Introduction to Supersymmetry

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These are preliminary notes written for the second part of the course "Advanced Topics in the Theory of the Fundamental Interactions", held by Brando Bellazzini and myself at the University of Padova in the academic year 2018-19. The second part of the course provides a basic introduction to supersymmetry. A main theme is to show how quantum corrections are greatly constrained by the general properties of supersymmetric theories. Methods of low-energy effective field theories will be emphasized.

References:

- A. Bilal "Introduction to Supersymmetry", hep-th/0101055
- M. Bertolini "Lectures on Supersymmetry", LINK
- J. Terning "Modern Supersymmetry", Oxford University Press, 2006
- J. Wess, J. Bagger "Supersymmetry and Supergravity", Princeton Univ. Press, 1992
- S. Weinberg "The Quantum Theory of Fields", Vol. III, Cambridge Univ. Press, 2005

The course follows mainly the first two references.

Contents

1 Motivation and hystorical remarks

1.1 What is supersymmetry

Supersymmetry (SUSY) is a *symmetry* that maps particles and fields of integer spin (bosons) into particles and fields of half-integer spin (fermions), and vice-versa. The generator Q, called the *supercharge*, acts as:

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle\ , \qquad Q\,|\text{fermion}\rangle = |\text{boson}\rangle\ . \tag{1.1}$$

Since it changes the spin of a particle, and thus its spacetime properties, supersymmetry is a spacetime symmetry.

Note that Q is *fermionic* and will thus satisfy anticommutation relations, as opposed to the commutation relations satisfied by the usual bosonic symmetry generators. In particular, the anticommutator of two supercharges generates a spacetime translation,

$$\{Q, \overline{Q}\} \sim \gamma^\mu P_\mu\ . \tag{1.2}$$

This means that the supersymmetry transformations are not independent of the Poincaré transformations. In fact, we will see that supersymmetry is a non-trivial (i.e. it is not a direct product) extension of the Poincaré group.

Each bosonic state has a fermionic superpartner, and vice-versa. Together the superpartners are unified into a *supermultiplet*. These form the basic representations of supersymmetry.

After having studied the basic properties of the supersymmetry algebra and its representations, we will discuss how supersymmetry is realized in field theory.

1.2 Why to study supersymmetry

There has been no experimental evidence for supersymmetry so far. Maybe it is realized in Nature at energy scales higher than those probed in current experiments (as we will see later, we know that it has to be broken at our energy scales), maybe it is not realized at all. However, there are many good reasons for studying it. These are in part phenomenological and in part purely theoretical. Here we summarize the main ones.

- Since it describes bosons and fermions at the same time, supersymmetry is a unifying framework with the potential of encompassing matter and radiation together.

- It is the only way to evade the Coleman-Mandula no-go theorem. Under reasonable assumptions, this theorem states that in relativistic QFT's there are no non-trivial extensions of the Poincaré algebra by ordinary Lie algebras [see QFT2 course by prof. Lechner]. Supersymmetry evades the Coleman-Mandula theorem because it based on a superalgebra, which is not an ordinary Lie algebra.

- Radiative corrections are suppressed, due to cancellations between fermion loops and boson loops. In particular, supersymmetry removes the quadratic divergences. This has important penomenological implications. In the Standard Model, the bare mass of the Higgs particle is $m_H \sim 100\text{GeV}$; the measured value is $m_H \sim 125\text{GeV}$ and is thus very close to the bare mass. However one would a priori expect large quantum corrections. Indeed the Yukawa coupling $-\lambda_f \, H \, \bar{f} \, f$ induces a one-loop correction to the Higgs propagator, and thus to the Higgs mass, as

$$\Delta m_H^2 \sim -\lambda_f^2 \, \Lambda^2 \ , \tag{1.3}$$

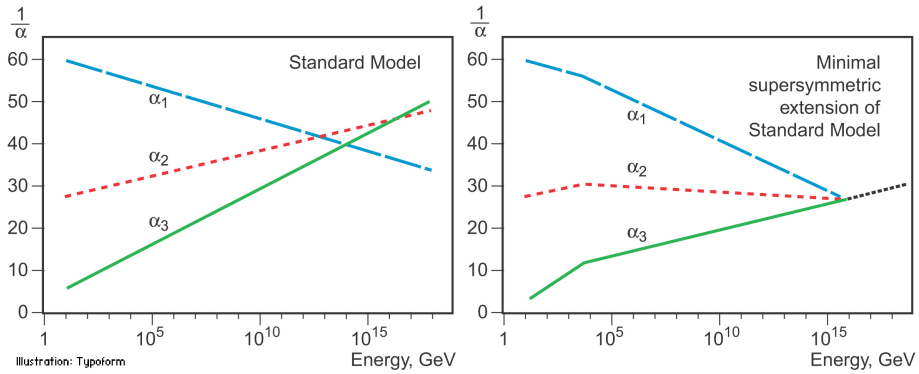
where Λ is the UV cutoff beyond which the Standard Model breaks down as an effective theory. For the correction to be not too large, Λ should be of the order of the TeV scale. However, there has been no compelling experimental reason so far for fixing $\Lambda \sim$ TeV, and the cutoff may also be much higher. It is therefore hard to explain why the Higgs mass receives little quantum corrections without invoking a huge fine-tuning. This is known as the hierarchy problem: the experimental value of the Higgs mass is unnaturally smaller than its natural theoretical value.
Supersymmetry helps in solving this problem. Consider a complex scalar s being the supersymmetry partner of the fermion f considered above. In the supersymmetric extension of the Standard Model this would couple to the Higgs as $-\lambda_s |H|^2 |s|^2$. This would correct the Higgs propagator at one-loop as

$$\Delta m_H^2 \sim \lambda_s \, \Lambda^2 \ . \tag{1.4}$$

Since one has $\lambda_s = \lambda_f^2$, there is an exact cancellation between bosons and fermions running in the loops. This is a consequence of supersymmetry and does not require invoking any fine tuning. It follows that the Higgs mass is stabilized at its tree level value! This is the basic reason why supersymmetry helps in solving the hierarchy problem.

- This nice behavior is encoded in what is called "non-renormalization theorems". These state that certain quantities computed at tree or one-loop level are protected against radiative corrections, so that the result is actually valid at all orders in perturbation theory.

- In the Standard Model, the three gauge couplings of the SU(3) $\times$ SU(2) $\times$ U(1) evolve with the energy scale and *approximately* meet at the scale of 10^{15} GeV. In the minimal



supersymmetric extension of the Standard Model these coupling constants unify *precisely* at the scale of $\sim 10^{16}$ GeV (see Figure, where $\alpha_1, \alpha_2, \alpha_3$ are the U(1), SU(2), SU(3) coupling constants, respectively). This means that at such energy scale there could be just one type of gauge interaction with a larger gauge group, containing SU(3) $\times$ SU(2) $\times$ U(1). This is very appealing from the theoretical point of view and supports the idea of a Great Unified Theory (GUT) at such energy scale.

- Susy provides natural dark matter candidates. Dark matter is believed to make up $\sim$ 25% of the energy density of the universe. Among the additional particles predicted by supersymmetry, the lightest supersymmetric particle is fully stable and thus a possible dark matter candidate.

- It is a building block of string theory, which overcomes the difficulties with quantum gravity by replacing point particles with extended objects such as open and closed strings.

- Susy is a theoretical laboratory for strongly coupled gauge dynamics. Strongly coupled non-Abelian gauge theories exhibit interesting but poorly understood phenomena at low energies, such as confinement and the generation of a mass gap. The additional constraints imposed by supersymmetry allow to say much more about the emergent degrees of freedom and the structure of the effective theory at low energies. Sometimes even exact results can be obtained. The hope is to use supersymmetry to learn qualitative features that also apply to more realistic models. For instance supersymmetric versions of QCD have given insight on the strong coupling dynamics that is responsible for quark confinement.

- Often supersymmetry uncovers beautiful mathematical structures.

1.3 Some history

Supersymmetry was born in the early seventies and has constantly been an active field of research since then. Here are some milestones:

- 1967 Coleman–Mandula no-go theorem.
- 1971 Gol'fand–Likhtman: susy algebra as a possible extension of the Poincaré group.
- 1971 Ramond, Neveu-Schwarz: susy in the two-dimensional worldsheet of string theory.
- 1973 Volkov–Akulov: first four-dimensional supersymmetric field theory, although supersymmetry was spontaneously broken and thus non-linearly realized (they were trying to explain the apparent vanishing mass of neutrinos by interpreting them as Goldstone particles).
- 1974 Wess–Zumino: first linear realization of supersymmetry in a four-dimensional field theory
- 1976 Ferrara-Freedman-Van Nieuwenhuizen: first theory where supersymmetry is locally realized. This automatically incorporates a graviton and was thus called *supergravity*. Supergravity may be seen as another motivation for supersymmetry, in the sense that a QFT with local supersymmetry automatically contains general relativity and is thus a step towards the unification of QFT with general relativity. However we know that supergravity theories are non-renormalizable. For this reason, they should be seen as a low-energy effective theory of a more complete theory. This more complete theory is string theory, which solves the problems of quantum gravity in a completely new framework, were particles are replaced by strings...but this would be another story.
- ...many developments...
- 1994 Seiberg–Witten theory
- ...many developments...

2 Preliminaries

We work in Minkowski$_4$. The metric is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, so we use a mostly minus signature convention.

Greek letters $\mu, \nu = 0, 1, 2, 3$ denote spacetime indices, while Latin letters $i, j = 1, 2, 3$ indicate space indices.

2.1 Lorentz and Poincaré groups

The Lorentz group is SO(1,3). The generators of the algebra are the three rotations J_i and the three boosts K_i. These satisfy the commutation relations:

$$[J_i, J_j] = i\,\epsilon_{ijk} J_k \ , \qquad [K_i, K_j] = -i\,\epsilon_{ijk} J_k \ , \qquad [J_i, K_j] = i\,\epsilon_{ijk} K_k \ . \tag{2.1}$$

The J_i are hermitian while the K_i are anti-hermitian. The combinations

$$J_i^{\pm} = \tfrac{1}{2}(J_i \pm i K_i) \tag{2.2}$$

are hermitian and satisfy

$$[J_i^{\pm}, J_j^{\pm}] = i\,\epsilon_{ijk} J_k^{\pm} \ , \qquad [J_i^{\pm}, J_j^{\mp}] = 0 \ , \tag{2.3}$$

thus they generate two commuting SU(2) algebras. In fact, one has the algebra isomorphism $\mathrm{SO}(1,3) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)^*$.

The Lorentz group SO(1,3) is also related by a homomorphism to $\mathrm{SL}(2,\mathbb{C})$, the group of 2×2 complex matrices with unit determinant. In order to see this, we introduce the matrices

$$\sigma_\mu = (\mathbb{1}\,,\,\sigma_i) \ , \tag{2.4}$$

where σ_i are the usual Pauli matrices, satisfying $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ (note that these have a lower index). The σ_μ form a basis for the 2×2 complex matrices. Given a four-vector vector x^μ, we can construct the 2×2 matrix $x^\mu \sigma_\mu$. This is an hermitian matrix and has determinant $x_\mu x^\mu$, which is a Lorentz invariant. Consider a Lorentz transformation acting on the four-vector as $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. We want to determine how it acts on the 2×2 matrix $x^\mu \sigma_\mu$. It must preserve the hermiticity (because it sends spacetime vectors into spacetime vectors) and the determinant (because it preserves the Lorentz norm $x_\mu x^\mu$). It follows that the action via a 2×2 matrix A

$$x^\mu \sigma_\mu \to A x^\mu \sigma_\mu A^\dagger \ , \tag{2.5}$$

with $\det A = 1$ (up to a phase) corresponds to a Lorentz transformation. We have thus realized the Lorentz transformations as complex 2×2 matrices of unit determinant, that is as elements of $\mathrm{SL}(2,\mathbb{C})$.[1]

The Poincaré group is the semi-direct product of the Lorentz group and the group of translations in spacetime. Denoting by P_μ the generators of translations, we have the addi-

[1] See Bertolini's lectures, section 2.1, for a more detailed explanation and in particular for the precise relation between A and Λ. The precise relation between SO(1,3) and $\mathrm{SL}(2,\mathbb{C})$ is $\mathrm{SO}(1,3) \simeq \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

tional commutation relations:

$$
\begin{aligned}
[P_\mu, P_\nu] = 0 \ , \qquad [J_i, P_j] &= i\,\epsilon_{ijk} P_k \ , \qquad [J_i, P_0] = 0 \ , \\
[K_i, P_j] &= -i\,P_0 \ , \qquad [K_i, P_0] = -i\,P_i \ .
\end{aligned} \tag{2.6}
$$

The generators of the Lorentz group can be repackaged into generators $M_{\mu\nu} = -M_{\nu\mu}$ as:

$$
M_{0i} = K_i \ , \qquad M_{ij} = \epsilon_{ijk} J_k \ , \tag{2.7}
$$

so that the Poincaré algebra reads:

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0 \ , \\
&[M_{\mu\nu}, M_{\rho\sigma}] = -i\,\eta_{\mu\rho} M_{\nu\sigma} + i\,\eta_{\mu\sigma} M_{\nu\rho} + i\,\eta_{\nu\rho} M_{\mu\sigma} - i\,\eta_{\nu\sigma} M_{\mu\rho} \ , \\
&[M_{\mu\nu}, P_\rho] = -i\,\eta_{\rho\mu} P_\nu + i\,\eta_{\rho\nu} P_\mu \ .
\end{aligned} \tag{2.8}
$$

This clearly shows the semi-direct product structure.

2.2 Spinors

Supersymmetry involves a lot of spinor algebra. In four dimensions, this is conveniently dealt with using a *two-component* spinor notation. After the training necessary to get used to it, this notation makes the computations involving spinors faster.

The two-component notation uses the basic representations of $\mathrm{SL}(2,\mathbb{C})$. A spinor is a defined as a two-component object $\psi = \binom{\psi_1}{\psi_2}$, where ψ_1, ψ_2 are anti-commuting Grassmann variables. It transforms under an element $\mathcal{M} \in \mathrm{SL}(2,\mathbb{C})$ as

$$
\psi_\alpha \to \psi'_\alpha = \mathcal{M}_\alpha{}^\beta \psi_\beta \ , \qquad \alpha, \beta = 1, 2. \tag{2.9}
$$

Since for $\mathrm{SL}(2,\mathbb{C})$ the representations associated with $\mathcal{M}$ and $\mathcal{M}^*$ are not equivalent (there is no matrix C so that $\mathcal{M} = C\mathcal{M}^* C^{-1}$), we can also introduce a different type of spinor, which transforms as

$$
\overline{\psi}_{\dot\alpha} \to \overline{\psi}'_{\dot\alpha} = \mathcal{M}^{*}{}_{\dot\alpha}{}^{\dot\beta}\, \overline{\psi}_{\dot\beta} \ . \tag{2.10}
$$

This is called a dotted spinor, while ψ_α is called undotted spinor.

Note that for a generic $\mathcal{M}$ we can write

$$
\begin{aligned}
\mathcal{M} &= e^{(\beta_j + i\omega_j)\sigma_j} \ , \\
\mathcal{M}^* &= e^{(\beta_j - i\omega_j)\sigma_j^*} \ ,
\end{aligned} \tag{2.11}
$$

where again σ_j are the Pauli matrices. This shows how the $\mathrm{SL}(2,\mathbb{C})$ matrices are expressed in terms of the generators of the spin-$\frac{1}{2}$ representation of the $\mathrm{SU}(2) \times \mathrm{SU}(2)^*$ algebra that we have encountered before. $\mathcal{M}$ is constructed exponentiating the J^+ generators while $\mathcal{M}^*$ is constructed exponentiating the J^- generators. Therefore undotted spinors are a $(\frac{1}{2},0)$ representation of $\mathrm{SU}(2)\times\mathrm{SU}(2)^*$ while dotted spinors form a $(0,\frac{1}{2})$ representation.

