It will be noticed that this is the same time as would be required for the ball to fall straight down from rest: the horizontal motion has no effect on the vertical motion. The value of x at $t = t_1$ is therefore

$$v_0 t_1 = 6.0 \text{ m s}^{-1} \times 2.0 \text{ s} = 12 \text{ m},$$

and this is the distance from the bottom of the wall at which the ball lands.

Example 2. A projectile has a range of 50 m and reaches a maximum height of 10 m. What is the elevation of the projectile?

Solution. If we divide Eq. (4.9) by (4.10) we get

$$\frac{H}{R} = \frac{v_0^2 \sin^2\theta/2g}{v_0^2\, 2 \sin\theta \cos\theta/g} = \tfrac{1}{4} \tan\theta.$$

Therefore

$$\tan\theta = 4H/R = 4 \times 10 \text{ m}/50 \text{ m} = 0.8,$$

which yields

$$\theta = 38^\circ\, 40'.$$

4.7 THE PHYSICS OF PROJECTILE MOTION

The interaction of mathematics and physics in this simple example of two-dimen- sional motion is worthy of attention. The physical assumption is that objects near the earth have a constant acceleration downwards. The other derived results, such as the equation of the trajectory and the expressions for the maximum height and the range, are mathematical deductions from this simple physical assumption. They are aspects of the physical situation not immediately obvious in the bald statement $\ddot{\mathbf{r}} = \mathbf{g}$, but they are, nevertheless, logically equivalent to it. In making the statement about the constancy of the acceleration, the physicist does so in the belief that he is making a statement about reality—a belief based, perhaps, on observation of falling bodies. When the logical process of mathematical analysis is applied, the derived results can be no more true than the statement itself. If some of these derived results are tested by experiment and found not to correspond with reality, then the conclusion is inescapable that the original statement did not correspond with reality either. In the process of deducing the consequences of any physical assumption, all the resources of mathematics can be utilized, and sometimes the complexity involved is frightening to the uninitiated. So much so that they tend to confuse the truth content with the mathematical detail. But, no matter how sophisticated the mathematics, the physical content never increases; if one wishes to alter the predicted consequences, one must alter the physical assumptions.

An excellent example of this process of modifying the physical statement of a problem, in the light of experimental evidence that conflicts with results derived from a previous statement, is to be found in the problem of the flight of a golf-ball. The assumption that the acceleration of the ball is constant and equal to

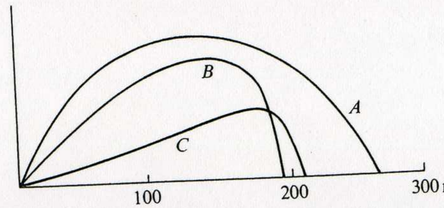

Fig. 4.19. The trajectory of a golf-ball. Curve A shows the path if only gravity acts on the ball; curve B the path if air resistance is also considered; and curve C the path when the effects of backspin are taken into account.

9.8 m s^{-2} downwards is not compatible with the observed facts, for it leads to predicted times of flight that are very much shorter than those actually observed and to a terminal speed that is equal to the speed on leaving the tee. This would make a golf-ball a lethal weapon indeed. Changing the physical assumptions to include the effect of air resistance modifies these results considerably. As shown in Fig. 4.19, the trajectory is no longer symmetrical, which is in agreement with observed trajectories, and the calculated terminal speed comes out to be much more reasonable. But the times of flight for angles of elevation comparable to those observed ($< 15°$) all turn out to be much too short; only two or three seconds. No amount of juggling with the parameters of the problem can give a satisfactory fit with the observed facts. In particular, it is not unknown for good drives to bend upwards for a good part of their total carry, and this implies, as we have seen, an upward acceleration toward the concave side of the curve. Hence, the conclusion is forced on the physicist that there is another influence at work, and this can be none other than the spin of the ball. When this is taken into account, good agreement can be obtained with observation. The lift on a ball with back- spin is quite analogous to the lift on an aerofoil. The importance of backspin in the golf drive was first realised by P. G. Tait in 1896. His approach, briefly indicated above, was that of the true physicist, and his sequence of assumption, mathematical deduction of consequences, checking with observation, reformula- tion of assumption, . . . , and so on, was a perfect example of scientific method, even if the application was to what heretics might regard as only a game. As a moral tailpiece, it might be added that, although he knew more about the physics of golf than any man of his day, Tait himself was only a mediocre player of the game.

