Gauge Transformations in Cosmology*

The perturbative approach is a fundamental tool in General Relativity (GR), where exact solutions are most often too idealised to properly represent the realm of natural phenomena. Unfortunately, the invariance of GR under diffeomorphisms (two solutions of Einstein's equations are physically equivalent if they are diffeomorphic to each other) makes the very definition of perturbations gauge dependent.
A gauge choice is an identification between points of the perturbed (i.e. physical) and the background spacetimes; generic perturbations are not invariant under a gauge transformation $\Longrightarrow$ gauge problem.
A change in the correspondence (between points of the physical and background points), keeping the background coordinates fixed, is called a gauge transformation, to be distinguished from a coordinate transformation, which changes the labeling of points in the background and physical spacetimes together.
The perturbation in some quantity is the

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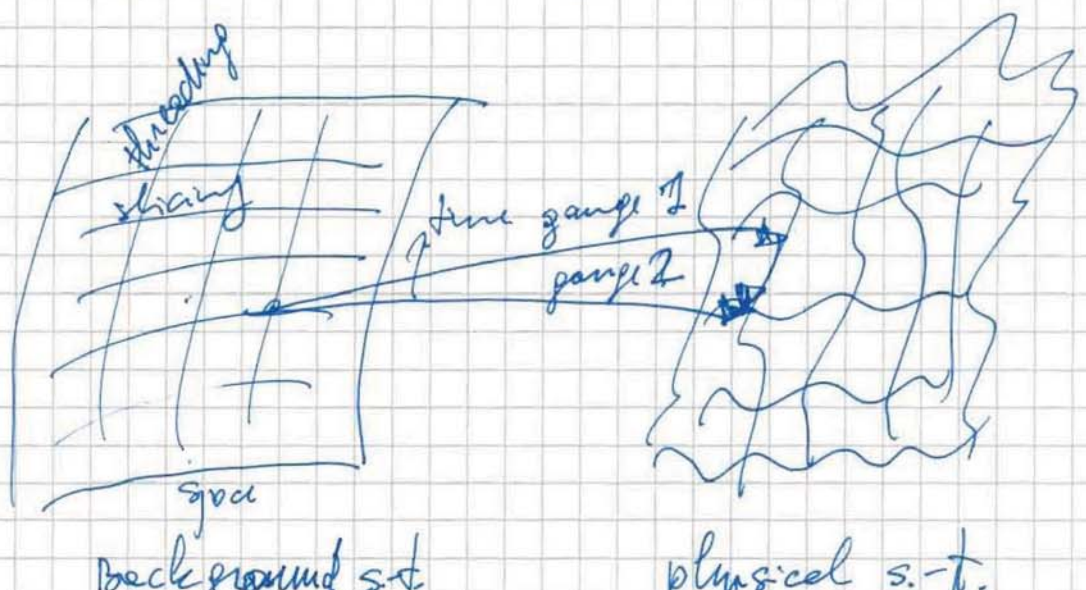
N.B.: in some points of these notes there are some typos that I have pointed out and corrected.

difference between the value it has at a point in the physical spacetime and the value it has at the corresponding point in the background space-time.

A gauge transformation induces a coordinate transformation in the physical space-time, but it also changes the point in the background space-time corresponding to a given point in the physical space-time. Thus, even if a quantity is a scalar under coordinates transformations, the value of the perturbation in that quantity will not be invariant under gauge-transformations if the quantity is non-zero and position-dependent in the background.



Background s-t   physical s.-t.

A choice of coordinates defines a threading of space-time into lines (corresponding to fixed spatial coordinates) and slicing into hypersurfaces (corresponding to fixed time).

There are two approaches to calculate how perturbations change under a small coordinate or gauge transformation. For the active view we study how perturbations change under a mapping, where the map directly induces the transformation of the perturbed quantities. In the passive view the relation between the two coordinate systems is specified, and we calculate how the perturbations are changed under this coordinate transformation.
In the passive approach the transformation is taken at the same physical point, whereas in the active approach the transformation of the perturbed quantities is evaluated at the same coordinate point.

Let us now consider a general infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$$

where ξ^μ (and its derivatives) are infinitesimally small.
Now remember that tensors transform as follows:

scalars $\phi'(x') = \phi(x)$

cov. vectors $V'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu(x)$

contra. vectors $V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$

$$T'^{\mu\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} T^{\rho\sigma}(x)$$

tensors $$T'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} T_{\rho\sigma}(x)$$

$$T'^{\mu}{}_{\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} T^{\rho}{}_{\sigma}(x)$$

The partial derivatives occurring in the tensor transformation rules are

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu} - \frac{\partial \xi^{\mu}(x)}{\partial x^{\nu}}$$

$$\frac{\partial x^{\nu}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu} + \frac{\partial \xi^{\nu}(x)}{\partial x^{\mu}} + O(\xi^2)$$

We are concerned with solutions of Einstein's equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. Since Einstein's equations are generally covariant, and $g_{\mu\nu}$ is a solution for an energy momentum tensor $T_{\mu\nu}(x)$, it follows that $g'_{\mu\nu}(x)$ is a solution for $T'_{\mu\nu}(x)$, where

$$g'_{\mu\nu}(x) = g'_{\mu\nu}(x') + \frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}}\xi^{\lambda}(x) + O(\xi^2)$$

$$= g_{\mu\nu}(x) + g_{\lambda\nu}(x)\frac{\partial \xi^{\lambda}(x)}{\partial x^{\mu}} + g_{\lambda\mu}(x)\frac{\partial \xi^{\lambda}(x)}{\partial x^{\nu}} + \frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}}\xi^{\lambda}(x)$$

and likewise for $T'_{\mu\nu}(x)$. We can identically rewrite these results in covariant form, as follows:

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathcal{L}_\xi g_{\mu\nu}(x)$$

$$T'_{\mu\nu}(x) = T_{\mu\nu}(x) + \mathcal{L}_\xi T_{\mu\nu}(x)$$

where $\mathcal{L}_\xi$ indicates the Lie derivative along the vector ξ, namely:

$$\mathcal{L}_\xi g_{\mu\nu} \equiv \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

$$\mathcal{L}_\xi T_{\mu\nu} \equiv T_{\lambda\mu}\xi^\lambda{}_{;\nu} + T_{\lambda\nu}\xi^\lambda{}_{;\mu} + T_{\mu\nu;\lambda}\xi^\lambda$$

(note that $\mathcal{L}_\xi g$ has the same form of $\mathcal{L}_\xi T$, except that $g_{\mu\nu}$ has zero covariant derivatives), the Lie derivative can be generalised to arbitrary tensors as:

$$\mathcal{L}_\xi S \equiv S_{;\lambda}\xi^\lambda = S_{,\lambda}\xi^\lambda$$

$$\mathcal{L}_\xi V_\mu \equiv V_\lambda \xi^\lambda{}_{;\mu} + V_{\mu;\lambda}\xi^\lambda$$

$$\mathcal{L}_\xi V^\mu \equiv -V^\lambda\xi^\mu{}_{;\lambda} + V^\mu{}_{;\lambda}\xi^\lambda$$

$$\mathcal{L}_\xi T^{\mu\nu} \equiv -T^{\lambda\nu}\xi^\mu{}_{;\lambda} - T^{\mu\lambda}\xi^\nu{}_{;\lambda} + T^{\mu\nu}{}_{;\lambda}\xi^\lambda$$

$$\mathcal{L}_\xi T^\mu{}_\nu \equiv -T^\lambda{}_\nu\xi^\mu{}_{;\lambda} + T^\mu{}_\lambda\xi^\lambda{}_{;\nu} + T^\mu{}_{\nu;\lambda}\xi^\lambda$$

In general, the effect of an infinitesimal coordinate transformation of any tensor T is that the new tensor equals the old one at the same coordinate

point plus the Lie derivative $\mathcal{L}_\xi T$.
The Lie derivative operator obeys the same properties as ordinary or covariant derivatives.
Let us now recall that:

$$\phi_{;\mu} = \phi_{,\mu}$$

$$V_{\mu;\nu} = V_{\mu,\nu} - \Gamma^\rho_{\mu\nu} V_\rho$$

$$V^\mu_{;\nu} = V^\mu_{,\nu} + \Gamma^\mu_{\rho\nu} V^\rho$$

$$T_{\mu\nu;\lambda} = T_{\mu\nu,\lambda} - \Gamma^\rho_{\mu\lambda} T_{\rho\nu} - \Gamma^\rho_{\lambda\nu} T_{\mu\rho}$$

$$T^{\mu\nu}{}_{;\lambda} = T^{\mu\nu}{}_{,\lambda} + \Gamma^\mu_{\rho\lambda} T^{\rho\nu} + \Gamma^\nu_{\lambda\rho} T^{\mu\rho}$$

$$T^\mu{}_{\nu;\lambda} = T^\mu{}_{\nu,\lambda} + \Gamma^\mu_{\rho\lambda} T^\rho{}_\nu - \Gamma^\rho_{\lambda\nu} T^\mu{}_\rho$$

etc...

Let us also recall the expression for the Christoffel (or affine connection) symbols:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(g_{\nu\sigma,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma} \right)$$

→ <u>Connection between active & passive approach</u>

A basic assumption of perturbation theory is the existence of a parametric family of solutions of the field equations, to which the unperturb. background spacetime (FRW in our case) belongs. One then deals with a 1-parameter family of models $\mathcal{M}_\lambda$; λ is real, and

(say) $\lambda = 0$ identifies the background $\mathcal{M}_0$.
On each $\mathcal{M}_\lambda$ there are ~~tensor~~ fields T_λ representing the physical and geometrical quantities (e.g. the metric, the stress-energy tensor of a fluid, a scalar field, etc...). The parameter λ is used for Taylor expanding ~~these~~ T_λ. The physical spacetime $\mathcal{M}_\lambda$ can eventually be identified by $\lambda = 1$. The aim of perturbation theory is to construct an approximated solution to $\mathcal{M}_\lambda$.
Each one-to-one correspondence between points of $\mathcal{M}_0$ and points of $\mathcal{M}_\lambda$ is thus a 1-parameter function of λ: we can represent two such "point-identification maps" as ψ_λ and φ_λ. Suppose that coordinates x^μ have been assigned on the background $\mathcal{M}_0$, labelling the different points. A one-to-one correspondence, e.g. ψ_λ, carries these coordinates over $\mathcal{M}_\lambda$, and defines a choice of gauge: therefore, it is natural to call the correspondence itself a "gauge".
As already stressed above, a change in this correspondence, keeping the background coordinates fixed, is called gauge transformation.
Thus, let p be any point in $\mathcal{M}_0$, with coordinates $x^\mu(p)$, and let use use ~~the~~ gauge ψ_λ:

$$O = \psi_\lambda(p)$$

is the point in $\mathcal{M}_\lambda$ corresponding to p, to which ψ_λ assigns the same coordinate labels. However, we could as well use a different gauge φ_λ and think of O as the point of $\mathcal{M}_\lambda$ corresponding to a different point q in the background, with coordinates $\tilde{x}^\mu$: then

$$O = \psi_\lambda(p) = \varphi_\lambda(q)$$

Thus, the change of the correspondence, i.e. the gauge transformation, may actually be seen as a one-to-one correspondence between different points in the background. Since we start from a point p in $\mathcal{M}_0$, we carry it over to $O = \psi_\lambda(p)$ in $\mathcal{M}_\lambda$ and then we come back to q in $\mathcal{M}_0$ with φ_λ^{-1}, i.e. $q = \varphi_\lambda^{-1}(O)$, the overall gauge transformation is also a function of λ, call it Φ_λ, and is given by composing φ_λ^{-1} with ψ_λ, so that we can write

$$q = \Phi_\lambda(p) = \varphi_\lambda^{-1}(\psi_\lambda(p))$$

We then have that the coordinates of q $\tilde{x}^\mu(\lambda, q) = \Phi_\lambda^\mu(x^\alpha(p))$ are 1-parameter functions of those of p, $x^\alpha(p)$. Such a transformation, which in one given coordinate system moves each point to another, is often

called an "active coordinate transformation", as opposed to passive ones, which change coordinate labels to each point.
The discussion above involves the comparison of tensors at different points p and q; this requires a transport law from q to p. This gives us two tensors at p: T itself and the transported one, which can be directly compared.
What we are going to see is the Lie dragging of a tensor by a vector field (already implicitely defined before).
Let a coordinate system x^μ be defined on $\mathcal{M}$, together with a vector field ξ. From

$$\frac{dx^\mu}{d\lambda} = \xi^\mu \qquad (1)$$

ξ generates on $\mathcal{M}$ a congruence of curves $x^\mu(\lambda)$; λ is the parameter along the congruence.
Given a point p, this will always lie on one of these curves, and we can take p to correspond to $\lambda = 0$ on this. The coordinates of a second point q at a parameter distance λ from p on the same curve will be given by

$$\tilde{x}^\mu(\lambda) = x^\mu + \lambda \xi^\mu + \cdots$$

where x^μ are the coordinates of p and $\tilde{x}^\mu$ those of q, approximated at first order in λ.

Eq. (1) is an active coordinate transformation. At the same time, we may think that a new coordinate system y^μ has been defined on $\mathcal{M}$, in such a way that the y coordinates at q coincide with the x coordinates at p, namely

$$y^\mu(q) = x^\mu(p) = x^\mu(q) - \lambda \xi^\mu(x(p)) + \cdots$$
$$= x^\mu(q) - \lambda \xi^\mu(x(q)) + \mathcal{O}(\lambda^2) \qquad (2)$$

In practice, we have defined in this way at every point a "passive coordinate transformation", which at first-order reads

$$y^\mu(\lambda) = x^\mu - \lambda \xi^\mu \qquad (3)$$

Suppose now that a tensor field is given on $\mathcal{M}$, e.g. a vector field Z with components Z^μ in x-coordinates. In the same way that we defined a new coordinate system y^μ via ~~Eq~~. (1) by the action of ξ, we can define a new vector field $\tilde{Z}$, with components $\tilde{Z}^\mu$ in x coordinates such that these components at the coordinate point $x^\mu(p)$ are equal to the components Z'^μ the old vector Z has in the y coordinates at the coordinate point $y(q)$:

$$\tilde{Z}^\mu(x(p)) \equiv Z'^\mu(y(q)) = \frac{\partial y^\mu}{\partial x^\nu}\bigg|_{x(q)} Z^\nu(x(q)) \qquad (*)$$

The last equality in this equation is just the ordinary (passive) transformation between the components of Z in the two coordinate systems x and y: we need it in order to relate $\tilde{Z}$ and Z in a single system (the x-frame here), thus eventually obtaining a covariant relation. Indeed, replacing eq. (3) into eq. (4) and a first-order expansion in λ about $x(p)$ in the RHS gives (as already seen)

$$\tilde{Z}^\mu(\lambda) = Z^\mu + \lambda \mathcal{L}_\xi Z^\mu \qquad (5)$$

$$\mathcal{L}_\xi Z^\mu = Z^\mu{}_{,\nu} \xi^\nu - \xi^\mu{}_{,\nu} Z^\nu \qquad (6)$$

This law is the "Lie-dragging".

- Higher-order gauge transformations

One has to realise that eq. (1) is just the first-order Taylor expansion about $x(p)$ of the solution of the ordinary differential equation

$$\frac{dx^\mu}{d\lambda} = \xi^\mu$$

defining the congruence $x^\mu(\lambda)$ associated w. ξ. The exact solution is the Taylor series:

$$x^\mu(q) = x^\mu(p) + \lambda \xi^\mu(x(p)) + \frac{\lambda^2}{2} \xi^\mu{}_{,\nu} \xi^\nu(x(p)) + \dots$$

having used $dx^\mu/d\lambda = \xi^\mu$; $d^2x^\mu/d\lambda^2 = \xi^\mu{}_{,\nu} \xi^\nu$ etc.