In order to raise and lower the spinor indices, we introduce the antisymmetric matrices:

$$\epsilon^{\alpha\beta}=\epsilon^{\dot\alpha\dot\beta}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\ ,\qquad\qquad \epsilon_{\alpha\beta}=\epsilon_{\dot\alpha\dot\beta}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\ . \tag{2.12}$$

These satisfy $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma}=\delta_\alpha^\gamma$, etc. They are Lorentz invariant since $\epsilon_{\alpha\beta}=\mathcal{M}_\alpha{}^\gamma\mathcal{M}_\beta{}^\delta\epsilon_{\gamma\delta}$, etc. On the spinors they act as:

$$\psi^\alpha=\epsilon^{\alpha\beta}\psi_\beta\ ,\qquad \psi_\alpha=\epsilon_{\alpha\beta}\psi^\beta\ ,\qquad \bar\psi^{\dot\alpha}=\epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}\ ,\qquad \bar\psi_{\dot\alpha}=\epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\ . \tag{2.13}$$

The convention here is that indices are always contracted putting the epsilon tensor on the left.

We will use the convention that ψ_α is a column-array, ψ^α is a row-array, $\bar\psi^{\dot\alpha}$ is a column-array and $\bar\psi_{\dot\alpha}$ is a row-array. More generally, lower undotted indices label column-arrays, while upper undotted indices label row-arrays. The opposite convention applies to dotted indices.

By comparing how they transform under $\mathrm{SL}(2,\mathbb{C})$, we can identify $(\psi_\alpha)^*=\bar\psi^{\dot\alpha}$ (both are column-arrays) and $\bar\psi_{\dot\alpha}=(\psi_\alpha)^\dagger$ (both are row-arrays), $(\bar\psi^{\dot\alpha})^\dagger=\psi^\alpha$ etc. Also, $(\mathcal{M}_\alpha{}^\beta)^*=(\mathcal{M}^{*-1T})^{\dot\alpha}{}_{\dot\beta}=\epsilon^{\dot\alpha\dot\gamma}(\mathcal{M}^*)_{\dot\gamma}{}^{\dot\delta}\epsilon_{\dot\delta\dot\beta}$. The last expression implies that the dotted spinor $\bar\psi^{\dot\alpha}$ transforms in a representation of $\mathrm{SL}(2,\mathbb{C})$ equivalent to the one of $\bar\psi_{\dot\alpha}$, the equivalence matrix being $\epsilon_{\dot\alpha\dot\beta}$. Explicitly, $\bar\psi^{\dot\alpha}$ transforms as:

$$\bar\psi^{\dot\alpha}\to\bar\psi'^{\dot\alpha}=(\mathcal{M}^{*-1T})^{\dot\alpha}{}_{\dot\beta}\bar\psi^{\dot\beta}\ . \tag{2.14}$$

Note from (2.11) that

$$\mathcal{M}^{*-1T}=e^{(-\beta_j+i\omega_j)\sigma_j}\ . \tag{2.15}$$

So the spinors ψ_α and $\bar\psi^{\dot\alpha}$ transform in the same way under rotation but with an opposite sign in the boost parameter [cf. notes on two-component spinors by B. Bellazzini].

The convention for contracting the spinor indices is:

$$\chi\,\psi=\chi^\alpha\psi_\alpha\ ,\qquad \bar\chi\,\bar\psi=\bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha}\ . \tag{2.16}$$

Namely, undotted indices are contracted with the “NorthWest to SouthEast” convention, while dotted indices are contracted with the “SouthWest to NorthEast” convention (let us repeat that this rule does not apply when raising or lowering indices with the epsilon tensor).

It is not hard to check that expressions where all the indices are contracted following these rules are indeed Lorentz invariant.

We also introduce the 2×2 counterpart of the gamma matrices. These are the following sigma matrices:

$$\sigma_\mu = (\mathbb{1}\,,\,\sigma_i)\ ,\qquad \bar{\sigma}_\mu = (\mathbb{1}\,,\,-\sigma_i)\ , \tag{2.17}$$

where the σ_μ are those already seen before.[2] The index structure of the sigma's is $(\sigma^\mu)_{\alpha\dot{\alpha}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$. We can now construct Lorentz four-vectors such as $\psi\sigma^\mu\bar{\chi}$ and $\bar{\psi}\bar{\sigma}^\mu\chi$.

♦ **Exercise.** Check that:
1) ψ^α transforms as $\psi'^\alpha = \psi^\beta(\mathcal{M}^{-1})_\beta{}^\alpha$;
2) $\chi\,\psi$ is a Lorentz invariant;
3) $\chi\,\psi = \psi\,\chi$ (recall that the spinor components are anti-commuting);
4) the following useful identities involving the sigma matrices:

$$\begin{aligned}
&\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}_\mu^{\dot{\beta}\beta} = 2\delta_\alpha^\beta\delta_{\dot{\alpha}}^{\dot{\beta}}\ .\\
&\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\,\eta_{\mu\nu}\ ,\\
&(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} \;=\; \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma_\mu)_{\beta\dot{\beta}}\ ,\qquad (\sigma_\mu)_{\alpha\dot{\alpha}} \;=\; \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta}\ ;
\end{aligned} \tag{2.18}$$

5) $\chi\sigma^\mu\bar{\psi}$ is a Lorentz four-vector;
6) $\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\,\bar{\sigma}^\mu\chi$.
7) $(\chi\sigma^\mu\bar{\psi})^\dagger = \psi\sigma^\mu\bar{\chi}$. ♦

Relation with four-component spinors

Let us make the connection with the four-component notation you may be more familiar with. Dirac spinors transform in the reducible $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz algebra and are given by $\Psi = \binom{\psi_\alpha}{\bar{\chi}^{\dot{\alpha}}}$. The gamma matrices in the Weyl basis are:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\ ,\qquad \gamma_5 = -i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\ . \tag{2.19}$$

We see that a spinor $\Psi_L = \binom{\psi_\alpha}{0}$ satisfies $\gamma_5\Psi_L = \Psi_L$ and is thus *left-handed*, while a spinor $\Psi_R = \binom{0}{\bar{\chi}^{\dot{\alpha}}}$ satisfies $\gamma_5\Psi_R = -\Psi_R$ and is thus *right-handed*. The action of the gamma-matrices on spinors is:

$$\gamma^\mu\Psi \;=\; \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot{\alpha}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} & 0 \end{pmatrix}\begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \;=\; \begin{pmatrix} (\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \\ (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\psi_\alpha \end{pmatrix} \tag{2.20}$$

[2] The matrices with the spacetime index up are: $\sigma^\mu = \big(\mathbb{1}\,,\,\sigma^i\big) = (\mathbb{1}\,,\,-\sigma_i)$, $\bar{\sigma}^\mu = \big(\mathbb{1}\,,\,-\sigma^i\big) = (\mathbb{1}\,,\,\sigma_i)$.

so we recover the action of the sigma's given above.

The Dirac conjugation gives $\overline{\Psi} \equiv \Psi^\dagger \gamma_0 = \left(\chi^\alpha, \bar{\psi}_{\dot{\alpha}}\right)$.

A Majorana spinor is a Dirac spinor with $\chi_\alpha = \psi_\alpha$, namely it is of the form $\binom{\psi_\alpha}{\bar{\psi}^{\dot{\alpha}}}$. Indeed, the Majorana condition is $\Psi^c = \Psi$, where the charge conjugation is defined as $\Psi^c = C\overline{\Psi}^T$, where the charge conjugation matrix satisfies $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ and can be taken to be $C = i\gamma^2\gamma^0$.

A Majorana mass term in the Lagrangian is a mass term built using a single Majorana spinor $\Psi = \binom{\psi_\alpha}{\bar{\psi}^{\dot{\alpha}}}$ and in two components notation reads

$$\overline{\Psi}\,\Psi = \psi\psi + \bar{\psi}\,\bar{\psi} = \psi\psi + h.c. \tag{2.21}$$

On the other hand, a Dirac mass term uses $\Psi = \binom{\psi_\alpha}{\bar{\chi}^{\dot{\alpha}}}$ and in two components notation reads

$$\overline{\Psi}\,\Psi = \chi\psi + \bar{\psi}\,\bar{\chi} = \psi\chi + h.c. \tag{2.22}$$

Finally, the Lorentz generators are

$$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \begin{pmatrix} i\sigma^{\mu\nu} & 0 \\ 0 & i\bar{\sigma}^{\mu\nu} \end{pmatrix} \ , \tag{2.23}$$

where

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \ , \qquad \bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu) \ . \tag{2.24}$$

Therefore the Lorentz algebra acts via $i(\sigma^{\mu\nu})_\alpha{}^\beta$ on left-handed spinors ψ_β and via $i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$ on right-handed spinors $\bar{\chi}^{\dot{\beta}}$. We can write:

$$\mathcal{M} = e^{\alpha_{\mu\nu}\sigma^{\mu\nu}} \ , \qquad \mathcal{M}^{*-1T} = e^{\alpha_{\mu\nu}\bar{\sigma}^{\mu\nu}} \ , \tag{2.25}$$

where the parameters $\alpha_{\mu\nu}$ are real.

⧫ **Exercise.** Check that the $\alpha_{\mu\nu}$ are related to the rotation and boost parameters ω_i, β_i in (2.11), (2.15) as: $\beta_i = \alpha_{0i}$, $\omega_i = -\frac{1}{2}\epsilon_{ijk}\alpha_{jk}$.

3 Supersymmetry algebra and its representations

3.1 Coleman-Mandula theorem

Under reasonable assumptions, the Coleman-Mandula theorem shows that in a relativistic QFT the only possible Lie algebra of symmetry generators consists of the generators P_μ and $M_{\mu\nu}$ of the Poincaré group, plus ordinary internal symmetry generators that commute with

P_μ and $M_{\mu\nu}$ and whose eigenvalues are independent of both momentum and spin.[3] In other words, the spacetime and internal symmetries can only be combined in a trivial way.

No-go theorems are always based on some assumptions, and sometimes they can be evaded by carefully revisiting and possibly relaxing part of such assumptions. Rather than being the last word on a subject, they have often been the starting point for new discoveries.

The Coleman–Mandula theorem forbids non-trivial extensions of the Poincaré group by ordinary Lie algebras. Lie algebras are generated by operators that satisfy commutation relations and take bosons into bosons and fermions into fermions. The theorem does not hold for more general algebras where some of the generators are fermionic: these satisfy anticommutation relations rather than commutation relations, and thus take bosons into fermions, and vice-versa. Haag, Lopuszański and Sohnius showed that the algebra associated with supersymmetry, called the superalgebra, is the only consistent realization of this more general algebra involving both commutators and anticommutators.

Note. One of the assumptions of the Coleman-Mandula theorem is that for any m there is only a finite number of particles with mass less than m. In fact another exception to the theorem is provided by theories with only massless particles, such as *conformal field theories*. Conformal field theories are governed by the conformal symmetry algebra, which non-trivially extends the Poincaré group by the dilatation operator D and the special conformal generators K_μ.

3.2 Superalgebra

Supersymmetry extends the Poincaré algebra by introducing fermionic generators Q^I_α, $\overline{Q}^I_{\dot\alpha}$, $I = 1, \ldots, N$. If $N = 1$, that is we have just one fermionic generator, we talk about minimal, or unextended supersymmetry. If $N > 1$, we talk about *extended* supersymmetry.

The commutation relations of the Poincaré generators $M_{\mu\nu}$, P_μ with the fermionic generators are taken to be:

$$\begin{aligned}
[P_\mu, Q^I_\alpha] &= 0 \ , \\
[P_\mu, \overline{Q}^I_{\dot\alpha}] &= 0 \ , \\
[M_{\mu\nu}, Q^I_\alpha] &= i\, (\sigma_{\mu\nu})_\alpha{}^\beta\, Q^I_\beta \ , \\
[M_{\mu\nu}, \overline{Q}^{\,I\dot\alpha}] &= i\, (\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\, \overline{Q}^{\,I\dot\beta} \ .
\end{aligned} \tag{3.1}$$

The first two tell us that the supersymmetry generators commute with the translations. The other two are just telling that the Lorentz algebra acts on Q^I_α (or $\overline{Q}^{I\dot\alpha}$) as on any other

[3] See Weinberg III, Chapter 24 for a more precise statement of the theorem and its proof.

undotted (or dotted) spinor. Moreover, the fermionic generators satisfy the anticommutation relations:

$$\begin{aligned}
\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\} &= 2\,\sigma^\mu_{\alpha\dot\beta} P_\mu\, \delta^{IJ} \ , \\
\{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta}\, Z^{IJ} \ , \\
\{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\} &= \epsilon_{\dot\alpha\dot\beta}\, (Z^{IJ})^* \ ,
\end{aligned} \tag{3.2}$$

where $Z^{IJ} = -Z^{JI}$ commute with all generators of the supersymmetry algebra and are called central charges. Note that they can only exist in the case of extended $N > 1$ supersymmetry, that is the $N = 1$ supersymmetry algebra has no central charges.

Note that the first expression in (3.2) is consistent with the fact that the anticommutator of Q^I_α and $\overline{Q}^J_{\dot\beta}$ must transform in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group, that is as a four-vector (this is because Q^I_α transforms in the $(\frac{1}{2}, 0)$ representation while $\overline{Q}^J_{\dot\beta}$ transforms in the $(0, \frac{1}{2})$).

We also observe that since $M_{12} = J_3$ and $\sigma_{12} = \bar\sigma_{12} = -\frac{i}{2}\sigma_3$, from the third and fourth expressions in (3.1) we have that

$$[J_3, Q^I_1] = \frac{1}{2} Q^I_1 \ , \qquad [J_3, Q^I_2] = -\frac{1}{2} Q^I_2 \ , \qquad [J_3, \overline{Q}^I_{\dot 1}] = -\frac{1}{2}\overline{Q}^I_{\dot 1} \ , \qquad [J_3, \overline{Q}^I_{\dot 2}] = \frac{1}{2}\overline{Q}^I_{\dot 2} \ . \tag{3.3}$$

It follows that Q^I_1 and $\overline{Q}^I_{\dot 2}$ raise the z-component of the spin by half a unit, while Q^I_2 and $\overline{Q}^I_{\dot 1}$ lower it by half a unit.

3.3 Representations of the superalgebra

For this lecture see Bertolini's notes, Sections 3.1 and 3.2.

Irreducible representations of supersymmetry are called supermultiplets. The main topics of the lecture are:

- review of the representations of the Poincaré group;
- three general properties of susy representations:
    1. all states in a supermultiplet have the same mass,
    2. in a supersymmetric theory the energy P_0 of any state is non-negative,
    3. a supermultiplet always contains an equal number of bosonic and fermionic states;
- construction of massless supermultiplets (for $N = 1$, $N = 2$ and $N = 4$ susy);
- construction of massive supermultiplets ($N = 1$ and $N = 2$) and shortening conditions in the case of extended supersymmetry ($N = 2$).

4 The simplest supersymmetric field theory

As a first example of an $N = 1$ supersymmetric field theory we discuss the original (free) Wess-Zumino model.

This is made by a complex scalar ϕ, a Majorana fermion ψ_α, $\bar{\psi}^{\dot{\alpha}}$ and a complex auxiliary field F. The Lagrangian is

$$\mathcal{L} = \partial_\mu \bar{\phi}\, \partial^\mu \phi + \tfrac{i}{2} \left(\partial_\mu \psi\, \sigma^\mu \bar{\psi} - \psi\, \sigma^\mu \partial_\mu \bar{\psi} \right) + \bar{F} F \ . \tag{4.1}$$

We immediately note that F is an auxiliary field that could be eliminated ("integrated out") using its algebraic equation of motion, that in this case reads $F = 0$. Nevertheless, it is convenient to keep it in order to show off-shell closure of the supersymmetry algebra, as we are going to see.

