{t_m}(x_t^i, v_t^j) & \mathrm{Cov}_{t_m}(v_t^i, v_t^j) \end{pmatrix} \tag{B.5}$$

that are respectively

$$\begin{aligned}
\mathrm{Cov}_{t_m}(x_t^i, x_t^j) =& \langle x^i x^j \rangle_{t_m,t} - \langle x^i \rangle_{t_m,t} \langle x^j \rangle_{t_m,t} = \\
=& \mathrm{Cov}(x^i_{t_m}, x^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \left(1 - \kappa^j \chi^j(t-t_m)\right) + \\
& + m \mathrm{Cov}_{t_m}(x^i_{t_m}, v^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \chi^j_x(t-t_m) + \\
& + m \mathrm{Cov}_{t_m}(x^j_{t_m}, v^i_{t_m}) \left(1 - \kappa^j \chi^j(t-t_m)\right) \chi^i_x(t-t_m) + \\
& + m^2 \mathrm{Cov}_{t_m}(v^i_{t_m}, v^j_{t_m}) \chi^i_x(t-t_m) \chi^j_x(t-t_m) + \\
& + k_B T^i \mathcal{C}_{ij}(t,t) = \\
=& \mathrm{Cov}(x^i_{t_m}, x^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \left(1 - \kappa^j \chi^j(t-t_m)\right) + \\
& + m \mathrm{Cov}_{t_m}(x^i_{t_m}, v^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \chi^j_x(t-t_m) + \\
& + m \mathrm{Cov}_{t_m}(x^j_{t_m}, v^i_{t_m}) \left(1 - \kappa^j \chi^j(t-t_m)\right) \chi^i_x(t-t_m) + \\
& + m^2 \mathrm{Cov}_{t_m}(v^i_{t_m}, v^j_{t_m}) \chi^i_x(t-t_m) \chi^j_x(t-t_m) + \\
& + k_B T^i \delta_{ij} \left[2\chi^i(t-t_m) - \kappa^i \left(\chi^i(t-t_m)\right)^2 - m \left(\chi^i_x(t-t_m)\right)^2 \right],
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
\mathrm{Cov}_{t_m}(x_t^i, v_t^j) =& \langle x^i v^j \rangle_{t_m,t} - \langle x^i \rangle_{t_m,t} \langle v^j \rangle_{t_m,t} = \\
=& - \kappa^j \mathrm{Cov}(x^i_{t_m}, x^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \chi^j_x(t-t_m) + \\
& + m \mathrm{Cov}_{t_m}(x^i_{t_m}, v^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \chi^j_v(t-t_m) + \\
& - m\kappa^j \mathrm{Cov}_{t_m}(x^j_{t_m}, v^i_{t_m}) \chi^j_x(t-t_m) \chi^i_x(t-t_m) + \\
& + m^2 \mathrm{Cov}_{t_m}(v^i_t, v^j_t) \chi^i_x(t-t_m) \chi^j_v(t-t_m) + \\
& + k_B T^i \left(\partial_{t'} \mathcal{C}_{ij}(t', t'') \right|_{t'=t''=t} = \\
=& - \kappa^j \mathrm{Cov}(x^i_{t_m}, x^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \chi^j_x(t-t_m) + \\
& + m \mathrm{Cov}_{t_m}(x^i_{t_m}, v^j_{t_m}) \left(1 - \kappa^i \chi^i(t-t_m)\right) \chi^j_v(t-t_m) + \\
& - m\kappa^j \mathrm{Cov}_{t_m}(x^j_{t_m}, v^i_{t_m}) \chi^j_x(t-t_m) \chi^i_x(t-t_m) + \\
& + m^2 \mathrm{Cov}_{t_m}(v^i_{t_m}, v^j_{t_m}) \chi^i_x(t-t_m) \chi^j_v(t-t_m) + \\
& + k_B T^i \delta_{ij} \Big[\chi^i_x(t-t_m) - \kappa^i \chi^i_x(t-t_m) \chi^i(t-t_m) - \\
& - m \chi^i_v(t-t_m) \chi^i_x(t-t_m) \Big],
\end{aligned} \tag{B.7}$$