In practice, we can generally write

$$\tilde{x}^\mu(\lambda) = x^\mu + \lambda \xi^\mu + \frac{\lambda^2}{2} \xi^\mu{}_{,\nu} \xi^\nu + \cdots$$

$$= \exp[\lambda \mathcal{L}_\xi] x^\mu$$

and

$$y^\mu(\lambda) = x^\mu - \lambda \xi^\mu + \frac{\lambda^2}{2} \xi^\mu{}_{,\nu} \xi^\nu + \cdots$$

As a consequence, we have

$$\tilde{Z}^\mu(\lambda) = [\exp[\lambda \mathcal{L}_\xi] Z]^\mu$$

$$= Z^\mu + \lambda \mathcal{L}_\xi Z^\mu + \frac{\lambda^2}{2} \mathcal{L}_\xi^2 Z^\mu + \cdots$$

and, with a more abstract notation

$$\tilde{T}(\lambda) = \exp[\lambda \mathcal{L}_\xi] T$$

Since in the background space-time σ_0 at each point, we now have a field representing T_λ for a given gauge, for different gauges we can compare these fields with the background value T_0 and define perturbations. In the first gauge the total perturbation is

$$\Delta T(\lambda) \equiv T(\lambda) - T_0$$

and in the second:

$$\Delta \tilde{T}(\lambda) = \tilde{T}(\lambda) - T_0.$$

This non-uniqueness is the gauge dependence of the perturbations.

Cosmological perturbations

Let's assume a flat FLRW background metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

which we can also write using conformal time τ (such that $d\tau = dt/a(t)$; $a(\tau) = a(t(\tau))$)

$$ds^2 = a^2(\tau)[-d\tau^2 + dx^2 + dy^2 + dz^2)$$

We can write our perturbed metric tensor ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$) as

$$g_{00} = -a^2(\tau)\left(1 + 2\sum_{r=1}^{\infty} \frac{1}{r!}\psi^{(r)}\right)$$

$$g_{0i} = g_{i0} = a^2(\tau)\sum_{r=1} \frac{1}{r!}\omega_i^{(r)}$$

$$g_{ij} = a^2(\tau)\left\{\left[1 - 2\sum_{r=1}^{\infty}\frac{1}{r!}\phi^{(r)}\right]\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}\chi_{ij}^{(r)}\right\}$$

where $\chi^{(r)\,i}_{\ i} = 0$. Here and in what follows Latin indices are raised and lowered using δ^{ij} and δ_{ij} respectively.

We can define scalar, vector and tensor parts of perturbations, where scalar (longitudinal) parts are those related to a scalar potential, vector parts are related to transverse (divergenceless or solenoidal) vector fields and tensor parts to transverse, trace-free tensors.

In our case, the shift $\omega_i^{(r)}$ can be decomposed as (Helmholtz theorem)

$$\omega_i^{(r)} = \partial_i \omega^{(r)\parallel} + \omega_i^{(r)\perp}$$

where $\omega_i^{(r)\perp}$ is a solenoidal vector, i.e.

$$\partial^i \omega_i^{(r)\perp} = 0.$$

Similarly, the traceless part of the spatial metric can be decomposed as (at any order)

$$\chi_{ij}^{(r)} = D_{ij}\chi^{(r)\parallel} + \partial_i \chi_j^{(r)\perp} + \partial_j \chi_i^{(r)\perp} + \chi_{ij}^{(r)T},$$

where $\chi_{ij}^{(r)\parallel}$ is a suitable function, $\chi_i^{(r)\perp}$ is a solenoidal vector field and $\partial^i \chi_{ij}^{(r)\perp} = 0$.
Hereafter

$$D_{ij} \equiv \partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2$$

Let us now consider that the source of our Einstein's equation is provided by a ~~perfect~~ fluid, characterized by a stress-energy tensor

$$T_{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu}$$

where u^μ is the four-velocity of fluid elements, with

$$u_\mu u^\mu = -1$$

and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the projection tensor on hypersurfaces orthogonal to u^μ,

$$h_{\mu\nu} u^\mu = 0.$$

Here ρ = energy density; p = isotropic pressure.

We can then write for the energy density

$$\rho = \rho_{(0)} + \sum_{r=1}^{\infty} \frac{1}{r!} \delta^r \rho$$

(perturbations in the pressure can be obtained recalling that

$w = p_0/\rho_0$ and $c_s^2 = dp/d\rho$. ~~up to~~ "entropy perturbations"; see below)

For the four velocity u^μ of the matter, we can write

$$u^\mu = \frac{1}{a}\left(\delta_0{}^\mu + \sum_{r=1}^{\infty} \frac{1}{r!} v_{(r)}{}^\mu\right)$$

Because of the normalization condition for u^μ, at any order the time component of $v_{(r)}^{(0)}$ is related to the "lapse" perturbation $\psi_{(r)}$. E.g. at first order, we obtain

$$v_{(1)}^0 = -\psi_{(1)}.$$

The velocity perturbation $v_{(r)}^i$ can also be split into a scalar and vector (vortical) part

$$v_{(r)}^i = \partial^i v_{(r)}{}^{\parallel} + v_{(r)\perp}^i . \quad (\partial^i v_{i\perp} = 0)$$

In what follows, we will also consider the case of a scalar field as source, in which case we can obviously write

$$\Phi = \Phi_{(0)} + \sum_{r=1}^{\infty} \frac{1}{r!} \delta\Phi_{(r)}$$

Finally, as we have seen, the gauge transformations are determined by the vectors $\xi_{(r)}$ (if we consider gauge transformations beyond the linear order, for which we would only need $\xi_{(1)}$). We have also here

$$\xi^0_{(r)} \equiv \alpha^{(r)}$$

and

$$\xi^i_{(r)} \equiv \partial^i \beta^{(r)} + d^i_{(r)}$$

with $\partial_i d_{(r)}^i = 0$.