The supersymmetry transformations are:

$$\begin{aligned} \delta\phi &= \sqrt{2}\, \epsilon\psi \ , \\ \delta\psi_\alpha &= \sqrt{2} i \, (\sigma^\mu \bar{\epsilon})_\alpha\, \partial_\mu \phi - \sqrt{2}\, \epsilon_\alpha F \ , \\ \delta F &= \sqrt{2} i\, \partial_\mu \psi\, \sigma^\mu \bar{\epsilon} \ , \end{aligned} \tag{4.2}$$

where the spinors ϵ_α and $\bar{\epsilon}^{\dot{\alpha}}$ are the supersymmetry parameters (one being the complex conjugate of the other). Their components are Grassmann variables, meaning that they anticommute.

♦ **Exercise.**

1. Obtain the supersymmetry variation of the complex conjugate fields $\bar{\phi}$, $\bar{\psi}^{\dot{\alpha}}$, $\bar{F}$.

2. Show that the Lagrangian is invariant up to total derivative terms (hint: use integrations by parts).

3. Consider the following additional term in the Lagrangian:

$$\mathcal{L}_m = -m\phi F - \tfrac{1}{2}\, m\psi\psi + h.c. \tag{4.3}$$

and show that after F is integrated out, this reduces to standard mass terms for the complex scalar and the Majorana spinor:

$$\mathcal{L}_m = -m^2 \bar{\phi}\phi - \tfrac{1}{2}\, m\psi\psi - \tfrac{1}{2}\, m\bar{\psi}\bar{\psi} \ . \tag{4.4}$$

Notice that the scalar and spinor masses are the same, as prescribed by supersymmetry.

4. Verify that $\mathcal{L}_m$ is susy-invariant. ♦

We want to check that the supersymmetry variations given above indeed realize the superalgebra on the fields ϕ, ψ, F.

The variation δ is related to the $N = 1$ supercharges as:

$$\delta = i\epsilon Q + i\bar{\epsilon}\bar{Q} \tag{4.5}$$

and we use the convention that as an abstract operator it acts on any field f as:

$$\delta f = [f, i\epsilon Q + i\bar{\epsilon}\bar{Q}] \ . \tag{4.6}$$

Note that since both the supersymmetry parameter ϵ and the supercharge Q are anticommuting objects, the variation δ should be regarded as a commuting object (that is, it "passes through" anticommuting variables without acquiring a minus sign). Here we are following the conventions of Bertolini's lectures (Section 4.2).[4]

The commutator of two supersymmetry variations δ_1, δ_2 acting on any field f gives:

$$[\delta_1, \delta_2]f \ = \ [[f, i\epsilon_2 Q + i\bar{\epsilon}_2\bar{Q}], i\epsilon_1 Q + i\bar{\epsilon}_1\bar{Q}] - 1 \leftrightarrow 2 \ = \ [f, [\epsilon_1 Q + \bar{\epsilon}_1\bar{Q}, \epsilon_2 Q + \bar{\epsilon}_2\bar{Q}]] \ . \tag{4.7}$$

We compute:

$$\begin{aligned} [\epsilon_1 Q + \bar{\epsilon}_1\bar{Q}, \epsilon_2 Q + \bar{\epsilon}_2\bar{Q}] &= -\epsilon_1^\alpha \epsilon_2^\beta \{Q_\alpha, Q_\beta\} - \bar{\epsilon}_1^{\dot\alpha}\bar{\epsilon}_2^{\dot\beta}\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} + \big(\epsilon_1^\alpha \bar{\epsilon}_2^{\dot\beta} - \epsilon_2^\alpha \bar{\epsilon}_1^{\dot\beta}\big)\{Q_\alpha, \bar{Q}_{\dot\beta}\} \\ &= 2\big(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1\big)P_\mu \ . \end{aligned} \tag{4.8}$$

where in the second line we used the superalgebra (3.2) with $N = 1$. It follows that

$$[\delta_1, \delta_2]f = 2\big(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1\big)\,[f, P_\mu] \ . \tag{4.9}$$

Since the translation operator acts as $[f, P_\mu] = -i\partial_\mu f$, we conclude that the commutator of two susy variations has to be

$$[\delta_1, \delta_2]f = -2i\big(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1\big)\partial_\mu f \ , \tag{4.10}$$

namely it is the spacetime derivative along the real vector $-2i\big(\epsilon_1\sigma^\mu\bar{\epsilon}_2 - \epsilon_2\sigma^\mu\bar{\epsilon}_1\big)$. Note that this is a linear action.

Let us then verify that the supersymmetry variations (4.2) indeed satisfy (4.10). We start from the complex scalar ϕ. We have:

$$\begin{aligned} \delta_1\delta_2\,\phi &= \sqrt{2}\,\epsilon_2\,\delta_1\psi \\ &= 2i\,\epsilon_2\sigma^\mu\bar{\epsilon}_1\,\partial_\mu\phi - 2\epsilon_2\epsilon_1 F \ . \end{aligned} \tag{4.11}$$

[4]Note the extra i factor in eq. (4.5) compared to what we had done in class.

It follows that

$$[\delta_1,\delta_2]\phi = \delta_1\delta_2\,\phi - (1\leftrightarrow 2)$$
$$= -2i\,(\epsilon_1\sigma^\mu\bar\epsilon_2 - \epsilon_2\sigma^\mu\bar\epsilon_1)\,\partial_\mu\phi - 2(\epsilon_2\epsilon_1 - \epsilon_1\epsilon_2)F\ . \tag{4.12}$$

Since $\epsilon_2\epsilon_1 = \epsilon_1\epsilon_2$, the term proportional to F vanishes. We have thus verified (4.10).

Next we consider the spinor field ψ. We have:

$$\delta_1\delta_2\,\psi_\alpha = \sqrt{2}i\,(\sigma^\mu\bar\epsilon_2)_\alpha\,\partial_\mu(\delta_1\phi) - \sqrt{2}\,\epsilon_{2\alpha}\,\delta_1 F$$
$$= 2i\,(\sigma^\mu\bar\epsilon_2)_\alpha\,(\epsilon_1\partial_\mu\psi) - 2i\epsilon_{2\alpha}\,(\partial_\mu\psi\,\sigma^\mu\bar\epsilon_1)\ , \tag{4.13}$$

where we have emphasized in blue color the term coming from the variation of F.

Therefore we can write

$$[\delta_1,\delta_2]\,\psi = 2i\,(\sigma^\mu\bar\epsilon_2)_\alpha\,(\epsilon_1\partial_\mu\psi) + 2i\epsilon_{1\alpha}\,(\partial_\mu\psi\,\sigma^\mu\bar\epsilon_2) - (1\leftrightarrow 2)\ . \tag{4.14}$$

Now we apply the general identity[5]

$$\xi_\alpha\chi_\beta = \xi_\beta\chi_\alpha + \epsilon_{\alpha\beta}\,\xi^\gamma\chi_\gamma \tag{4.15}$$

to the first term (this identity is easily checked by assigning 1,2 values to α,β). We obtain:

$$2i\,(\sigma^\mu\bar\epsilon_2)_\alpha\,(\epsilon_1\partial_\mu\psi) = -2i\,\epsilon_1^\beta(\sigma^\mu\bar\epsilon_2)_\alpha\,\partial_\mu\psi_\beta = -2i\,\epsilon_1^\beta(\sigma^\mu\bar\epsilon_2)_\beta\,\partial_\mu\psi_\alpha - 2i\epsilon_{1\alpha}(\sigma^\mu\bar\epsilon_2)^\gamma\partial_\mu\psi_\gamma$$
$$= -2i\,(\epsilon_1\sigma^\mu\bar\epsilon_2)\,\partial_\mu\psi_\alpha - 2i\epsilon_{1\alpha}(\partial_\mu\psi\sigma^\mu\bar\epsilon_2)\ , \tag{4.16}$$

where in the second line we have just rearranged the spinors in the last term. Hence (4.14) becomes:

$$[\delta_1,\delta_2]\,\psi = -2i\,(\epsilon_1\sigma^\mu\bar\epsilon_2)\,\partial_\mu\psi_\alpha - 2i\epsilon_{1\alpha}(\partial_\mu\psi\sigma^\mu\bar\epsilon_2) + 2i\epsilon_{1\alpha}\,(\partial_\mu\psi\,\sigma^\mu\bar\epsilon_2) - (1\leftrightarrow 2)\ , \tag{4.17}$$

and we see that the second and third terms precisely cancel out. The surviving term realizes the algebra (4.10). This clearly shows the importance of the blue term, coming from the variation of F, to realize the supersymmetry algebra. Had we set $F=0$ from scratch using its equation of motion, the blue term would have not been there. In this case we could still have achieved closure of the superalgebra, at the expense of using the equation of motion $\partial_\mu\psi\,\sigma^\mu = 0$ of the spinor field. This is a general fact: without the auxiliary fields the supersymmetry algebra only closes on-shell, that is using the (fermion field) equations of motion. In general it is preferable to work with an off-shell realization of supersymmetry if it

[5] This comes from antisymmetrizing the Fierz identity $\xi_\alpha\chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\xi\chi) + \frac{1}{2}(\xi\sigma_{\mu\nu}\chi)\sigma^{\mu\nu}{}_\alpha{}^\gamma\epsilon_{\gamma\beta}$.

exists. A main reason is that off-shell closure of the algebra is independent of the Lagrangian (and thus of the interactions between the fields) while on-shell closure requires the equations of motion and thus depends on the specific Lagrangian under consideration. Notice that the four off-shell degrees of freedom of the bosonic fields ϕ, F do match the four off-shell degrees of freedom of the fermionic field ψ. When we go on-shell, the auxiliary field F carries no degrees of freedom while ϕ carries two degrees of freedom, which again matches the two on-shell degrees of freedom of ψ.

We still have to check the algebra on F. This is easily done:

$$\begin{aligned} \delta_1\delta_2 F &= \sqrt{2}i\,\partial_\mu(\delta_1\psi^\alpha)(\sigma^\mu\bar\epsilon_2)_\alpha \\ &= -2(\sigma^\nu\bar\epsilon_1)^\alpha(\sigma^\mu\bar\epsilon_2)_\alpha\,\partial_\mu\partial_\nu\phi - 2i\,(\epsilon_1\sigma^\mu\bar\epsilon_2)\partial_\mu F \end{aligned} \tag{4.18}$$

and therefore:

$$[\delta_1,\delta_2]\,F = -2i\,(\epsilon_1\sigma^\mu\bar\epsilon_2 - \epsilon_2\sigma^\mu\bar\epsilon_1)\partial_\mu F\ . \tag{4.19}$$

where the term containing $\partial_\mu\partial_\nu\phi$ does not contribute to the commutator as it is symmetric under $1 \leftrightarrow 2$. This concludes our proof that the supersymmetry variations (4.2) satisfy the superalgebra (4.10).

5 Superspace and Superfields. Supersymmetric actions

Here we present just the essential formulae. For more details see Sections 4.2, 4.3, 4.4, 4.5 of Bertolini's lectures.

5.1 Superspace

We want to construct supersymmetric quantum field theories. In order to do this we need to work with representations of the supersymmetry algebra on fields. A convenient and systematic way to do this uses superspace and superfields, that is fields defined in superspace.

We will restrict for now to $N = 1$ supersymmetry, and thus present what is known as $N = 1$ superspace. The notion of superspace for extended supersymmetry is more complicated and one often still uses $N = 1$ superspace to describe $N > 1$ supersymmetric quantum field theories.

The definition of superspace starts from the idea that in the same way as P_μ generates the space-time translations along the ordinary coordinates x^μ, the $N = 1$ supersymmetry generators Q_α and $\bar{Q}_{\dot\alpha}$ generate translations along some new, anticommuting Grasmannian coordinates θ and $\bar\theta_{\dot\alpha}$. *Superspace* is thus an extension of the ordinary spacetime by these Grassmannian directions, and has coordinates $(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha})$. *Superfields* are simply fields in superspace.

Before going on, we list here some properties of the Grassmannian coordinates θ_α, $\bar\theta_{\dot\alpha}$ that are not hard to prove:

$$\begin{aligned}
&\theta_\alpha\theta_\beta = -\theta_\beta\theta_a \quad \Longrightarrow \quad \theta_\alpha\theta_\beta\theta_\gamma = 0 \ , \\
&\theta^\alpha\theta^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\theta\theta \ , \qquad \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta \ , \\
&\theta\sigma^\mu\bar\theta\,\theta\sigma^\nu\bar\theta = \tfrac{1}{2}\theta\theta\,\bar\theta\bar\theta\,\eta^{\mu\nu} \ , \\
&\theta\psi\,\theta\chi = -\tfrac{1}{2}\theta\theta\,\psi\chi \ .
\end{aligned} \tag{5.1}$$

From these properties, it follows that the most general $N=1$ superfield is Taylor-expanded in the Grassmannian coordinates as:

$$\begin{aligned}
Y(x,\theta,\bar\theta) = f(x) \ + \ \theta\psi(x) \ + \ \bar\theta\bar\chi(x) \ + \ \theta\theta\, m(x) \ + \ \bar\theta\bar\theta\, n(x) \\
+ \ \theta\,\sigma^\mu\bar\theta\, v_\mu(x) \ + \ \theta\theta\,\bar\theta\bar\lambda(x) \ + \ \bar\theta\bar\theta\,\theta\rho(x) \ + \ \theta\theta\,\bar\theta\bar\theta\, d(x) \ .
\end{aligned} \tag{5.2}$$

Each term in this expansion is an ordinary field, hence a superfield is a finite collection of ordinary fields. We will see that this construction allows to realize different representations of the supersymmetry algebra on fields.

We will also need derivative and integration in the θ variables. The derivative $\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$, $\bar\partial_{\dot\alpha} \equiv \frac{\partial}{\partial\bar\theta^{\dot\alpha}}$ is defined as

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta \ , \qquad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}^{\dot\beta} \ , \qquad \partial_\alpha\bar\theta^{\dot\beta} = \bar\partial_{\dot\alpha}\theta^\beta = 0 \ . \tag{5.3}$$

For a single Grassmann variable θ the integration is defined as:

$$\int \mathrm{d}\theta(a+\theta b) = b \qquad \Longrightarrow \qquad \int \mathrm{d}\theta = \partial_\theta \ . \tag{5.4}$$

In $N=1$ superspace, we take:

$$\mathrm{d}^2\theta = \frac{1}{2}\mathrm{d}\theta^1\mathrm{d}\theta^2 \ , \qquad \mathrm{d}^2\bar\theta = \frac{1}{2}\mathrm{d}\bar\theta^{\dot 2}\mathrm{d}\bar\theta^{\dot 1} \tag{5.5}$$

so that

$$\int \mathrm{d}^2\theta\,\theta\theta = \int \mathrm{d}^2\bar\theta\,\bar\theta\bar\theta = 1 \ , \qquad \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\,\theta\theta\,\bar\theta\bar\theta = 1 \ . \tag{5.6}$$

Using the relations above one can see that the derivative with respect to the Grassmann coordinates satisfies

$$\left(\frac{\partial}{\partial\theta^\alpha}\right)^\dagger = +\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \ . \tag{5.7}$$

We would like to realize the action of supersymmetry generators on fields via differential operators, pretty much as $P_\mu = -i\partial_\mu$. Let us just state the result (the proof can be found e.g. in Bertolini's lectures):

$$\begin{aligned} Q_\alpha &= -\,i\,\partial_\alpha - (\sigma^\mu\bar\theta)_\alpha\,\partial_\mu \\ \bar Q_{\dot\alpha} &= +\,i\,\bar\partial_{\dot\alpha} + (\theta\sigma^\mu)_{\dot\alpha}\,\partial_\mu \ . \end{aligned} \tag{5.8}$$

It is easy to check that these realize the supersymmetry algebra (3.1), (3.2), in particular $\{Q_\alpha, \bar Q_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu$. The action on a superfield is:

$$(i\epsilon Q + i\bar\epsilon\bar Q)Y(x,\theta,\bar\theta) = \delta_{\epsilon,\bar\epsilon}Y(x,\theta,\bar\theta) \equiv Y(x+\delta x, \theta+\delta\theta, \bar\theta+\delta\bar\theta) - Y(x,\theta,\bar\theta) \ , \tag{5.9}$$

with

$$\delta x^\mu = i\theta\sigma^\mu\bar\epsilon - i\epsilon\sigma^\mu\bar\theta \ , \qquad \delta\theta^\alpha = \epsilon^\alpha \ , \qquad \delta\bar\theta^{\dot\alpha} = \bar\epsilon^{\dot\alpha} \ . \tag{5.10}$$

Namely, a supersymmetry transformation is a particular translation in superspace.