4.8 UNIFORM CIRCULAR MOTION

Consider a particle traveling round a circular path with constant speed v. The time to go once round is called the *period* of the motion and is written T. Since the circumference of a circle is $2\pi r$, the relation between speed, radius, and period is

$$T = 2\pi r/v. \qquad (4.12)$$

The reciprocal of the period is called the frequency and will be written in this book as f. Other common symbols for the frequency are n and ν. The angular velocity or angular frequency of the particle is defined as the rate at which the radius from the center of the circle to the particle sweeps out angle. Since by definition it sweeps out 2π radians in time T, the angular velocity is simply

$$\omega = \frac{2\pi}{T} = 2\pi f. \tag{4.13}$$

Since the period is a time, it has units s, therefore both f and ω have units s^{-1}. It is conventional to write the units of ω as rad s^{-1}, and the units of f as simply s^{-1}, or c/s, standing for "cycles per second". The cycle per second is also known as the hertz (Hz). The radian unit of angle is discussed in Appendix A. It has no dimensions.

Substituting the value of T from Eq. (4.12) into (4.13) gives

$$\omega = v/r, \quad \text{or} \quad v = \omega r. \tag{4.14}$$

Consider the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ for two successive instants of time, as shown in Fig. 4.20. These vectors are, of course, both of length r, equal to the radius of the circle, and so there is no change in the magnitude of $\mathbf{r}$ with time. But there is a change of direction, and this means that the vector $\mathbf{r}$ is changing with time. The displacement of the particle between the two instants is the difference

$$\Delta\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = P_1P_2.$$

It will be apparent that as the interval of time is shrunk to zero in the usual way, then this vector displacement will tend to become perpendicular to the radius. Since the velocity vector is defined by the relation

$$\mathbf{v} \equiv \dot{\mathbf{r}} \equiv \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t},$$

the conclusion is that, for circular motion, the velocity vector is perpendicular to the position vector. The velocity vector is *always* along the tangent line to the path, but it is only perpendicular to the position vector for a circular path, since this is a unique geometrical property of a circle. This is true whether the speed is constant or not. The rule for obtaining the velocity from the position vector may be formulated as follows.

a) *The magnitude of the rate of change of position in uniform circular motion is obtained by multiplying the magnitude of the position vector by the angular velocity* ω (*Eq.* 4.14).

b) *The direction of the rate of change of position is perpendicular to the position vector, the sense being given by a rotation of the position vector in the direction of motion.*

This rule is illustrated in Fig. 4.21. The velocity vector and the position vector have been drawn from a common point, which may be taken as the center of the circle. Vectors may always be slid about parallel to themselves on the page: all that matters is that the magnitude and the direction be right.

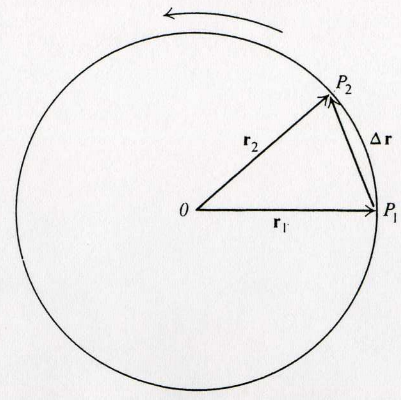

Fig. 4.20. The displacement of a particle which is undergoing uniform circular motion.

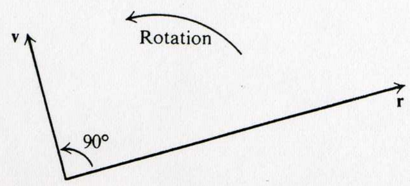

Fig. 4.21. The relation of the position vector and the velocity vector in uniform circular motion.