Linear gauge transformations

As we have seen we can perturb a tensor, T, by considering the congruence parametrised by λ, so that in two gauges x^μ and $\tilde{x}^\mu$ we have

$$T(\lambda) = T_0 + \lambda \delta T$$

$$\tilde{T}(\lambda) = T_0 + \lambda \delta \tilde{T}$$

On the other hand, the effect of a linear gauge transformation on a tensor T is obtained as

$$\tilde{T}(\lambda) = T(\lambda) + \lambda \mathcal{L}_{\xi_{(1)}} T$$

We then have

$$\tilde{T}(\lambda) \doteq T_0 + \lambda \delta\tilde{T} = T_0 + \lambda \delta T + \mathcal{L}_{\xi(1)} T,$$

hence

$$\boxed{\delta\tilde{T} = \delta T + \mathcal{L}_{\xi(1)} T_0} + \mathcal{O}(\lambda)$$

In what follows we will omit the index (1) indicating that quantities are at linear order. We have

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_\xi g^{(0)}_{\mu\nu}$$

which implies ($' \equiv d/d\tau$)

$$\tilde{\psi} = \psi + \alpha' + \frac{a'}{a}\alpha$$

$$\tilde{\omega}_i = \omega_i - \alpha_{,i} + \beta_{,i}' + d_i'$$

$$\tilde{\phi} = \phi - \frac{1}{3}\nabla^2\beta - \frac{a'}{a}\alpha$$

$$\tilde{\chi}_{ij} = \chi_{ij} + 2 D_{ij}\beta + d_{i,j} + d_{j,i}$$

For a scalar like ρ (but it also applies to p or ~~to~~ a scalar field Φ)

$$\delta\tilde{\rho} = \delta\rho + \rho_{(0)}'\alpha$$

For the four velocity $\delta\tilde{u}^\mu = \delta u^\mu + \mathcal{L}_\xi u^\mu_{(0)}$, so

$$\tilde{v}^0 = v^0 - \frac{a'}{a}\alpha - \alpha' \; ; \; \tilde{v}^i = v^i - \beta'^{,i} - d^{i\prime}$$

As we have seen, the vector ξ^μ generating the gauge transformation involves two scalars (α, β). and one divergence-free vector d^i. This holds at any order in perturbation theory. Hence the various gauges are defined by suitable choices of ~~these~~ two scalars and one vector.

Let us consider here some popular gauge choices.

1) Poisson gauge

The Poisson gauge is defined by the choice

$$\omega^{\parallel} = 0$$
$$\chi^{\parallel} = 0$$
$$\chi_i^{\perp} = 0$$

This generalizes the so-called longitudinal or (conformal) Newtonian gauge in which vector and tensor perturbations are not considered (not that this is not a gauge-choice but a dynamical statement).

2) Synchronous gauge

This is defined by the choice

$$\psi = 0$$

If we also take $\omega^{\parallel} = \omega_i^{\perp} = 0$ this is called synchronous and time-orthogonal gauge. In this gauge the proper time for

observers at fixed spatial coordinates coincides with cosmic time in the FRW background, i.e. $dt = a(\tau)\, d\tau$.
It could be easily seen that the synchronous gauge is plagued by residual gauge freedom (somehow similar to the Gribov ambiguity in electrodynamics)

3) Comoving gauge

The comoving gauge is defined by the condition that the 3-velocity of the fluid vanishes, i.e.

$$v^i = 0 \Rightarrow v'' = v_i^{\perp} = 0$$

If we also require orthogonality of the constant-τ hypersurfaces to the 4-velocity, ($T^0{}_i = 0$) this gives

$$v'' + \omega'' = 0$$

(zero momentum). Notice that we cannot require simultaneously $v_i^{\perp} = 0$ & $\omega_i^{\perp} = 0$ as a gauge condition (but it can be a dynamical requirement).

4) Spatially flat gauge

The spatially flat or uniform curvature gauge is identified by the condition that one selects spatial hypersurfaces on which the induced 3-metric of spatial hypersurfaces is left unperturbed by scalar or vector perturbations, which requires

$$\varphi = \chi^{\parallel} = \chi_i^{\perp} = 0$$

5) <u>Uniform-density gauge</u>

This gauge is defined by the condition

$$\delta\rho = 0$$

which leaves freedom on one scalar and one vector perturbation.

<u>Gauge invariance at linear order</u>

In order to find the quantities which are invariant at linear order it is convenient to separate scalar, vector and tensor modes in the metric transformation rules. We have

$$\begin{cases} \omega^{\parallel} \to \tilde{\omega}^{\parallel} = \omega^{\parallel} - \alpha + \beta' \\ \omega_i^{\perp} \to \tilde{\omega}_i^{\perp} = \omega_i^{\perp} + d_i' \\ \chi_{ij}^{T} \to \tilde{\chi}_{ij}^{T} = \chi_{ij}^{T} \end{cases} , \quad \begin{cases} \chi^{\parallel} \to \tilde{\chi}^{\parallel} = \chi^{\parallel} + 2\beta \\ \chi_i^{\perp} \to \tilde{\chi}_i^{\perp} = \chi_i^{\perp} + d_i \end{cases}$$

$\Rightarrow$ tensor modes are gauge-invariant at linear order

and for the 3-velocity:

$$\begin{cases} v^{\parallel} \to \tilde{v}^{\parallel} = v^{\parallel} - \beta' \\ v_{\perp}^{i} \to \tilde{v}_{\perp}^{i} = v_{\perp}^{i} - d^{i\prime} \end{cases}$$

To find gauge-invariant quantities one then looks for linear combinations of these quantities.