The general superfield (5.2) contains too many field components to provide an irreducible representation of the superalgebra. In order to obtain irreducible representations, we need to reduce the number of components by imposing some constraints. If the constraint is susy-preserving, the constrained object will still be a superfield and thus will provide a susy-invariant action by the construction above. At this scope, we introduce the covariant derivatives

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\,(\sigma^\mu\bar\theta)_\alpha\,\partial_\mu \ , \\ \bar D_{\dot\alpha} &= \bar\partial_{\dot\alpha} + i\,(\theta\sigma^\mu)_{\dot\alpha}\,\partial_\mu \ , \end{aligned} \tag{5.11}$$

where it should be noticed that $\bar D_{\dot\alpha} = (D_\alpha)^\dagger$, which follows recalling that $(\partial_\alpha)^\dagger = \bar\partial_{\dot\alpha}$ and $(\partial_\mu)^\dagger = -\partial_\mu$.

These have the property of anti-commuting with $Q_\alpha, \bar Q_{\dot\alpha}$ (check them!):

$$\begin{aligned} &\{D_\alpha, D_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu = -2\sigma^\mu_{\alpha\dot\beta}P_\mu \ , \\ &\{D_\alpha, D_\beta\} = \{D_\alpha, Q_\beta\} = \{D_\alpha, \bar Q_{\dot\beta}\} = 0 \qquad \text{and similarly for } \bar D_{\dot\alpha} \ . \end{aligned} \tag{5.12}$$

This implies that

$$\delta_{\epsilon,\bar\epsilon}\left(D_\alpha Y\right) = D_\alpha\left(\delta_{\epsilon,\bar\epsilon}Y\right) \ , \tag{5.13}$$

so if Y is a superfield, $D_\alpha Y$ is also a superfield. In other words, D_α, $\bar{D}_{\dot{\alpha}}$ can be used to impose a susy-invariant constraint on the general superfield.

In the next two subsections we will introduce two important examples of superfields with less components than the general superfield: the chiral superfields and the real superfield.

5.2 Chiral superfields

A *chiral superfield* Φ is a superfield satisfying the condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0 \ . \tag{5.14}$$

Similarly, and *anti-chiral superfield* Ψ satisfies

$$D_\alpha \Psi = 0 \ . \tag{5.15}$$

Notice that if Φ is chiral, then $\bar{\Phi}$ is anti-chiral.

Let us express the chiral superfield in terms of its ordinary field components. At this scope, it is useful to perform the change of superspace coordinates:

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{5.16}$$

In these variables the covariant derivatives read

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i(\sigma^\mu\bar{\theta})_\alpha \frac{\partial}{\partial y} \ , \qquad \bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \ , \tag{5.17}$$

which implies

$$\bar{D}_{\dot{\alpha}}\theta_\beta = \bar{D}_{\dot{\alpha}} y^\mu = 0 \ , \qquad D_\alpha\bar{\theta}_{\dot{\beta}} = D_\alpha \bar{y}^\mu = 0 \ . \tag{5.18}$$

Therefore the condition $\bar{D}_{\dot{\alpha}}\Phi = 0$ means that the chiral multiplet explicitly depends only on (y^μ, θ_α) and not on $\bar{\theta}_{\dot{\alpha}}$. The components can be written as:

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\,\theta\psi(y) - \theta\theta\, F(y) \ . \tag{5.19}$$

Taylor-expanding y^μ around x^μ we get the expression of the chiral superfields in terms of the original superspace coordinates:

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\,\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \theta\theta\, F(x) - \tfrac{i}{\sqrt{2}}\,\theta\theta\,\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \tfrac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) \ , \tag{5.20}$$

which is also the same as $\Phi(x,\theta,\bar{\theta}) = e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}\Phi(x,\theta)$.

The independent components are ϕ, ψ, F, which corresponds precisely to the off-shell degrees of freedom of the chiral multiplet. Indeed the chiral superfield is the superfield realizing the chiral supermultiplet.

Of course, a similar story holds for the anti-chiral multiplet. This is seen by just taking the hermitian conjugate in the formulae above.

♦ **Exercise.**

1. Show that in the $(y, \theta, \bar{\theta})$ coordinates, the supersymmetry generators read

$$Q_\alpha = -i\partial_\alpha \ , \qquad \bar{Q}_{\dot{\alpha}} = i\,\bar{\partial}_{\dot{\alpha}} + 2(\theta\sigma)_{\dot{\alpha}}\,\frac{\partial}{\partial y^\mu} \ . \tag{5.21}$$

2. Check that given the chiral superfield $\Phi = (\phi, \psi, F)$, its susy variation

$$\delta_{\epsilon,\bar{\epsilon}}\Phi = \left(i\epsilon Q + i\bar{\epsilon}\bar{Q}\right)\Phi \tag{5.22}$$

yields precisely the transformation of the components given in (4.2). In order to see this, it is convenient to work in the $(y, \theta, \bar{\theta})$ coordinates and use (5.21).

3. Derive the corresponding transformation for an anti-chiral superfield. In this case it is convenient to write the generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$ in terms of $(\bar{y}^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. ♦

5.3 Vector superfields

The chiral superfield does not contain a vector field v_μ (as its vector component is $\sim \partial_\mu\phi$), hence it cannot be used to define gauge interactions. On the other hand, the general superfield Y does contain a vector v_μ, but this is generally complex; moreover we have already noticed that the general superfield contains too many components to provide an irreducible representation of supersymmetry. We thus define a *vector (or real) superfield* V by imposing the reality condition

$$\bar{V} = V \ . \tag{5.23}$$

Recalling the expansion (5.2) of a general superfield, this condition gives the expansion:

$$\begin{aligned} V(x,\theta,\bar{\theta}) = \ & C(x) \ + \ i\theta\chi(x) \ - \ i\bar{\theta}\bar{\chi}(x) \ + \ \theta\,\sigma^\mu\bar{\theta}\,v_\mu(x) \ + \ \tfrac{i}{2}\,\theta\theta\,(M(x) + iN(x)) \\ & - \ \tfrac{i}{2}\,\bar{\theta}\bar{\theta}\,(M(x) - iN(x)) \ + \ i\theta\theta\,\bar{\theta}\left(\bar{\lambda}(x) + \tfrac{i}{2}\,\sigma^\mu\partial_\mu\chi(x)\right) \\ & - \ i\bar{\theta}\bar{\theta}\,\theta\left(\lambda(x) + \tfrac{i}{2}\,\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \ + \ \tfrac{1}{2}\,\theta\theta\,\bar{\theta}\bar{\theta}\,\left(D(x) - \tfrac{1}{2}\,\partial^2 C(x)\right) \ , \end{aligned} \tag{5.24}$$

where the real fields (C, M, N, v_μ, D) define 8 bosonic degrees of freedom while (χ, λ) give 8 fermionic degrees of freedom. These are still too many to describe an $N = 1$ gauge vector supermultiplet. The redundant components can be eliminated by introducing a supersymmetric version of the gauge transformations and of the gauge-fixing condition. The transformation

$$V \ \rightarrow \ V + \Phi + \bar{\Phi} \ , \tag{5.25}$$

where Φ is a chiral superfield, implies in particular $v_\mu \to v_\mu - \partial_\mu(2\,\mathrm{Im}\phi)$, hence it can be seen as a supersymmetric generalization of the gauge transformation.

By a suitable choice of the components of Φ, we can transform away the components χ, C, M, N of the vector superfield (in addition to imposing an ordinary gauge-fixing condition on v_μ). This choice is called Wess-Zumino gauge, and it reduces the vector superfield to

$$V_{\rm WZ} \;=\; \theta\sigma^\mu\bar\theta v_\mu(x) + i\theta\theta\,\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\,\theta\lambda(x) + \tfrac{1}{2}\,\theta\theta\,\bar\theta\bar\theta\,D(x)\ . \tag{5.26}$$

Dealing with a vector superfield becomes particularly simple in WZ gauge. In particular, it is not hard to check that

$$(V_{\rm WZ})^2 = \tfrac{1}{2}\,\theta\theta\,\bar\theta\bar\theta\,v_\mu v^\mu\ , \qquad (V_{\rm WZ})^n = 0\,,\ n \geq 3 \tag{5.27}$$

(do it as an exercise, you just need to use the third line of (5.1). This property will be very useful when we will construct gauge actions.

Note that $V_{\rm WZ}$ contains $4_B + 4_F$ degrees of freedom (after the ordinary gauge fixing). These are the off-shell degrees of freedom of a vector supermultiplet. We will see that the real scalar field D is auxiliary, hence on-shell (that is after imposing the equations of motion of all fields) we have $2_B + 2_F$ degrees of freedom, which match those of the massless vector multiplet.

5.4 Susy invariant actions

For an action to be susy-invariant, the Lagrangian must be a Poincaré scalar density of mass dimension 4, transforming as a total spacetime derivative under supersymmetry transformation.

It is very easy to construct susy-invariant actions in superspace. For any superfield $Y(x,\theta,\bar\theta)$, the superspace integral

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\ Y(x,\theta,\bar\theta) \tag{5.28}$$

is in fact a susy-invariant action. This is easily proven as follows. The integration measure is invariant under translations in superspace:

$$\int \mathrm{d}\theta\,\theta = \int \mathrm{d}(\theta+\xi)\,(\theta+\xi) = 1\ . \tag{5.29}$$

This implies that

$$\delta_{\epsilon,\bar\epsilon}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\ Y(x,\theta,\bar\theta) = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\ \delta_{\epsilon,\bar\epsilon}Y(x,\theta,\bar\theta)\ . \tag{5.30}$$

Using (5.8), (5.9), we get

$$\delta_{\epsilon,\bar{\epsilon}} Y = \epsilon^\alpha \partial_\alpha Y + \bar{\epsilon}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}} Y + \partial_\mu \left[-i \left(\epsilon \sigma^\mu \bar{\theta} - \theta \sigma^\mu \bar{\epsilon} \right) Y \right] . \tag{5.31}$$

Integration in $\mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}$ kills the first two terms and leaves just the last term, which is a total derivative and thus vanishes upon integrating in $\mathrm{d}^4 x$. We have thus proven susy-invariance of our superspace integral,

$$\delta_{\epsilon,\bar{\epsilon}} \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; Y(x, \theta, \bar{\theta}) = 0 . \tag{5.32}$$

This gives a very powerful way to construct supersymmetric invariant actions. Since the product of two (or more) superfields is still a superfield, the superfield that appears in the action formula may also be a rather complicated polynomial in other superfields.

In addition, we want the spacetime Lagrangian density that is obtained upon integration in $\mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}$ is a real scalar density, and this poses some constraints on the superfield to be integrated.

Finally, we also need the Lagrangian to have mass dimension $[M]^4$. Now, $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ have dimension $[M]^{-1/2}$ (this can be deduced e.g. by comparing the dimensions of ϕ and $\theta\psi$ in a chiral superfield). This means that if a superfield has dimension $[Y]$ (this is defined as the dimension of its bottom component), then the top component proportional to $\theta^2\bar{\theta}^2$ has dimension $[Y] + 2$. Therefore to make a dimension 4 Lagrangian we need $[Y] = 2$. This can also be seen from (5.28): $\int \mathrm{d}\theta$ and $\int \mathrm{d}\bar{\theta}$ have dimension $[M]^{1/2}$ since integration in the Grassmann coordinates is equivalent to a derivative. Therefore $\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}$ has dimension $[M]^2$ and to obtain a dimension 4 Lagrangian we need $[Y] = 2$.

5.5 Lagrangian for a chiral superfield

We now construct a supersymmetric Lagrangian for the chiral multiplet Φ. First note that $\bar{\Phi}\Phi$ is real superfield and a space-time scalar. Then assume that the bottom component of Φ, that is ϕ, has mass dimension 1, so that the bottom component of $\bar{\Phi}\Phi$, that is $\bar{\phi}\phi$, has mass dimension 2. It follows that

$$\mathcal{L} = \int \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; \bar{\Phi}\Phi \tag{5.33}$$

is a good susy-invariant Lagrangian. Expanding in components, we find

$$\mathcal{L} = \partial_\mu \bar{\phi} \, \partial^\mu \phi + \tfrac{i}{2} \left(\partial_\mu \psi \, \sigma^\mu \bar{\psi} - \psi \, \sigma^\mu \partial_\mu \bar{\psi} \right) + \bar{F} F + \text{total derivative} . \tag{5.34}$$

This is exactly the Lagrangian (4.1) of the Wess-Zumino model. We have thus obtained an alternative (and much faster!) proof that the Lagrangian (4.1) is susy-invariant.

♦ **Exercise.** Show that (5.33) indeed yields (5.34) (this may also be done by first working in the $(y^\mu, \bar{y}^\mu, \theta, \bar{\theta})$ coordinates and then Taylor-expanding y^μ and $\bar{y}^\mu$ around x^μ).

The kinetic Lagrangian $\mathcal{L}$ seen above can be generalized in two different ways, by still using a single chiral superfield Φ. The first yields more general kinetic terms for Φ, while the second provides mass and interaction terms. Let us see them in turn.

Take

$$K(\Phi, \bar{\Phi}) = \sum_{n,m=1}^{\infty} c_{mn}\, \bar{\Phi}^m\, \Phi^n\ , \qquad \text{with } c_{mn} = c^*_{nm}\ . \tag{5.35}$$

We also assume that the dimension of the coefficients is $[c_{mn}] = [M]^{2-(m+n)}$. Then K is a real, scalar superfield with $[K] = 2$. This is called the *Kähler potential*. Therefore

$$\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, K(\Phi, \bar{\Phi}) \tag{5.36}$$

is a more general kinetic term Lagrangian than the one seen before. Note that this still defines a two-derivative Lagrangian. The fact that for $m, n > 1$ the coefficients c_{mn} have negative mass dimension means that they can appear in a supersymmetric but non-renormalizable theory. This should be thought as a low-energy effective theory valid up to some cutoff scale Λ. Then we will have $c_{mn} \sim \Lambda^{2-(m+n)}$, that is the additional terms with respect to the canonical $\bar{\Phi}\Phi$ will be suppressed by inverse powers of the cutoff. If we want a renormalizable kinetic term, then we need to restrict to $m = n = 1$, that is $K = \bar{\Phi}\Phi$.

Notice that in (5.35) the sum starts from $m = n = 1$, meaning that we did not include a possible $\Phi + \bar{\Phi}$ term. This is because its $\theta^2\bar{\theta}^2$ component turns out to be a total spacetime derivative and thus does not contribute to the action. In fact, this implies that the Kähler potential K' defined as

$$K'(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})\ , \tag{5.37}$$

where Λ is a chiral superfield, gives the same action and is thus physically equivalent to K. This property indicates that the Kähler potential is not really a function of $\Phi, \bar{\Phi}$, but rather a "gauge" connection in a suitably defined bundle.