As the particle moves with constant speed round the circle, the position vector (which is simply the directed line from the center to the particle) rotates with constant angular velocity ω, and the velocity vector also rotates with constant angular velocity ω but is 90° ahead of the position vector by the rule above. In Fig. 4.22 the position vectors at two instants of time have been drawn, as in Fig. 4.20, and the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for the same two instants have also been drawn from the center of the circle. The angles P_1OP_1' and P_2OP_2' are both 90°.

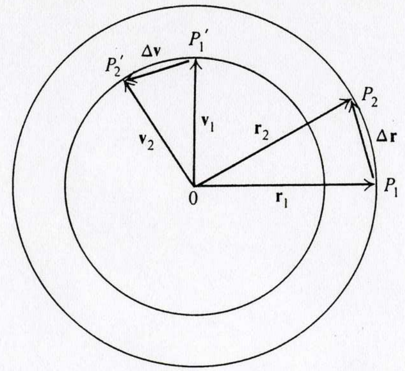

Fig. 4.22. The displacement and the change of velocity of a particle undergoing uniform circular motion.

The change in velocity is $\Delta\mathbf{v} = P_1'P_2' = \mathbf{v}_2 - \mathbf{v}_1$, and so the average acceleration over the interval is $\Delta\mathbf{v}/\Delta t = (\mathbf{v}_2 - \mathbf{v}_1)/(t_2 - t_1)$. As the interval Δt is shrunk to zero, $\Delta\mathbf{v}$ becomes perpendicular to $\mathbf{v}$ in exactly the same way as $\Delta\mathbf{r}$ becomes per- pendicular to $\mathbf{r}$. Note that the magnitude of the velocity does not change with time, only its direction. It follows that the instantaneous acceleration defined by

$$\mathbf{a} \equiv \dot{\mathbf{v}} \equiv \ddot{\mathbf{r}} \equiv \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t}$$

is perpendicular to the velocity, just as the instantaneous velocity is perpendicular to the position vector. Study of Fig. 4.22 will show that the triangles OP_1P_2 and $OP_1'P_2'$ are similar, and therefore, by the properties of similar triangles, the ratios of corresponding sides are equal. That is,

$$\Delta r/r = \Delta v/v,$$

since $r_1 = r_2 = r$, and $v_1 = v_2 = v$. Dividing both sides of this equation by Δt gives

$$\frac{1}{r}\frac{\Delta r}{\Delta t} = \frac{1}{v}\frac{\Delta v}{\Delta t}.$$

Proceeding to the limit by shrinking the interval Δt to zero leads directly to

$$v/r = a/v.$$

This gives the following alternative expressions for the magnitude of the acceleration, by the use of Eq. (4.14):

$$a = v^2/r = \omega^2 r = \omega v. \tag{4.15}$$

The direction of the acceleration is perpendicular to the velocity just as the velocity is perpendicular to the position vector, and study of Fig. 4.22 shows that the acceleration leads the velocity by 90° just as the velocity leads the position vector by 90°. Thus the rule for obtaining the acceleration from the velocity is the same as the rule for obtaining the velocity from the position vector. The magnitude of the acceleration is the angular velocity times the magnitude of the velocity; the direction is 90° ahead of the velocity. Figure 4.23 shows the three vectors $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{a}$ for the same instant, all drawn from the center of the circle. The acceleration is 180° ahead of the position vector, which is another way of saying that it acts toward the center of the circle. It is called the *centripetal* acceleration. The magnitudes of the three vectors are constant in time, but the vectors themselves are not, since their directions are constantly changing. The general rule for finding the rate of change of any vector in uniform circular motion should now be clear from the above discussion; and if, for example, one wanted to find the rate of change of the acceleration, then, clearly, one would find the magnitude by the product ωa, and the direction by rotating once again through 90° in the sense of the rotation of the particle. This would give the vector shown dotted in Fig. 4.23.

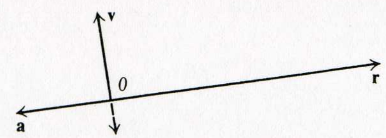

Fig. 4.23. The relation of the position vector, the velocity vector, and the acceleration vector in uniform circular motion.