A set of gauge-invariant variables has been obtained by Bardeen (1980) and we will describe them here.

→ <u>Scalar quantities</u>

Considering purely geometric quantities, only two independent gauge-independent quantities can be constructed from the metric tensor amplitudes alone, since there are two gauge functions and four metric tensor amplitudes. By inspection of the transformation laws, these are conveniently taken as

$$2\Phi_A \equiv 2\psi + 2\omega'' + 2\frac{a'}{a}\omega'' - \left(\chi'' + \frac{a'}{a}\chi''\right)$$

and

$$-2\Phi_H \equiv -2\phi - \frac{1}{3}\nabla^2\chi'' + 2\frac{a'}{a}\omega'' - \frac{a'}{a}\chi''$$

which in the gauge where $\omega'' = \frac{1}{2}\chi''$ would reduce to $\Phi_A = \psi$, i.e. to the lapse function and to $-\Phi_H = -\phi - \frac{1}{6}\nabla^2\chi''$.

which is proportional to the spatial curvature (Φ_H is usually called Bardeen's gauge invariant gravitational potential).

The simplest gauge-invariant velocity (scalar mode) is

$$2V_S \equiv 2v'' + \chi''$$

The energy density perturbation amplitude δ

must be combined with other quantities to produce a gauge-invariant measure of the density perturbation, and one obvious criterion is that the gauge-invariant quantities reduce to $\delta\rho$ as soon as the perturbation comes inside the particle horizon, $k^{-1}a'/a \ll 1$. First, consider

$$\mathcal{E}_m = \delta\rho + \rho_0'(v'' + \omega'')$$

which equals the energy density perturbation in the gauge where $v'' = -\omega''$, which is just the condition that the matter world-lines are orthogonal to the $\tau =$ constant spacelike hypersurfaces. Thus $\mathcal{E}_m$ is the natural choice of gauge-invariant energy-density perturbation amplitude from the point of view of the matter. It is the density perturbation relative to the spacelike hypersurface which represents everywhere the matter local rest frame.

An alternative gauge-invariant density perturbation is

$$2\mathcal{E}_g = 2\delta\rho + \rho_0'(2\omega'' - \chi'^{\prime\prime})$$

One can see that $\mathcal{E}_g$ measures the energy-density perturbation relative to the hypersurface where normal unit vectors have zero shear. This geometrically selected hypersurface is as close as possible to a "Newtonian" time slicing.

→ Vector quantities

The only gauge-invariant combination of vector ~~tensor~~ geometric perturbations is

$$\Psi_i = \omega_i^{\perp} - \chi_i'^{\perp}$$

check sign! sign is OK

which represents a "frame-dragging" term (related, e.g., to the Lense-Thirring effect). The only other possibility is to consider the matter velocity, which gives rise to the vector gauge-invariant part.

$$V_s^{\,i} = v_{\perp}^{\,i} + \chi_{\perp}'^{\,i}$$

check sign! sign is OK

which is related to the vorticity tensor

$$\omega_{\mu\nu} \equiv \frac{1}{2} h_{\mu}{}^{\rho}\left(u_{\rho;\sigma} - u_{\sigma;\rho}\right) h^{\sigma}{}_{\nu} .$$

→ Tensor quantities

As we have seen above, at linear order, tensor perturbations are automatically gauge-invariant.

→ General rule

As we have seen, gauge transformation at linear order involve the Lie derivative along the vector ξ of the background quantity. Hence a tensor quantity T is gauge-invariant if it is zero at the background level, i.e. $T_0 = 0$.

Evolution of Cosmological Perturbations

To study the evolution of cosmological perturbations in General Relativity, we have to perturb the Einstein's equations and the stress-energy continuity equations (and the Klein-Gordon equation of motion, if a scalar field is involved).
First of all, we perturb the connection coefficients, recalling that, for the spatially flat case, the only non-zero connection coefficients are $\Gamma^0_{00} = a'/a$, $\Gamma^i_{0j} = (a'/a)\delta^i_j$ and $\Gamma^0_{ij} = (a'/a)\delta_{ij}$.
We have (at first order), for scalar perturbations

Attenzione: nelle formule seguenti, fino a pag. 25 inclusa dovete sostituire a phi --> psi, e viceversa.

$$\delta\Gamma^0_{00} = \phi'$$

$$\delta\Gamma^0_{0i} = \partial_i\phi + \frac{a'}{a}\partial_i\omega$$

$$\delta\Gamma^i_{00} = \frac{a'}{a}\partial^i\omega^{\parallel} + \partial^i\omega'^{\parallel} + \partial^i\phi$$

$$\delta\Gamma^0_{ij} = -2\frac{a'}{a}\phi\,\delta_{ij} - \partial_i\partial_j\omega^{\parallel} - 2\frac{a'}{a}\psi\,\delta_{ij} - \psi'\delta_{ij} + \frac{a'}{a}D_{ij}\chi^{\parallel} + \frac{1}{2}D_{ij}\chi'^{\parallel}$$

$$\delta\Gamma^i_{0j} = -\psi'\delta^i_j + \frac{1}{2}D^i_j\chi'^{\parallel}$$

$$\delta\Gamma^i_{jk} = -\partial_j\psi\,\delta^i_k - \partial_k\psi\,\delta^i_j + \partial^i\psi\,\delta_{jk} - \frac{a'}{a}\partial^i\omega^{\parallel}\delta_{jk} + \frac{1}{2}\partial_j D^i_k\chi^{\parallel} + \frac{1}{2}\partial_k D^i_j\chi^{\parallel} - \frac{1}{2}\partial^i D_{jk}\chi^{\parallel}$$