Let us now discuss the second option, introducing mass and non-derivative interaction terms in a supersymmetric way. When dealing with chiral superfields, there exists a different way to define a supersymmetric action. Consider a chiral superfield $W(\Phi)$ obtained by taking products of Φ. While its integral in the full superspace vanishes, the following integral in half superspace is real and non-vanishing:

$$\mathcal{L}_{\rm int} = \int \mathrm{d}^2\theta\, W(\Phi) + \int \mathrm{d}^2\bar\theta\, \overline{W}(\Phi) \tag{5.38}$$

Since the supersymmetry variation of the F-term in a chiral multiplet is a spacetime derivative, this action is guaranteed to be susy invariant. The function $W(\Phi)$ is called the *superpotential*. Notice that in order to give a physical action it must be $[W] = 3$. In addition, in order to be a chiral superfield, W must be a *holomorphic* function of Φ (just evaluate $D_{\dot\alpha}W(\Phi,\bar\Phi)$ to see that it cannot depend on $\bar\Phi$). As we will see, this simple property has far reaching consequences.

The superpotential is also constrained by R-symmetry. Take a chiral superfield of R-charge r; in order to indicate this we use the notation $R[\Phi] = r$. By definition, this means that its bottom component has R-charge r: $R[\phi] = r$. Since the supercharges have R-charge $R[Q_\alpha] = -1$, $R[\bar Q_{\dot\alpha}] = 1$ and the corresponding parameters have R-charge $R[\epsilon_\alpha] = 1$, $R[\bar\epsilon_{\dot\alpha}] = -1$, from the variations (4.2) it follows that the remaining components ψ, F of Φ have $R[\psi] = r - 1$ and $R[F] = r - 2$. From the structure of the chiral superfield we deduce that the Grassmannian coordinates have R-charge:

$$R[\theta] = 1\ , \qquad R[\bar\theta] = -1\ , \qquad R[\mathrm{d}\theta] = -1\ , \qquad R[\mathrm{d}\bar\theta] = 1\ . \tag{5.39}$$

It follows that if the the R-symmetry is a symmetry of the Lagrangian (this may be or may be not true) then the superpotential must have R-charge 2:

$$R[W] = 2\ . \tag{5.40}$$

(On the other hand, for such theories the Kähler potential must have R-charge 0).

The expression of (5.38) in terms of the chiral superfield components is (check this!):

$$W(\Phi) = W(\phi) + \sqrt{2}\,\frac{\partial W}{\partial\phi}\,\theta\psi - \theta\theta\left(\frac{\partial W}{\partial\phi}F + \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\,\psi\psi\right)\ , \tag{5.41}$$

where the derivatives of the superpotential are evaluated at $\Phi = \phi$. Therefore,

$$\mathcal{L}_{\rm int} = -\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\,\psi\psi + h.c. \tag{5.42}$$

where the right hand side is evaluated at x^μ. Assuming a canonical Kähler potential $K = \bar\Phi\Phi$, the full Lagrangian then is:

$$\begin{aligned}\mathcal{L} &= \int \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\, K(\Phi,\bar\Phi) + \int \mathrm{d}^2\theta\, W(\Phi) + \int \mathrm{d}^2\bar\theta\, \overline{W}(\Phi) \\ &= \partial_\mu\bar\phi\,\partial^\mu\phi + \tfrac{i}{2}\left(\partial_\mu\psi\,\sigma^\mu\bar\psi - \psi\,\sigma^\mu\partial_\mu\bar\psi\right) + \bar F F + \left(-\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi\partial\phi}\,\psi\psi + h.c.\right)\ .\end{aligned} \tag{5.43}$$

As in the free Wess-Zumino model, the field F is still auxiliary. Integrating it out using its equation of motion gives

$$\bar{F} = \frac{\partial W}{\partial \phi} \ , \qquad F = \frac{\partial \overline{W}}{\partial \bar{\phi}} \tag{5.44}$$

and thus

$$\mathcal{L} = \partial_\mu \bar{\phi}\, \partial^\mu \phi + \frac{i}{2}\left(\partial_\mu \psi\, \sigma^\mu \bar{\psi} - \psi\, \sigma^\mu \partial_\mu \bar{\psi}\right) - \left|\frac{\partial W}{\partial \phi}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial \phi \partial \phi}\, \psi\psi - \frac{1}{2}\frac{\partial^2 \overline{W}}{\partial \bar{\phi} \partial \bar{\phi}}\, \bar{\psi}\bar{\psi} \ . \tag{5.45}$$

Notice that there is a non-trivial scalar potential,

$$V(\phi, \bar{\phi}) = \left|\frac{\partial W}{\partial \phi}\right|^2 \ . \tag{5.46}$$

Susy invariant actions obtained by full superspace integrals are called D-terms, while those obtained by half superspace integrals are called F-terms.

We could then consider more complicated susy-invariant Lagrangians using n chiral multiplets Φ_i, $i = 1, \ldots, n$. This would lead us to consider more general Kähler potential $K(\Phi^i, \bar{\Phi}_i)$ and superpotential $W(\Phi_i)$. The Lagrangian has the same form as in the case of one field:

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, K(\Phi^i, \bar{\Phi}_i) + \int \mathrm{d}^2\theta\, W(\Phi^i) + \int \mathrm{d}^2\bar{\theta}\, \overline{W}(\bar{\Phi}_i) \ . \tag{5.47}$$

♦ **Exercise.** As a first thing, notice that if Y is a general superfield, then $\bar{D}^2 Y$ is chiral, since $\bar{D}_\alpha \bar{D}^2 \equiv 0$. Then show that any integral in full superspace can be written as an integral in half superspace as:

$$\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, Y = \frac{1}{4}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \bar{D}^2 Y \ . \tag{5.48}$$

The converse is not true, for instance the half-superspace integral $\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \Phi^n$ for a chiral superfield Φ cannot be written as a full superspace integral. The half superspace integrals that can written as full superspace integrals should not really be seen as F-terms, but rather as D-terms.

♦ **Exercise.** Consider the Lagrangian (5.43), with $K = \bar{\Phi}\Phi$. Using the result of the previous exercise, show that imposing that the action is extremized under a variation of Φ gives the superfield equation of motion

$$\frac{1}{4}\overline{D}^2 \bar{\Phi} + \frac{\partial W}{\partial \Phi} = 0 \tag{5.49}$$

[for more explanations, see Bertolini's lectures around eqs. (5.3), (5.4)]. Notice that this is a chiral superfield. Work out its components and convince yourself that these are equivalent to the equations of motion of the fields F, ψ, ϕ. In particular, its bottom component is the equation of motion for F,

$$\bar{F} - \frac{\partial W}{\partial \phi} = 0 \ . \tag{5.50}$$

This illustrates that the equations of motion of the fields in a supermultiplet form themselves the components of a supermultiplet and are therefore related to each other by supersymmetry variations. You can also check this explicitly: start from (5.50) and verify using (4.2) that its susy variation gives the equation of motion for ψ; variation of the latter then gives the ϕ equation of motion as an F-component. Further variations give redundant equations. ♦

6 Interacting Wess-Zumino model and holomorphy

Let us take a single chiral superfield Φ and require renormalizability of the theory. Then the most general superpotential is

$$W(\Phi) = \tfrac{1}{2} m \Phi^2 + \tfrac{1}{3} \lambda \Phi^3 \ . \tag{6.1}$$

Together with the Kähler potential $K = \bar{\Phi}\Phi$, this defines the interacting Wess-Zumino model. Notice that the $\frac{1}{2}m\Phi^2$ term in the superpotential yields precisely the mass terms (4.3), which on-shell become (4.4). Let us take a look at the interaction term $\frac{1}{3}\lambda\Phi^3$. From (5.45) we obtain the Lagrangian:

$$\mathcal{L}_\lambda = -\lambda^2 |\phi|^4 - \lambda \phi \psi \psi - \lambda \bar{\phi} \bar{\psi} \bar{\psi} \ , \tag{6.2}$$

where here λ is taken real. We see that the coefficient of the quartic self-interaction of the scalar field is related to the Yukawa couplings of the scalar and fermion fields. This implies that the one-loop corrections to the scalar propagator due to these interaction terms are both proportional to λ^2 and exactly cancel out (recall that fermion loop comes with a minus sign compared to boson loops). This property does not hold just at one-loop: in fact the superpotential (6.1) turns out to be exact at tree level! Although this result was originally obtained using a diagrammatic technique (supergraphs), in the following we will prove it by adopting a more modern approach which uses the *spurion method* and the holomorphy of the superpotential. This will give us the opportunity to illustrate a general property of supersymmetry, that is how *holomorphy in the couplings provides a simple derivation of very powerful non-renormalization theorems.*

The idea of the spurion method in supersymmetric theories is to promote any parameter in the Lagrangian to be the VEV of a superfield. In particular, if we focus on the superpotential term in the Lagrangian, each coupling, which may be complex or not, can be thought of as the bottom component VEV of a chiral superfield. The latter is assumed very heavy and thus frozen at its VEV. The theory is viewed as an effective theory of a parent UV theory where these heavy fields have been integrated out, so that only their VEVs remain in the Lagrangian and can be treated as spurion fields. Often this trick allows to enhance the symmetries of the Lagrangian. These spurionic symmetries constrain the quantum corrections and thus the possible effective operators that are generated when one adopts the Wilsonian approach and integrates out the physics between one scale and another.

This point of view makes it clear that the F-term Lagrangian is not only holomorphic in the fields, but also in the couplings. The Wilsonian effective action should also display such holomorphic dependence on the UV couplings (while this is not true for the 1PI effective action). This means that quantum corrections to the tree-level superpotential are constrained by holomorphy in the couplings, in addition to the usual spurious symmetries introduced by the spurion methods.

Let us illustrate this further with a very simple example. Let us assume that the tree-level superpotential W_{tree} contains a term $\lambda\mathcal{O}_{-1}$. Regarding λ as the VEV of a superfield, we can introduce a spurious U(1) symmetry under which λ has charge 1 while $\mathcal{O}_{-1}$ has charges -1. Imagine that we want to know how an operator $\mathcal{O}_{-10}$ can appear through quantum corrections. The usual spurion analysis would lead us to consider terms in the effective superpotential of the form:

$$\Delta W \sim \lambda^{10}\mathcal{O}_{-10} + \lambda^{11}\bar{\lambda}\mathcal{O}_{-10} + \ldots + \lambda^{10}e^{-|\lambda|^2}\mathcal{O}_{-10} + \ldots \ , \qquad (6.3)$$

which are all uncharged under the spurious U(1) symmetry. In addition, here we are assuming that the classical limit $\lambda \to 0$ must be smooth and therefore no negative powers of λ can appear. Now, the requirement that the spurion field λ only appears holomorphically in the quantum-corrected superpotential introduces a new, drastic constraint, implying that only the first term is admittable.

We are thus discovering a general feature of supersymmetric theories: *combining holomorphy of the superpotential with the spurion method and with smoothness requirements in various weak-coupling limits allows to strongly constrain the effective superpotential terms that are generated by quantum corrections.*

After these general considerations, let us then come to our Wess-Zumino model and start from the tree-level superpotential

$$W_{\rm tree} = \tfrac{1}{2} m\Phi^2 + \tfrac{1}{3}\lambda\Phi^3 \ . \tag{6.4}$$

We ask what is the form of the effective superpotential $W_{\rm eff}$ after quantum corrections have been taken into account. We use the spurion method and promote m and λ to spurionic chiral superfields. This allows us to introduce a spurious U(1) flavor symmetry and a spurious U(1) R-symmetry. By flavor symmetry we denote a symmetry whose generator commutes with the supercharges; hence both the supercharges and the Grasmannian coordinates $\theta, \bar\theta$ are uncharged under a flavor symmetry. As seen above, the R-symmetry instead acts non-trivially on the supercharges (by preserving the susy algebra) and thus on the susy parameters and on the Grassmannian coordinates of superspace. For the fields in (6.4) we take the following charges under the two symmetries:

	$U(1)_R$	U(1)
Φ	1	1
m	0	-2
λ	-1	-3

(6.5)

so that the superpotential has R-charge 2 and flavor charge 0. Of course, the symmetries are spurious since they are spontaneously broken once the spurion fields m and λ acquire a non-vanishing bottom-component VEV. We now discuss the effective superpotential in a Wilsonian sense. This should be holomorphic in Φ, m, λ and must still have R-charge 2 and flavor charge 0. The most general form satisfying these conditions is:

$$W_{\rm eff} = \sum_{n=-\infty}^{\infty} a_n\, \lambda^n m^{1-n} \Phi^{n+2} \ = \ m\Phi^2\, f\left(\frac{\lambda\Phi}{m}\right) \ , \tag{6.6}$$

where $f_{\rm tree} = \frac{1}{2} + \frac{1}{3}\frac{\lambda\Phi}{m}$. We now consider the classical limit $\lambda \to 0$; in this limit we should recover the tree level result and therefore there cannot be negative powers of λ, which would make $W_{\rm eff}$ diverge. Hence $n \geq 0$ and moreover $a_0 = \frac{1}{2}$ and $a_1 = \frac{1}{3}$ so that $f_{\rm tree}$ is recovered. Taking the massless limit $m \to 0$ at the same time as $\lambda \to 0$ (so that the theory is still weakly coupled) in such a way that $m/\lambda \to 0$ and requiring smoothness of the Wilsonian effective action implies $n \leq 1$ (the Wilsonian effective action does not suffer from IR divergences associated with $m = 0$ particles because we do not integrate down to zero momenta, contrarily to the 1PI effective action). This completely fixes $f = f_{\rm tree}$ and therefore

$$W_{\rm eff} = \tfrac{1}{2} m\Phi^2 + \tfrac{1}{3}\lambda\Phi^3 = W_{\rm tree} \ , \tag{6.7}$$

that is the tree-level superpotential is already the full quantum superpotential. This means that our superpotential is not renormalized at any order in perturbation theory and even non-perturbatively. That is, (6.7) is an *exact* result.

This result can be generalized to any model containing only chiral superfields. This means that in the absence of gauge interactions, the tree-level superpotential is already the exact quantum superpotential.

Supersymmetry also has nice features in the presence of gauge interactions. In order to illustrate these, we need to introduce the supersymmetric gauge Lagrangian.

Integrating out[6]

In order to illustrate further the power of holomorphy, let us consider another example, where we have two chiral fields H and L and a superpotential

$$W = \frac{1}{2}MH^2 + \frac{1}{2}\lambda L^2 H \ . \tag{6.8}$$

We want to integrate out the massive (heavy) field H and obtain the effective superpotential for the massless field L. This will be valid at scales lower than the H-mass M. Again we adopt the spurion method, promote the couplings to chiral superfields and enhance the global symmetries to

$$\begin{array}{cccc} & \mathrm{U}(1)_a & \mathrm{U}(1)_b & \mathrm{U}(1)_R \\ H & 1 & 0 & 1 \\ L & 0 & 1 & \frac{1}{2} \\ M & -2 & 0 & 0 \\ \lambda & -1 & -2 & 0 \end{array} \tag{6.9}$$

where $\mathrm{U}(1)_a$ and $\mathrm{U}(1)_b$ are spurious symmetries while $\mathrm{U}(1)_R$ is a true R-symmetry. The effective superpotential must respect the above symmetries and again be holomorphic in M and λ (as well as L of course). The only possibility is:

$$W_{\text{eff}} = a\,\frac{\lambda^2 L^4}{M} \ , \tag{6.10}$$

where the constant a is not fixed by the present argument and can be determined by a perturbative computation at tree level.