$$
\begin{aligned}
\mathrm{Cov}_{t_m}(v^i_{t_m}, v^j_{t_m}) =& \langle v^i v^j\rangle_{t_m,t} - \langle v^i\rangle_{t_m,t}\langle v^j\rangle_{t_m,t} = \\
=& \kappa^j\kappa^i \mathrm{Cov}(x^i_{t_m}, x^j_{t_m})\chi^i(t-t_m)\chi^j_x(t-t_m) + \\
& - m\kappa^i \mathrm{Cov}_{t_m}(x^i_{t_m}, v^j_{t_m})\chi^i_x(t-t_m)\chi^j_v(t-t_m) + \\
& - m\kappa^j \mathrm{Cov}_{t_m}(x^j_{t_m}, v^i_{t_m})\chi^j_x(t-t_m)\chi^i_v(t-t_m) + \\
& + m^2 \mathrm{Cov}_{t_m}(v^i_{t_m}, v^j_{t_m})\chi^i_v(t-t_m)\chi^j_v(t-t_m) + \\
& + k_B T^i \left(\partial_{t'}\partial_{t''}\mathcal{C}_{ij}(t',t'')\right|_{t'=t''=t} = \\
=& \kappa^j\kappa^i \mathrm{Cov}(x^i_{t_m}, x^j_{t_m})\chi^i(t-t_m)\chi^j_x(t-t_m) + \\
& - m\kappa^i \mathrm{Cov}_{t_m}(x^i_{t_m}, v^j_{t_m})\chi^i_x(t-t_m)\chi^j_v(t-t_m) + \\
& - m\kappa^j \mathrm{Cov}_{t_m}(x^j_{t_m}, v^i_{t_m})\chi^j_x(t-t_m)\chi^i_v(t-t_m) + \\
& + m^2 \mathrm{Cov}_{t_m}(v^i_{t_m}, v^j_{t_m})\chi^i_v(t-t_m)\chi^j_v(t-t_m) + \\
& + k_B T^i \delta_{ij}\left[\frac{1}{m} - \kappa^i\left(\chi^i_x(t'-t_m)\right)^2 - m\left(\chi^i_v(t'-t_m)\right)^2\right].
\end{aligned}
\tag{B.8}
$$

References

[1] Deffner S and Campbell S 2017 *J. Phys. A: Math. Theor.* **50** 453001 URL https://doi.org/10.1088%2F1751-8121%2Faa86c6

[2] Okuyama M and Ohzeki M 2018 *Phys. Rev. Lett.* **120** URL http://dx.doi.org/10.1103/PhysRevLett.120.070402

[3] Shanahan B, Chenu A, Margolus N and del Campo A 2018 *Phys. Rev. Lett.* **120** URL http://dx.doi.org/10.1103/PhysRevLett.120.070401

[4] Shiraishi N, Funo K and Saito K 2018 *Phys. Rev. Lett.* **121** URL http://dx.doi.org/10.1103/PhysRevLett.121.070601

[5] Funo K, Shiraishi N and Saito K 2019 *New J. Phys.* **21** 013006 URL https://doi.org/10.1088%2F1367-2630%2Faaf9f5

[6] Horowitz J M and Gingrich T R 2020 *Nature Phys.* **16** 15–20

[7] Barato A C and Seifert U 2015 *Phys. Rev. Lett.* **114** 158101

[8] Seifert U 2018 *Physica A* **504** 176–191 14th International Summer School on Fundamental Problems in Statistical Physics, Bruneck, Italy, July 16-29, 2017

[9] Pietzonka P, Barato A C and Seifert U 2016 *J. Stat. Mech.* 124004

[10] Polettini M, Lazarescu A and Esposito M 2016 *Phys. Rev. E* **94** 052104

[11] Gingrich T R, Horowitz J M, Perunov N and England J L 2016 *Phys. Rev. Lett.* **116** 120601

[12] Nardini C and Touchette H 2018 *Eur. Phys. J. B* **91** 16

[13] Pietzonka P, Ritort F and Seifert U 2017 *Phys. Rev. E* **96** 012101

[14] Gingrich T R, Rotskoff G M and Horowitz J M 2017 *J. Phys. A: Math. Theor.* **50** 184004

[15] Horowitz J M and Gingrich T R 2017 *Phys. Rev. E* **96** 020103

[16] Proesmans K and Van den Broeck C 2017 *EPL* **119** 20001

[17] Macieszczak K, Brandner K and Garrahan J P 2018 *Phys. Rev. Lett.* **121** 130601