The background Ricci tensor has the only non-zero components $R_{00} = -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2$ and $R_{ij} = \left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right]\delta_{ij}$.
Its perturbations read:

$$\delta R_{00} = \frac{a'}{a}\nabla^2\omega'' + \nabla^2\omega''' + \nabla^2\phi + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}\phi'$$

$$\delta R_{0i} = \frac{a'}{a}\partial_i\omega'' + \left(\frac{a'}{a}\right)^2\partial_i\omega'' + 2\partial_i\psi' + 2\frac{a'}{a}\partial_i\phi + \frac{1}{2}\partial_k D^k{}_i\chi'$$

$$\delta R_{ij} = \Big[-\frac{a'}{a}\phi' - 5\frac{a'}{a}\psi' + 2\frac{a''}{a}\phi - 2\left(\frac{a'}{a}\right)^2\phi - 2\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi - \psi'' + \nabla^2\psi - \frac{a'}{a}\nabla^2\omega''\Big]\delta_{ij} - \partial_i\partial_j\omega''' + \frac{a'}{a}D_{ij}\chi''' + \frac{a''}{a}D_{ij}\chi'' + \left(\frac{a'}{a}\right)^2 D_{ij}\chi'' + \frac{1}{2}D_{ij}\chi'' + \partial_i\partial_j\psi - \partial_i\partial_j\phi - 2\frac{a'}{a}\partial_i\partial_j\omega'' + \frac{1}{2}\partial_k\partial_i D^k{}_j\chi'' + \frac{1}{2}\partial_k\partial_j D^k{}_j\chi'' - \frac{1}{2}\nabla^2 D_{ij}\chi''.$$

The background Ricci scalar reads $R = \frac{6}{a^2}\frac{a''}{a}$, while its perturbation is

$$\delta R = \frac{1}{a^2}\Big(-6\frac{a'}{a}\nabla^2\omega'' - 2\nabla^2\omega''' - 2\nabla^2\phi - 6\psi'' - 6\frac{a'}{a}\phi' - 18\frac{a'}{a}\psi' - 12\frac{a''}{a}\phi + 4\nabla^2\psi + \partial_k\partial^i D^k{}_i\chi''\Big).$$

→ The gauge-invariant curvature perturbation on comoving hypersurfaces.

At linear order, the intrinsic spatial curvature on hypersurfaces of constant conformal time τ and for a flat universe is:

$$^{(3)}R = \frac{4}{a^2} \nabla^2 \hat{\phi}$$

where, for simplicity, we have defined

$$\hat{\phi} \equiv \phi + \frac{1}{6} \nabla^2 \chi''$$

The quantity $\hat{\phi}$ is often referred to as "curvature" perturbation. This quantity is not gauge-invariant, in fact

$$\hat{\phi} \to \tilde{\hat{\phi}} = \hat{\phi} - \frac{a'}{a} \alpha$$

However, the combination

$$\boxed{-\zeta = \hat{\phi} + \frac{a'}{a} \frac{\delta \rho}{\rho_0'}}$$

is clearly gauge-invariant. It is called the "gauge-invariant curvature perturbation of comoving hypersurfaces" and is very often used in connection w. inflation because it is conserved on super-horizon scales and if non-adiabatic pressure perturbations are absent.

Perturbations of the stress-energy tensor

If the source of Einstein's equations is given by a perfect fluid, then we have

$$-\cancel{\delta}T^0{}_0 = \rho_0(\tau) + \delta\rho \equiv \rho_0(\tau)(1+\delta)$$

The space-components of the stress-tensor $T^i{}_j$ are represented by an isotropic pressure

$$p_0 + \delta p = \frac{1}{3} T^i{}_i = p_0(1+\Pi_L)$$

and a traceless anisotropic stress, with

$$T^i{}_j = p_0(\tau)\left[(1+\Pi_L)\,\delta^i{}_j + \underbrace{\Pi_T{}^i{}_j}_{\equiv D^i{}_j \Pi_T}\right],$$

N.B.: during my lecture I have indicated the anisotropic stress tensor just with P_i^j; here the T in P_T stands for "traceless".

where the trace-less 3-tensor $\Pi_T{}^i{}_j$ represents the anisotropic stress (which, strictly speaking, represents an imperfection of the fluid).
One can also define the non-adiabatic pressure perturbation (also called "entropy perturbation")

$$\delta P_{nad} = p_0(\tau)\Pi_L - \rho_0 \frac{dp_0}{d\rho_0}\delta$$

$$= \delta P - c_s^2 \delta\rho .$$

We finally have

$$\delta T^0{}_i = (\rho_0 + p_0)(\partial_i v'' + \partial_i \omega'')$$

$$\delta T^i{}_0 = -(\rho_0 + p_0)\partial^i v''$$

Equations of motion for perturbations

From the energy constraint one obtains (00 comp.)