The same result can be obtained by integrating out H. This means that we treat it as a constant field frozen at its VEV (as it should be at scales much lower than M). Thanks to

[6] See Section 9.5 of Bertolini or Section 8.3 of Terning.

the fact that the superpotential above is already the full quantum superpotential, the VEV of H is straightforwardly determined using the classical equation of motion. This gives

$$0 = \frac{\partial W}{\partial H} = MH + \frac{1}{2}\lambda L^2 \quad \longrightarrow \quad H = -\frac{\lambda}{2M}L^2 \ . \tag{6.11}$$

Substituting this in the superpotential yields

$$W_{\text{eff}} = -\frac{1}{8}\frac{\lambda^2 L^4}{M} \ , \tag{6.12}$$

which is the same result obtained with the spurion analysis (with the coefficient a now being determined).

As a final comment, we observe that here we have not imposed smoothness of W_{eff} for $M \to 0$. This is for a simple reason: W_{eff} is only valid at energies lower than M, which plays the role of an UV cutoff. So it doesn't make sense to send $M \to 0$ in this effective theory. If we try to do this, we find a singularity. This should not be regarded as a pathology, it rather indicates that the effective theory needs to be modified for $M \to 0$. Indeed new light degrees of freedom should be included; these are those carried by the field H that we have integrated out.

7 Supersymmetric gauge theories

7.1 Abelian gauge theory

We would like to construct supersymmetric gauge interactions. Let us start from the Abelian case, that is we consider a gauge group $G = \text{U}(1)$. We have seen in Section 5.3 that a vector superfield V, subject to the gauge transformation (5.25), contains an (Abelian) gauge field v_μ. In order to construct a supersymmetric gauge Lagrangian using such superfield, we should as a first thing construct the superfield representing the supersymmetric extension of the field strength. So we should act on V with some differential operator; in order to obtain again a superfield, this should be constructed from the susy-covariant derivatives D_α and $\overline{D}_{\dot\alpha}$, which send superfields into superfields.

The wanted supersymmetrization of the field strength is achieved by:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \ , \qquad \overline{W}_{\dot\alpha} = -\frac{1}{4}DD\bar{D}_{\dot\alpha}V \ . \tag{7.1}$$

Notice that W_α is a chiral superfield, indeed $\bar{D}_{\dot\alpha}W_\alpha = \bar{D}_{\dot\alpha}\bar{D}^2 D_\alpha V = 0$ since $\bar{D}_{\dot\alpha}\bar{D}_{\dot\beta}\bar{D}^{\dot\beta} = 0$ identically. Moreover, it is invariant under gauge transformations

$$V \to V + \Phi + \bar{\Phi} \ . \tag{7.2}$$

Indeed,

$$\begin{aligned} W_\alpha \quad &\to \quad W_\alpha - \frac{1}{4}\bar{D}\bar{D}D_\alpha\big(\Phi + \bar{\Phi}\big) \;=\; W_\alpha + \frac{1}{4}\bar{D}^{\dot\beta}\bar{D}_{\dot\beta}D_\alpha\Phi \\ &= \quad W_\alpha + \frac{1}{4}\bar{D}^{\dot\beta}\{\bar{D}_{\dot\beta}, D_\alpha\}\Phi \;=\; W_\alpha + \frac{i}{2}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\,\bar{D}^{\dot\beta}\Phi \;=\; W_\alpha\ . \end{aligned} \tag{7.3}$$

Since W_α is gauge-invariant, we can work in a convenient gauge, which of course will be the Wess-Zumino gauge introduced in Section 5.3. The vector superfield in this gauge was given in (5.26). In order to work out the components of W_α, it is convenient to switch to the shifted coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$. In these coordinate, $V_{\rm WZ}$ reads:

$$V_{\rm WZ}(y,\theta,\bar\theta) \;=\; \theta\sigma^\mu\bar\theta v_\mu(y) + i\theta\theta\,\bar\theta\bar\lambda(y) - i\bar\theta\bar\theta\,\theta\lambda(y) + \tfrac{1}{2}\,\theta\theta\,\bar\theta\bar\theta\,(D(y) - i\partial_\mu v^\mu(y))\ , \tag{7.4}$$

where in order to see equivalence with (5.26) one needs to use $(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\eta^{\mu\nu}$. Acting with D_α and recalling (5.18), we obtain:

$$D_\alpha V_{\rm WZ} = (\sigma^\mu\bar\theta)_\alpha v_\mu + 2i\theta_\alpha\bar\theta\bar\lambda - i\bar\theta\bar\theta\lambda_\alpha + \theta_\alpha\,\bar\theta\bar\theta D + i(\sigma^{\mu\nu}\theta)_\alpha\bar\theta\bar\theta F_{\mu\nu} + \theta\theta\bar\theta\bar\theta\,(\sigma^\mu\partial_\mu\bar\lambda)_\alpha\ , \tag{7.5}$$

where all components still depend on y. Here,

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \tag{7.6}$$

is the field strength of the Abelian gauge field v_μ, so we are on the right track for constructing the supersymmetric extension of the field strength. Using $\bar{D}_{\dot\alpha}y^\mu = 0$ and $\bar{D}\bar{D}\bar\theta\bar\theta = -4$, we can go on and obtain:

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha\,F_{\mu\nu} + \theta\theta\,\left(\sigma^\mu\partial_\mu\bar\lambda\right)_\alpha\ . \tag{7.7}$$

Notice that this chiral superfield carries a spinor index α, so its bottom component is not a scalar field as in the chiral superfield Φ studied before, but the spin 1/2 gaugino field λ_α.

Since W_α is chiral, the half-superspace integral

$$\mathcal{L}_{\rm gauge} = \int {\rm d}^2\theta\,W^\alpha W_\alpha + h.c. \tag{7.8}$$

is Lorentz invariant, real and supersymmetric. Since $[\lambda_\alpha] = 3/2$, we have that $[W_\alpha] = 3/2$ and therefore we have a good dimension 4 Lagrangian. In components, we have:

$$\int {\rm d}^2\theta\,W^\alpha W_\alpha = -2i\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 - \frac{1}{2}(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta}F_{\mu\nu}F_{\rho\sigma}\ , \tag{7.9}$$

where we used ${\rm tr}(\sigma^{\mu\nu}) = 0$. Using further

$$(\sigma^{\mu\nu})^{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\ , \tag{7.10}$$

we arrive at

$$\int \mathrm{d}^2\theta\, W^\alpha W_\alpha = -\frac{1}{2} F_{\mu\nu}F^{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + D^2 + \frac{i}{4}\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\ . \tag{7.11}$$

Therefore we obtain for the supersymmetric Abelian gauge Lagrangian (up to total derivative terms):

$$\mathcal{L}_{\text{gauge}} = -F_{\mu\nu}F^{\mu\nu} - 4i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + 2D^2\ . \tag{7.12}$$

We conclude observing that

$$\int \mathrm{d}^2\theta\, W^\alpha W_\alpha = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, D^\alpha V\, W_\alpha\ , \tag{7.13}$$

meaning that the gauge Lagrangian can also be written as a full superspace integral. Because of this, it cannot really be considered an F-term and should instead be seen as a D-term. This will be important when we will discuss its renormalizations properties.

♦ **Exercise.** Using the superfield approach, work out the supersymmetry transformations for the components of the vector superfield.

7.2 Pure super-Yang-Mills theory

We would like to generalize the Abelian construction above to accommodate for non-Abelian interactions. Let us thus consider a general gauge group G of rank r. The gauge potential is

$$v_\mu = v_\mu^a\, T^a\ , \qquad a = 1, \ldots, \dim G\ , \tag{7.14}$$

where T^a are the generators of G in the adjoint representation. These are taken hermitian, $(T^a)^\dagger = T^a$. The gauge field strength reads

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu]\ . \tag{7.15}$$

Under an ordinary gauge transformation with parameter $u = u^a T^a$, these transform as:

$$v_\mu \to U^{-1} v_\mu U + 2i\, U^{-1}\partial_\mu U\ , \qquad F_{\mu\nu} \to U^{-1}F_{\mu\nu}U\ , \tag{7.16}$$

where the gauge covariant derivative reads

$$D_\mu = \partial_\mu - \frac{i}{2}[v_\mu, \cdot]\ . \tag{7.17}$$

Let us thus start by generalizing the vector superfield to

$$V = V^a T^a\ . \tag{7.18}$$

It is important to notice that all its components, and not just the gauge field v_μ, transform in the adjoint representation.

In this non-Abelian case, the basic object to consider is e^V rather than V iself (here e^V should be understood as a formal Taylor expansion $1 + V + \frac{1}{2}V^2 + \ldots$; notice this is also a superfield, since multiplication of superfields yields again a superfield). For the finite, non-Abelian gauge transformation in superspace we take:

$$e^V \to e^{i\bar{\Lambda}} e^V e^{-i\Lambda} \; , \tag{7.19}$$

where Λ is a chiral superfield. In the Abelian case and at first order in Λ, this reduces to the gauge transformation used in Subsection 7.1 (upon identifying Φ there with $-i\Lambda$ here). Again it is possible to impose the Wess-Zumino gauge, in which $(V)^n = 0$ for $n \geq 3$. This implies

$$e^V = 1 + V + \frac{1}{2}V^2 \; . \tag{7.20}$$

This makes it further clear why the Wess-Zumino gauge is particularly convenient. In what follows we will always work in this gauge.

The non-Abelian gauge superfield is:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}\left(e^{-V} D_\alpha e^V\right) \; , \qquad \overline{W}_{\dot\alpha} = +\frac{1}{4}DD\left(e^V \bar{D}_{\dot\alpha} e^{-V}\right) \; , \tag{7.21}$$

which to first order in V corresponds to the definition of the Abelian gauge superfield. We now prove that this transforms covariantly,

$$W_\alpha \to e^{i\Lambda}\, W_\alpha\, e^{-i\Lambda} \tag{7.22}$$

and we thus have a good definition for a field strength and $\mathrm{Tr}\, W^\alpha W_\alpha$ is gauge invariant. Notice that the transformed superfield in (7.22) is still chiral.

Proof. Under the gauge transformation (7.19), the gauge superfield transforms as:

$$\begin{aligned} W_\alpha \to &-\frac{1}{4}\bar{D}\bar{D}\left[e^{i\Lambda} e^{-V} e^{-i\bar{\Lambda}} D_\alpha\big(e^{-i\Lambda} e^V e^{i\bar{\Lambda}}\big)\right] \\ = &-\frac{1}{4}\bar{D}\bar{D}\left[e^{i\Lambda}\big(e^{-V} D_\alpha e^V e^{-i\Lambda} + D_\alpha e^{-i\Lambda}\big)\right] \\ = &-\frac{1}{4}e^{i\Lambda}\bar{D}\bar{D}\left(e^{-V} D_\alpha e^V\right) e^{-i\Lambda} \;=\; e^{i\Lambda} W_\alpha e^{-i\Lambda} \; , \end{aligned} \tag{7.23}$$

that is what we wanted to show. In order to reach the last line, we used that $\bar{D}_{\dot\alpha} e^{\pm i\Lambda} = 0$ because Λ is chiral and that $\bar{D}\bar{D}D_\alpha e^{-i\Lambda} = -\bar{D}^{\dot\beta}\{\bar{D}_{\dot\beta}, D_\alpha\} e^{-i\Lambda} = 0$. In the same way one can prove that $\overline{W}_{\dot\alpha}$ transforms as

$$\overline{W}_{\dot\alpha} \to e^{i\bar{\Lambda}}\, \overline{W}_{\dot\alpha}\, e^{-i\bar{\Lambda}} \; . \tag{7.24}$$

Next we expand the gauge superfield W_α in components, and in particular check whether it contains the correct non-Abelian field strength. Using (7.20) into (7.21), we have

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\bar{D}\bar{D}\left[\left(1 - V + \frac{1}{2}V^2\right) D_\alpha \left(1 + V + \frac{1}{2}V^2\right)\right] \\ &= -\frac{1}{4}\bar{D}\bar{D}D_\alpha V + \frac{1}{8}\bar{D}\bar{D}[V, D_\alpha V] \ . \end{aligned} \tag{7.25}$$

The first term has the same component expansion we already computed in the Abelian case. One can see that the second term gives

$$\frac{1}{8}\bar{D}\bar{D}\,[V, D_\alpha V] = \frac{1}{2}(\sigma^{\mu\nu}\theta)_\alpha [v_\mu, v_\nu] - \frac{i}{2}\theta\theta\,\sigma^\mu_{\alpha\dot{\beta}}[v_\mu, \bar{\lambda}^{\dot{\beta}}] \ . \tag{7.26}$$

Putting the two terms together, we see that the ordinary derivatives get promoted to covariant derivatives and we obtain the result:

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + \theta\theta\left(\sigma^\mu D_\mu \bar{\lambda}(y)\right)_\alpha \ , \tag{7.27}$$

where $F_{\mu\nu}$ and D_μ are precisely the field strength and the covariant derivative defined in (7.15) and (7.17). It should be recalled that all components are in the adjoint representation of the gauge group, that is $\lambda = \lambda^a T^a$, $D = D^a T^a$, as well as of course $F_{\mu\nu} = F^a_{\mu\nu} T^a$.

We conclude that the chiral superfield W_α provides the correct non-Abelian gauge field strength. It also includes the covariant derivative of the gaugino field.

♦ **Exercise.** Prove the component expansion (7.26) (*hint:* first compute $[V, D_\alpha V]$ and then use $\bar{D}\bar{D}\bar{\theta}\bar{\theta} = -4$).

The Lagrangian constructed using the superfield above does not contain the gauge coupling constant g explicitly. In order to introduce it and obtain canonically normalized kinetic terms, we first redefine our fields as

$$V \to 2gV \quad \Leftrightarrow \quad v_\mu \to 2gv_\mu \ , \quad \lambda \to 2g\lambda \ , \quad D \to 2gD \ . \tag{7.28}$$

It also follows that the superfield strength W_α is redefined as $W_\alpha \to 2gW_\alpha$. The ordinary field strength and the covariant derivatives now read

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu] \ , \qquad D_\mu = \partial_\mu - ig[v_\mu,] \ . \tag{7.29}$$

Independently of the rescaling just made, one also introduces the complexified coupling constant

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2} \ , \tag{7.30}$$

where $\theta_{\rm YM}$ will give rise to a new term, the θ-term, that we did not include in the Abelian case. The $N=1$ super-Yang-Mills Lagrangian can then be written as:

$$\begin{aligned}\mathcal{L}_{\rm SYM} &= \frac{1}{32\pi}\,{\rm Im}\left(\tau\int {\rm d}^2\theta\,{\rm Tr}\,W^\alpha W_\alpha\right)\\ &= {\rm Tr}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\,\lambda\sigma^\mu D_\mu\bar\lambda + \frac{1}{2}D^2\right) + \frac{\theta_{\rm YM}}{32\pi^2}\,g^2\,{\rm Tr}\,F_{\mu\nu}\tilde F^{\mu\nu}\ . \end{aligned} \tag{7.31}$$

where

$$\tilde F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{7.32}$$

is the dual field strength and the gauge group generators are normalized as ${\rm Tr}\,T^aT^b = \delta^{ab}$. So we have obtained not only the $N=1$ super-Yang-Mills kinetic Lagrangian, but also a new $\theta_{\rm YM}$-term. These are both supersymmetric.

♦ **Exercise.** Using the superspace approach, derive the supersymmetry transformations of the components in the gauge superfield $V_{\rm WZ}$.

7.3 General matter-coupled super-Yang-Mills theory

We now want to couple matter superfields to the pure super-Yang-Mills theory constructed above. We thus consider chiral superfields Φ^i, transforming in some representation R of the gauge group G, with the generators being represented by matrices $(T^a_R)^i{}_j$. This means that Φ^i transforms as

$$\Phi^i \to \left(e^{i\Lambda}\right)^i{}_j\Phi^j\ , \qquad \text{where } \Lambda = \Lambda^a T^a_R\ . \tag{7.33}$$

For the transformed field to remain chiral, we need that Λ is a chiral superfield. In other words, the transformation of Φ^i should not involve the anti-chiral superfield $\bar\Lambda$.