[18] Dechant A and Sasa S i 2018 *J. Stat. Mech.* 063209

[19] Ito S and Dechant A 2020 *Phys. Rev. X* **10**(2) 021056 URL https://link.aps.org/doi/10.1103/PhysRevX.10.021056

[20] Timpanaro A M, Guarnier G, Goold J and Landi G T 2019 *Phys. Rev. Lett.* **123** 090604

[21] Busiello D M and Pigolotti S 2019 *Phys. Rev. E* **100**(6) 060102 URL https://link.aps.org/doi/10.1103/PhysRevE.100.060102

[22] Dechant A 2019 *J. Phys. A: Math. Theor.* **52**

[23] Dechant A and Sasa S i 2020 *Proc. Natl. Acad. Sci.* **117** 6430–6436 URL https://www.pnas.org/content/117/12/6430
[24] Koyuk T and Seifert U 2020 *arXiv:2005.02312*
[25] Fischer P, Chun H M and Seifert U 2022 *Phys. Rev. E* **102** 012120
[26] Falasco G and Esposito M 2020 The dissipation-time uncertainty relation (*Preprint* arXiv:2002.03234)
[27] Pietzonka P, Barato A C and Seifert U 2016 *Phys. Rev. E* **93** 052145
[28] Maes C 2017 *Phys. Rev. Lett.* **119** 160601
[29] Garrahan J P 2017 *Phys. Rev. E* **95** 032134
[30] Chiuchiu D and Pigolotti S 2018 *Phys. Rev. E* **97** 032109
[31] Di Terlizzi I and Baiesi M 2019 *J. Phys. A: Math. Theor.* **52** 02LT03
[32] Van Vu T and Hasegawa Y 2019 *Phys. Rev. E* **100** 032130
[33] Barato A C, Chetrite R, Faggionato A and Gabrielli D 2019 *J. Stat. Mech.* 084017
[34] Van Vu T and Hasegawa Y 2019 *Phys. Rev. E* **100** 012134
[35] Van Vu T and Hasegawa Y 2020 *J. Phys. A: Math. Theor.* **53** 075001
[36] Potts P P and Samuelsson P 2019 *Phys. Rev. E* **100**(5) 052137 URL https://link.aps.org/doi/10.1103/PhysRevE.100.052137
[37] Seifert U 2019 *Annual Review of Condensed Matter Physics* **10** 171–192 URL https://doi.org/10.1146/annurev-conmatphys-031218-013554
[38] Li J, Horowitz J M, Gingrich T R and Fakhri N 2019 *Nature Comm.* **10** 1666
[39] Manikandan S K, Gupta D and Krishnamurthy S 2020 *Phys. Rev. Lett.* **124** 120603
[40] Otsubo S, Ito S, Dechant A and Sagawa T 2020 *Phys. Rev. E* **101**(6) 062106 URL https://link.aps.org/doi/10.1103/PhysRevE.101.062106
[41] Van Vu T, Vo V T and Hasegawa Y 2020 *Phys. Rev. E* **101**(4) 042138 URL https://link.aps.org/doi/10.1103/PhysRevE.101.042138
[42] Falasco G, Esposito M and Delvenne J C 2020 Unifying thermodynamic uncertainty relations URL https://doi.org/10.1088%2F1367-2630%2Fab8679
[43] Baiesi M, Maes C and Wynants B 2009 *Phys. Rev. Lett.* **103** 010602
[44] Baiesi M, Maes C and Wynants B 2009 *J. Stat. Phys.* **137** 1094–1116
[45] Baiesi M, Boksenbojm E, Maes C and Wynants B 2010 *J. Stat. Phys.* **139** 492–505
[46] Gomez-Solano J R, Petrosyan A and Ciliberto S 2011 *Phys. Rev. Lett.* **106**(20) 200602 URL https://link.aps.org/doi/10.1103/PhysRevLett.106.200602
[47] Berner J, Müller B, Gomez-Solano J R, Krüger M and Bechinger C 2018 *Nature Comm.* **9** 1–8
[48] Di Terlizzi I and Baiesi M 2020 Explicit solution of the generalised Langevin equation URL https://doi.org/10.1007/s10955-020-02639-4
[49] Kubo R 1966 *Rep. Prog. Phys.* **29** 255–284
[50] Goychuk I 2013 *Advances in Chemical Physics* **150** 187–253
[51] Nascimento E d S and Morgado W A 2019 *EPL (Europhysics Letters)* **126** 10002