$$3\frac{a'}{a}\left(\hat{\phi}' + \frac{a'}{a}\psi\right) - \nabla^2\left(\hat{\phi} + \frac{\dot{a}}{a}\sigma\right) = -4\pi G a^2 \delta\rho$$

where

$$\sigma \equiv \frac{1}{2}\chi'' - w'' \quad \text{(shear perturbation)}$$

The momentum constraint (0-i eq.) gives

$$\hat{\phi}' + \frac{a'}{a}\psi = -4\pi G a^2(\rho + p)V$$

where

$$V \equiv v'' + w''$$

The perturbed spatial Einstein eqs. yield

$$\hat{\phi}'' + 2\frac{a'}{a}\hat{\phi}' + \frac{a'}{a}\psi' + \left(2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right)\psi =$$

$$= 4\pi G a^2\left(\Pi_L + \frac{2}{3}\nabla^2\Pi_T\right)\rho_0$$

$$\sigma' + 2\left(\frac{a'}{a}\right)\sigma + \hat{\phi} - \psi = 8\pi G a^2 \Pi_T \rho_0$$

If we put ourselves in the longitudinal gauge, the latter eq. gives

$$\phi - \psi = 8\pi G a^2 \Pi_T \rho_0$$

which implies that if $\Pi_T = 0$ (no anisotropic stress)

$\phi = \psi$

Using the peculiarity of the longitudinal gauge and the form of the gauge-invariant potentials in such a gauge, we can write

$$-\Phi_H - \Psi_H = 8\pi G\, a^2 \Pi_T\, p_0$$

and

$$\Phi_H'' + 3\frac{a'}{a}\Phi_H' + \left(2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right)\Phi_H = -4\pi G a^2 \Pi_L\, p_0$$

if one has adiabatic perturbations, then $\delta p = c_s^2 \delta\rho$ where $c_s^2 \equiv p'/\rho'$

$$\boxed{\Phi_H'' + 3(1+c_s^2)\frac{a'}{a}\Phi_H' + \left[2\left(\frac{a'}{a}\right)' + (1+3c_s^2)\left(\frac{a'}{a}\right)^2 - c_s^2\nabla^2\right]\Phi_H = 0}$$

Energy and momentum conservation eqs.

$$\begin{cases} \delta\rho' + 3\frac{a'}{a}(\delta\rho + \delta p) - 3(\rho_0 + p_0)\phi' + (\rho_0 + p_0)\nabla^2(V + \sigma) = 0 \\ V' + (1 - 3c_s^2)\frac{a'}{a}V + \psi + \frac{1}{\rho_0 + p_0}\left(\delta p + \frac{2}{3}p_0\nabla^2\Pi_L\right) = 0 \end{cases}$$

The first of these equations allows to easily obtain the ζ conservation eq. on super-horizon scales. Indeed, on super-horizon scales, we have that the gradient terms can be neglected, hence

$$\delta\rho' + 3\frac{a'}{a}(\delta\rho + \delta p) - 3(\rho + p)\hat{\phi}' \simeq 0$$

If we have adiabatic perturbations, $\delta p = c_s^2 \delta\rho$,

$$\delta\rho' + 3\frac{a'}{a}(1+c_s^2)\delta\rho - 3(\rho+p)\hat{\phi}' \simeq 0$$

Let's assume the uniform density gauge, where $\delta\rho = 0$ and $\zeta = -\hat{\phi}$. In such a gauge, then

$$\boxed{\zeta' = 0} \quad \text{super-horizon scales}$$

which, being gauge-invariant is true in any gauge, provided $\delta p_{nad} = 0$.

the complete eq. reads:

$$\zeta' = -\frac{a'}{a}\frac{\delta p_{nad}}{\rho + p} - \Sigma_V$$

where

$$\Sigma_V \equiv \frac{1}{3}\nabla^2(V + \sigma)$$

Vector perturbations (*)

The case of vector perturbations is a very simple one. One doesn't have propagation equations but only a constraint equation, relating Ψ_i to the divergence-free velocity and a conservation eq. for the vorticity, which tells us that vorticity is conserved, along fluid trajectories, in the absence of dissipative effects (Kelvin's circulation theorem).

Tensor perturbations

For tensor perturbations, we obtain the GW equation

$$\chi''^{T}_{ij} + 2\frac{a'}{a}\chi'^{T}_{ij} - \nabla^2\chi^{T}_{ij} = 16\pi G\, p_0 a^2 \Pi_{T_{ij}}$$

where $\Pi_{T_{ij}}$ is the tensor-component of the stress (which is usually zero), so, for adiabatic perts.

$$\boxed{\chi''^{T}_{ij} + 2\frac{a'}{a}\chi'^{T}_{ij} - \nabla^2\chi^{T}_{ij} = 0}$$ gravitational waves

(*) From the (0-i) Einstein equations one obtains the equation for the Ψ_i quantity introduced earlier:

$$\nabla^2\Psi_i = 16\pi G a^2 (\rho_0 + p_0) V_{ic}$$

From the continuity equation for the fluid one gets:

$$\left[(\rho_0 + p_0)V_{ic}\right]' + 4\mathcal{H}(\rho_0 + p_0)V_{ic} = -\nabla_k\left(\Pi^k{}_{,i} + \Pi_i{}^{,k}\right) \qquad (\nabla_k \equiv \partial_k)$$

which is related to the Kelvin's circulation theorem.