Note that the canonical kinetic term $\bar\Phi\Phi$ discussed before would not be gauge-invariant. In fact it is straightforward to see that the correct gauge-invariant generalization of the kinetic term is

$$\bar\Phi\, e^V\Phi\ , \tag{7.34}$$

and the full matter Lagrangian reads

$$\mathcal{L}_{\rm matter} = \int {\rm d}^2\theta{\rm d}^2\bar\theta\,\bar\Phi\,e^V\Phi + \int {\rm d}^2\theta\,W(\Phi) + \int {\rm d}^2\bar\theta\,\overline{W}(\bar\Phi)\ , \tag{7.35}$$

Let us compute the D-term of this Lagrangian (7.35). Recalling that we work in Wess-Zumino gauge, we have

$$\bar\Phi\,e^V\Phi = \bar\Phi\Phi + \bar\Phi V\Phi + \frac{1}{2}\bar\Phi V^2\Phi\ . \tag{7.36}$$

We need to extract the $\theta\theta\bar{\theta}\bar{\theta}$ component of this superfield. We already discussed the first term, recall eq. (5.34). The other two terms can be calculated multiplying the superfields and focussing on the $\theta\theta\bar{\theta}\bar{\theta}$ piece; one can check that their effect is to make the ordinary derivatives in (5.34) covariant under the gauge symmetry, and to add some further interaction terms dictated by supersymmetry. Up to total spacetime derivatives, we end up with:

$$\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \bar{\Phi}\, e^V \Phi = \overline{D_\mu\phi} D^\mu\phi - i\psi\sigma^\mu D_\mu\bar{\psi} + \bar{F}F + \frac{i}{\sqrt{2}}\left(\bar{\phi}\lambda\psi - \bar{\psi}\bar{\lambda}\phi\right) + \frac{1}{2}\bar{\phi}D\phi\ , \tag{7.37}$$

where the covariant derivative is $D_\mu = \partial_\mu - \frac{i}{2}v_\mu^a T_R^a$. It should not be confused with the auxiliary field D appearing in the last term. Moreover, the Yukawa coupling between the gaugino λ and the matter fields ϕ, ψ is understood as:

$$\bar{\phi}\lambda\psi = \bar{\phi}_i\, \lambda^a (T_R^a)^i{}_j\, \psi^j\ , \tag{7.38}$$

and similarly for $\bar{\psi}\bar{\lambda}\phi$.

What about the superpotential in the Lagrangian (7.35)? Of course this must be gauge invariant, and we should ask how this can be achieved using only chiral superfields. For instance, let us consider $G = \mathrm{SU}(3)$ and try to construct a superpotential for the chiral superfield Φ (quark superfield) transforming in the $\mathbf{3}$ representation. The only renormalizable term we can write down is $\epsilon_{ijk}\Phi^i\Phi^j\Phi^k$, there is no way to write down a gauge invariant mass term quadratic in the fields. In order to obtain mass terms for matter superfields transforming non-trivially under the gauge group, we need a chiral field Φ transforming in the $\mathbf{3}$ (quark superfield) and another field $\tilde{\Phi}$, also *chiral*, transforming in the $\bar{\mathbf{3}}$ (anti-quark superfield), so that we can add gauge-invariant mass terms $\tilde{\Phi}\Phi$ to the superpotential. This is a general lesson: in order to have mass terms for colour charged matter fields, one has to introduce two sets of chiral superfields that transform in conjugate representations of the gauge group.

Fayet-Iliopoulos terms. There is one final ingredient we should add before being in the position to write the general matter-coupled super-Yang-Mills Lagrangian. This is given by the Fayet-Iliopoulos terms. Suppose the gauge group contains n $\mathrm{U}(1)_A$ factors, with $A = 1, 2, \ldots, n$. Associated to each of them we have an Abelian vector superfield V^A. Under the Abelian super-gauge transformation of the type $V \to V - i\Lambda + i\bar{\Lambda}$, the D-term of V^A transforms as a total derivative, $D^A \to D^A + \partial_\mu\partial^\mu(\ldots)$. We can write down the Lagrangian

$$\mathcal{L}_{\mathrm{FI}} = \sum_A \xi_A \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ V^A = \frac{1}{2}\sum_A \xi_A D^A\ , \tag{7.39}$$

where the ξ_A are called Fayet-Iliopoulos parameters. This is gauge invariant (up to total spacetime derivatives), real and supersymmetric (being the full superspace integral of a real superfield). Note that since $[D] = 2$, we should take $[\xi_A] = 2$.

The full Lagrangian. Putting all the terms discussed so far together, we can write a very general matter-coupled super-Yang-Mills Lagrangian. Assuming a canonical Kähler potential, and performing the redefinition $V \to 2gV$ (discussed above) that gives canonical kinetic terms for the gauge field, the most general Lagrangian is:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{SYM}} + \mathcal{L}_{\mathrm{matter}} + \mathcal{L}_{\mathrm{FI}} \\
&= \frac{1}{32\pi}\mathrm{Im}\left(\tau \int \mathrm{d}^2\theta\, \mathrm{Tr}\, W^\alpha W_\alpha\right) + 2g\sum_A \xi_A \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\; V^A \\
&+ \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, \bar\Phi\, e^{2gV}\Phi + \int \mathrm{d}^2\theta\, W(\Phi) + \int \mathrm{d}^2\bar\theta\, \overline{W}(\bar\Phi)\ .
\end{aligned} \tag{7.40}
$$

In components, it reads

$$
\begin{aligned}
\mathcal{L} = \mathrm{Tr}&\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\,\lambda\sigma^\mu D_\mu\bar\lambda + \frac{1}{2}D^2\right) + \frac{\theta_{\mathrm{YM}}}{32\pi^2}\, g^2\, \mathrm{Tr}\, F_{\mu\nu}\tilde F^{\mu\nu} + g\sum_A \xi_A D^A \\
&+ \overline{D_\mu\phi}\, D^\mu\phi - i\psi\sigma^\mu D_\mu\bar\psi + \bar F F + \sqrt{2}ig\left(\bar\phi\lambda\psi - \bar\psi\bar\lambda\phi\right) + g\bar\phi D\phi \\
&- \frac{\partial W}{\partial\phi^i}F^i - \frac{\partial \overline{W}}{\partial\bar\phi_i}\bar F_i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\,\psi^i\psi^j - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial\bar\phi_i\partial\bar\phi_j}\,\bar\psi_i\bar\psi_j\ .
\end{aligned} \tag{7.41}
$$

The equations of motion for the auxiliary fields F^i and D^a that follow from this Lagrangian are:

$$
\bar F_i = \frac{\partial W}{\partial\phi^i}\ , \qquad D^a = -g\bar\phi T^a\phi - g\,\xi^a\ , \tag{7.42}
$$

where it is understood that the Fayet-Iliopoulos parameters are non-zero only when $a = A$, that is when the index a labelling the generators of the gauge group G runs over its Abelian factors.

Integrating the auxiliary fields out, that is replacing their solution from the equation of motion in the Lagrangian, we arrive at the following on-shell Lagrangian:

$$
\begin{aligned}
\mathcal{L} \;=\; \mathrm{Tr}&\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\,\lambda\sigma^\mu D_\mu\bar\lambda\right) + \frac{\theta_{\mathrm{YM}}}{32\pi^2}\, g^2\, \mathrm{Tr}\, F_{\mu\nu}\tilde F^{\mu\nu} + \overline{D_\mu\phi}\, D^\mu\phi - i\psi\sigma^\mu D_\mu\bar\psi \\
&+ \sqrt{2}ig\left(\bar\phi\lambda\psi - \bar\psi\bar\lambda\phi\right) - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\partial\phi^j}\,\psi^i\psi^j - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial\bar\phi_i\partial\bar\phi_j}\,\bar\psi_i\bar\psi_j - \mathcal{V}(\phi,\bar\phi)\ .
\end{aligned} \tag{7.43}
$$

The function $\mathcal{V}(\phi,\bar{\phi})$ is the scalar potential

$$\mathcal{V}(\phi,\bar{\phi}) \;=\; \frac{\partial W}{\partial\phi^i}\frac{\partial\overline{W}}{\partial\bar{\phi}_i} + \frac{g^2}{2}\sum_a \left|\bar{\phi}_i (T^a_R)^i{}_j\phi^j + \xi^a\right|^2 \;\geq\; 0 \;, \tag{7.44}$$

where we have emphasized that it is non-negative. Note that is can also be written as

$$\mathcal{V} = \bar{F}F + \frac{1}{2}D^2\bigg|_{\text{on-shell}} \;. \tag{7.45}$$

We will discuss it further in the next Section, where we study the vacuum structure of supersymmetric theories.

7.4 Renormalization of the gauge coupling

We have seen in Section 6 that the effective superpotential of a theory with just chiral multiplets does not receive quantum corrections neither perturbatively, nor non-perturbatively. This property extends to any F-term, that is any half-superspace integral that cannot be rewritten as a full superspace integral. One may wonder whether a similar non-renormalization property holds for the gauge kinetic term $\left(\int d^2\theta\,\tau\,\mathrm{Tr}\,W^\alpha W_\alpha\right)$, that is also a half-superspace integral (although it is not exactly on the same footing as a superpotential term since we have seen it can be rewritten as a full superspace integral using $\bar{D}\bar{D}\simeq\int d^2\bar{\theta}$). Again exploiting the power of holomorphy, we will see that in fact when we lower the renormalization group scale the gauge kinetic term receives corrections only at one loop in perturbation theory.

In order to see this, we need to briefly discuss two properties of general Yang-Mills theories, not necessarily supersymmetric. The first property concerns the θ-term

$$S_\theta = \frac{\theta_{\mathrm{YM}}}{32\pi^2}\int d^4x\,\mathrm{Tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} \;, \tag{7.46}$$

where we recall that $\tilde{F}^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. You can check that the integrand can be written as a total derivative,

$$S_\theta = 2\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\mu\mathrm{Tr}\left[A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma\right] \tag{7.47}$$

and thus integrates to a boundary term, although of a gauge dependent quantity. It follows that S_θ does not affect the classical equations of motion. Also, it has no effect in perturbation theory and θ_{YM} does not get renormalized perturbatively. The value of the boundary term depends on the behavior of the gauge field at infinity. Convergence of the action requires that the field strength $F_{\mu\nu}\to 0$ for $|x|\to\infty$, however A_μ does not need to vanish: it can take any pure gauge configuration asymptotically. Now, there exist gauge transformations that are not continuosly connected to the identity (these are called large gauge transformations). The action S_θ does not vanish when the gauge field is given by one of such gauge transformations. Working in Euclidean signature, one can show that it actually evaluates to an integer (times $\theta_{\rm YM}$):

$$S_\theta = \frac{\theta_{\rm YM}}{32\pi^2}\int {\rm d}^4x\,{\rm Tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} = n\,\theta_{\rm YM}\ , \qquad \text{where } n\in\mathbb{Z}\ . \tag{7.48}$$

The integer n is a topological quantity since it is independent of local deformations of the field configurations, and is called the instanton number.[7] Since the action enters in the path integral as $\int\mathcal{D}\phi\,e^{iS_\theta}$, we conclude that a shift

$$\theta_{\rm YM}\to\theta_{\rm YM}+2\pi \tag{7.49}$$

is a symmetry of the theory as it does not change the path integral. This means that $\theta_{\rm YM}$ should be regarded as a periodic variable, with period 2π. For this reason it is called the θ-angle.

The second property we will need concerns the beta function of Yang-Mills theories. The gauge coupling runs with the renormalization group scale μ, the rate of its variation being controlled by the beta function $\beta=\mu\frac{\partial g(\mu)}{\partial\mu}$. A one-loop computation shows that for Yang-Mills theories the beta function is

$$\mu\frac{\partial g}{\partial\mu} = -\frac{b_1}{16\pi^2}\,g^3+\mathcal{O}(g^5)\ , \tag{7.50}$$

where b_1 is a numerical coefficient that depends on the details of the theory, and $\mathcal{O}(g^5)$ denotes higher order corrections. The solution of this equation is

$$\frac{1}{g^2(\mu)} = -\frac{b_1}{8\pi^2}\,\log\frac{\Lambda}{\mu}\ , \tag{7.51}$$

up to higher order corrections. Formally Λ arises in this formula as an integration constant. Physically it is interpreted as the scale where the one-loop coupling diverges; of course the one-loop approximation is not correct in such regime and thus Λ sets the scale at which higher loop corrections and non-perturbative effects should be taken into account. At any fixed scale μ_0, we can rewrite (7.51) as an expression for Λ:

$$\Lambda\equiv\mu_0\,e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}}\ . \tag{7.52}$$

[7] Instantons are solutions to the Euclidean equations of motion with non-vanishing action. They play an important role when evaluating the non-perturbative contributions to the path integral of gauge theories. A detailed discussion of instantons would require another full course.

Despite the apparence, Λ does not depend on the renormalization group scale. Indeed we can compute:

$$\frac{\partial\Lambda}{\partial\mu_0} = e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}} + \mu_0 \left[-\frac{8\pi^2}{b_1 g^3(\mu_0)} \frac{2}{\mu_0} \left(\frac{b_1}{16\pi^2}\, g^3 + \mathcal{O}(g^5) \right) \right] e^{-\frac{8\pi^2}{b_1 g^2(\mu_0)}} = 0 + \dots\ , \tag{7.53}$$

where the dots denote possible higher order corrections. However one can show that $\frac{\partial\Lambda}{\partial\mu_0} = 0$ holds at any order in perturbation theory: Λ is an intrinsic scale, independent of the renormalization group scale. So using (7.52) we can evaluate Λ at a given scale μ_0, knowing that it will remain the same when we lower the scale.

Armed with the two facts about Yang-Mills theories briefly illustrated above, we can tackle the question of gauge coupling renormalization in supersymmetric theories. Our main scope will be to illustrate how the additional properties of supersymmetry greatly constrain the quantum corrections. We consider pure $N = 1$ super-Yang-Mills theory with Lagrangian

$$\mathcal{L} = \frac{1}{16\pi i} \int \mathrm{d}^2\theta\, \tau\, \mathrm{Tr}\, W^\alpha W_\alpha + h.c.\ , \tag{7.54}$$

where we recall that the complexified gauge coupling is $\tau = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2}$ and we stress that we are *not* performing the rescaling (7.28); then the gauge fields are not canonically normalized and the coupling constant g appears in τ and nowhere else.

For pure $N = 1$ super-Yang-Mills theory with gauge group $G = \mathrm{SU}(N)$, the coefficient b_1 in (7.50) can be shown to be $b_1 = 3N$. Let us assume we start at some UV scale μ_0 and measure the intrinsic scale

$$|\Lambda| = \mu_0\, e^{-\frac{8\pi^2}{3N g^2(\mu_0)}}\ , \tag{7.55}$$

where $|\Lambda|$ denotes what we have called Λ in (7.51), (7.52). We have seen that this is actually independent of μ_0, so the same value would be obtained if it was measured at any other scale. We can further define a complexified intrinsic scale Λ as

$$\Lambda \equiv |\Lambda|\, e^{i\frac{\theta_{\rm YM}}{3N}} = \mu_0\, e^{\frac{2\pi i \tau(\mu_0)}{3N}}\ . \tag{7.56}$$

It is important to notice that since $\theta_{\rm YM}$ does not get renormalized, this remains an intrinsic scale, independent of μ_0.

We want to study the quantum corrections to the gauge kinetic term while we integrate down to some lower scale μ. Because of supersymmetry, the effective gauge kinetic term must have the form

$$\frac{\tau(\Lambda;\mu)}{16\pi i}\, \mathrm{Tr}\, W^\alpha W_\alpha\ + h.c.\ . \tag{7.57}$$

and we ask how the effective coupling τ depends on Λ and μ. We know that at one-loop the gauge coupling is

$$\frac{1}{g^2(\mu)} = -\frac{3N}{8\pi^2}\log\frac{|\Lambda|}{\mu} \ . \tag{7.58}$$

The one-loop complexified gauge coupling can then be written as a function of Λ and μ as

$$\tau_{\text{1-loop}}(\Lambda;\mu) = \frac{3N}{2\pi i}\log\frac{\Lambda}{\mu} \ . \tag{7.59}$$

Now we impose two crucial requirements:

1) as seen above, a shift in the theta-angle $\theta_{\rm YM} \to \theta_{\rm YM} + 2\pi$ must leave the physics invariant. This shift is the same as $\tau \to \tau + 1$ and correspondingly

$$\Lambda \to e^{\frac{2\pi i}{3N}}\,\Lambda \ . \tag{7.60}$$

So $\tau(\Lambda;\mu)$ must depend on Λ in such a way that when this is transformed as in (7.60), τ just shifts by 1. Since this is already achieved by the one-loop term (7.59), any additional term must be invariant under (7.60).

2) τ must be a holomorphic function of Λ. Indeed similarly to the coupling constants in a true superpotential term, Λ may be regarded as the VEV of a chiral superfield; therefore the term (7.57) should depend holomorphically on it.

From these two requirements it follows that the effective coupling must take the form

$$\tau(\Lambda;\mu) = \frac{3N}{2\pi i}\log\frac{\Lambda}{\mu} + f(\Lambda;\mu) \ , \tag{7.61}$$

with f holomorphic in Λ and invariant under (7.60). We also impose a further consistency requirement:

3) that f has a Taylor expansion including only positive powers of Λ. This is because for Λ very small, which is a weak coupling limit, we should get back the one-loop result.

This implies that the effective coupling (7.61) has the form

$$\tau(\Lambda;\mu) \ = \ \frac{3N}{2\pi i}\log\frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{3Nn} \ , \tag{7.62}$$

where n is an integer. The first term is the one-loop result while the other terms cannot be reproduced in perturbation theory: they are non-perturbative corrections due to the instantons seen above, where n is precisely the instanton number. We conclude that *the complexified gauge coupling τ is one-loop exact in perturbation theory, and only receives non-perturbative corrections.*

Note: for pure $N = 1$ super-Yang-Mills theory one can further show that $a_n = 0$, that is all non-perturbative corrections vanish and τ is really exact at one loop; on the other hand, for a matter-coupled super-Yang-Mills theory like the supersymmetric QCD that we will see below the non-perturbative corrections do not vanish.

We emphasize again that the result above is valid only in the so called "holomorphic scheme" where the gauge coupling appears only in the complex parameter τ. If we rescale the vector superfield by the gauge coupling g in order to achieve canonically normalized kinetic terms, then the gauge coupling expressed in this scheme receives corrections at higher loop order as well.

8 Vacuum structure

8.1 Supersymmetric vacua

Perturbative computations in quantum field theory are done by studying the field fluctuations around a stable configuration, that is usually taken to be the vacuum. We define the vacuum as a Lorentz invariant, stable (or sufficiently long-lived) state. Lorentz invariance implies that only scalar fields can take a non-zero vacuum expectation value; it also implies that this value must be constant. Hence the only term that contributes when evaluating the Hamiltonian in the vacuum is the scalar potential. On the other hand stability means minimal energy. Therefore vacua are in one-to-one correspondence with the (global or local) minima of the scalar potential.

Let us then look at vacua in supersymmetric theories. We have seen that the scalar potential for a general $N = 1$ theory is

$$\begin{aligned} \mathcal{V}(\phi,\bar{\phi}) \;&=\; \frac{\partial W}{\partial \phi^i}\frac{\partial \overline{W}}{\partial \bar{\phi}_i} + \frac{g^2}{2}\sum_a \left|\bar{\phi}_i (T^a_R)^i{}_j \phi^j + \xi^a\right|^2 \\ &= \bar{F}F + \frac{1}{2}D^2\bigg|_{\mathrm{on-shell}} \;\geq\; 0\ . \end{aligned} \tag{8.1}$$

Non-negativity of the scalar potential implies that the vacuum energy can never be negative. As seen when studying the general consequences of the superalgebra, this also follows from

$$\langle\Omega|P^0|\Omega\rangle \;\sim\; \sum_\alpha \left(||Q_\alpha|\Omega\rangle||^2 + ||\bar{Q}_{\dot\alpha}|\Omega\rangle||^2\right) \;\geq 0\ , \tag{8.2}$$

where $|\Omega\rangle$ can be any state, in particular the vacuum. This preserves supersymmetry if it is annihilated by the supercharges,

$$Q_\alpha|\Omega\rangle = \bar{Q}_{\dot\alpha}|\Omega\rangle = 0\ . \tag{8.3}$$

We conclude that a vacuum is supersymmetric if and only if it has zero energy. Conversely, supersymmetry is broken in the vacuum whenever the latter has positive energy. This means that supersymmetric vacua are in one-to-one correspondence with zeros of the scalar potential, and are thus characterized by the equations:

$$0 = D^a = -g\,\bar{\phi}T^a\phi - g\,\xi^a \;, \qquad 0 = \bar{F}_i = \frac{\partial W}{\partial \phi^i} \;, \tag{8.4}$$

These are called the D-term and F-term equation, respectively. While solving these equations one should mod out by gauge transformations, as solutions that are related by a gauge transformation describe the same state. The F-term equation means that supersymmetric vacua extremize the superpotential, when this is present. Notice that in general it is easier to extremize the superpotential rather than the full scalar potential (8.1). So it is easier to find supersymmetric vacua than generic vacua.

The set of solutions to the D-term and F-term equations (8.4) is called the *moduli space* of supersymmetric vacua. The scalar fields that parameterize it are flat directions of the scalar potential and are called *moduli.* Since these scalar fields don't feel any potential, their fluctuations around a given supersymmetric VEV correspond to massless fields. Therefore the moduli provide the lightest fields in the low energy effective theory around a given supersymmetric vacuum. Different VEV's for the moduli lead to physically inequivalent low-energy effective theories, since the spectrum of massive fields in general changes.

It is also important to notice that while in non-supersymmetric theories (or in a susy-breaking vacuum of a supersymmetric theory), the space of classical flat directions is generically lifted by quantum corrections (captured by the Coleman-Weinberg potential), in supersymmetric theories this cannot happen: if the vacuum energy is zero at tree level, it must remain zero at all orders in perturbation theory. This is because the quantum corrections that would generate a potential for the moduli are suppressed by cancellations between bosons and fermions running in the loops. This means that if a vacuum is supersymmetric at tree level, it will remain such at all orders in perturbation theory. In other words, *supersymmetry can only be broken either at tree-level, or by non-perturbative effects.*

We now study an example where the D-term and F-term equations have a solution, namely the theory admits supersymmetric vacua. Then we will move on to study vacua in which these conditions cannot be solved and supersymmetry is spontaneously broken.

The example of SQED

[See Bertolini's lectures, pages 95–99]. Let us consider SQED, the supersymmetric version of quantum elecrodynamics. This has gauge group U(1), N_f pairs of chiral superfields $Q^i, \tilde{Q}_i$

having opposite charge under U(1), and no superpotential nor Fayet-Iliopoulos terms. For definiteness we take the charges to be all equal to $+1$ for the Q^i and -1 for the $\tilde{Q}_i$. The Lagrangian is

$$\mathcal{L}_{\rm SQED} = \frac{1}{32\pi}{\rm Im}\left(\tau\int {\rm d}^2\theta\, W^\alpha W_\alpha\right) + \int {\rm d}^2\theta\, {\rm d}^2\bar{\theta}\left(\bar{Q}_i\, e^{2V} Q^i + \bar{\tilde{Q}}_i\, e^{-2V}\tilde{Q}^i\right) \ . \tag{8.5}$$

(The fields here are normalized so that the gauge coupling only appears in τ). Since there is no superpotential, supersymmetric vacua are characterized by the D-term equation

$$\bar{q}_i q^i - \bar{\tilde{q}}_i \tilde{q}^i \ = \ 0 \ , \tag{8.6}$$

where q^i and $\tilde{q}^i$ are the bottom component of Q^i and $\tilde{Q}^i$, respectively. In addition we should take into account the redundancy due to the gauge symmetry, which acts as

$$q^i \to e^{i\alpha} q^i \ , \qquad \tilde{q}^i \to e^{-i\alpha}\tilde{q}^i \ . \tag{8.7}$$

These are two real conditions. It follows that the complex dimension of the moduli space of supersymmetric vacua is

$$\dim_{\mathbb{C}} \mathcal{M} = 2N_f - 1 \ . \tag{8.8}$$

So although we started with $2N_f$ chiral multiplets, only $2N_f - 1$ are needed to describe the moduli space and (as we will discuss further below) the low-energy effective theory. Where has the remaining chiral multiplet gone? If q^i and $\tilde{q}^i$ have a non-zero VEV, then the gauge group is broken in the vacuum and the photon becomes massive via the Higgs mechanism. In this mechanism, the photon acquires its third polarization state by absorbing a real scalar field. However, for this to happen in a supersymmetric way it must be that an entire chiral multiplet is absorbed by the vector multiplet. Recall that the bosonic on-shell degrees of freedom of a massive vector multiplet are those of a massive vector field and a real scalar field, which correspond exactly to the bosonic on-shell degrees of freedom of a massless vector multiplet and a chiral multiplet. The fermionic degrees of freedom work accordingly: the Weyl fermion in the chiral multiplet provides the needed degrees of freedom to make the fermion in the vector multiplet (called the "photino") massive. This is the *super-Higgs mechanism*. We remark again that the supersymmetric vacua that make the moduli space are physically inequivalent, as the mass of e.g. the photon depends on the VEV of the scalar fields.

We also note that there is a special point in the moduli space: the origin $\langle q^i \rangle = \langle \tilde{q}^i \rangle = 0$. In this point the gauge symmetry is restored; correspondingly, the D-term and gauge invariance conditions are trivially satisfied and one has more massless degrees of freedom.

One says that the theory is un-Higgsed as in this special point there is no (super-)Higgs mechanism taking place.

In order to discuss the low-energy effective theory on the moduli space, let us focus on the simplest case $N_f = 1$, where the moduli space has complex dimension 1. The moduli space is conveniently described using the only independent gauge-invariant operator we can construct using Q and $\tilde{Q}$, which is

$$M = Q\tilde{Q} \tag{8.9}$$

(this is called the "meson"). Its VEV $\langle M\rangle = \langle Q\tilde{Q}\rangle = \langle q\tilde{q}\rangle$ parametrizes the moduli space. Moreover, the fluctuations of M around the VEV describe the massless degrees of freedom in the low-energy effective theory. Indeed in our $N_f = 1$ example, after projecting on the moduli space we can write

$$\bar{Q}Q = \bar{\tilde{Q}}\tilde{Q} = \sqrt{\bar{M}M}\ . \tag{8.10}$$

So the Kähler potential, which in the original UV theory is canonical, on the moduli space reads

$$K = \bar{Q}Q + \bar{\tilde{Q}}\tilde{Q} = 2\sqrt{\bar{M}M} \tag{8.11}$$

and is thus non-canonical. The scalar kinetic term that follows from this Kähler potential is:

$$\frac{1}{2}\int \mathrm{d}^4x \frac{1}{\sqrt{\bar{m}m}}\,\partial_\mu \bar{m}\,\partial^\mu m\ , \tag{8.12}$$

where m is the bottom component of M. Notice that this is singular at $m = 0$, that is at the origin of the moduli space. This should be no surprise: we have already seen in the "Integrating out" example in Section 6 that singularities showing up in the (Wilsonian) low-energy effective theory generically signal the appearance of extra light degrees of freedom that should be included in the description, and in this case we know that at the origin of the moduli space the vector multiplet becomes massless again as the theory is unHiggsed.

Let us also briefly discuss the case $N_f = 2$. Now the moduli space has complex dimension $2N_f - 1 = 3$. We can make four possible gauge-invariant meson operators, $M^i{}_j = Q^i\tilde{Q}_j$, $i, j = 1, 2$, but being constructed from the two vectors $Q^i, \tilde{Q}_j$, this is a rank-1 matrix and therefore obeys the constraint $0 = \det M \equiv M^1{}_1 M^2{}_2 - M^1{}_2 M^2{}_1$. So we really have three independent meson operators. Again these can be used to parameterize the three-dimensional moduli space. The Kähler potential on the moduli space is $K = \bar{Q}_i Q^i + \bar{\tilde{Q}}_i\tilde{Q}^i = 2\sqrt{\bar{M}^j{}_i M^i{}_j} = 2\sqrt{\mathrm{Tr}\,\bar{M}M}$ and again we have a singularity in the scalar kinetic terms at the origin of the moduli space, where the theory is un-Higgsed.

In this example we have learnt one more general lesson: the parametrization in terms of the independent gauge-invariant operators is very useful to characterize the moduli space of supersymmetric vacua and to describe the corresponding low-energy effective theory. This is another instance of how the symmetries — in this case the spontaneously broken gauge symmetry together with the unbroken gauge symmetry — constrain low-energy effective theories.

⧫ **Exercise.** In the $N_f = 1$ example described above, work out the mass of the photon at a generic point in the moduli space and check that it depends on the VEV of m.

⧫ **Exercise.** Consider the purely matter theory defined by $K = \bar{Q}Q$, $W = \frac{1}{2}mQ^2$. Determine whether there are supersymmetric vacua and, if so, compute the mass spectrum of the field fluctuations around them.

8.2 Supersymmetry breaking

Spontaneous supersymmetry breaking and the Goldstino

We discuss spontaneous supersymmetry breaking and prove the corresponding Goldstone theorem, showing the existence of a massless fermion field (the *Goldstino*) → see Bertolini's lectures, Chapter 7, pages 122–125.

There are two different ways one can obtain breaking of supersymmetry in a vacuum: by giving a vev to one (or more) F-term, or to one (or more) D-term. Let us discuss two simple models illustrating these different mechanisms.

F-term breaking: O'Raifeartaigh model

Assume the gauge group has no U(1) factors or anyway the Fayet-Iliopoulos parameters ξ^a vanish. Susy will necessarily be broken if no extrema of the scalar potential satisfy the F-term and D-term conditions. As long as the superpotential $W(\Phi^i)$ has no linear term, $\langle\phi^i\rangle = 0$ will always be a supersymmetric vacuum. Hence let us assume that there is a linear term in the superpotential, $W = a_i\Phi^i + \dots$. For this to be gauge invariant, we need that the representation of the gauge group under which Φ^i transforms contains at least one singlet.

As a concrete example of this mechanism, we can take a model with canonical Kähler potential, and superpotential given by

$$W = \frac{1}{2}hX\Phi_1^2 + m\Phi_1\Phi_2 - \mu^2 X \ , \tag{8.13}$$

where in this example the chiral superfields X, Φ_1, Φ_2 are all singlets of the gauge group.

The equations for the auxiliary fields F are:

$$\begin{aligned} \bar{F}_X &= \frac{1}{2}h\phi_1^2 - \mu^2 \ , \\ \bar{F}_1 &= hx\phi_1 + m\phi_2 \ , \\ \bar{F}_2 &= m\phi_1 \ . \end{aligned} \tag{8.14}$$

Clearly the first and the third equations cannot vanish simultaneously, hence there are no supersymmetric vacua. The scalar potential is

$$\begin{aligned} \mathcal{V} &= |F_X|^2 + |F_1|^2 + |F_2|^2 \\ &= |\tfrac{1}{2}h\phi_1^2 - \mu^2|^2 + |hx\phi_1 + m\phi_2|^2 + |m\phi_1|^2 \end{aligned} \tag{8.15}$$

For $|\mu| < |m|$, it has a minimum in

$$\langle\phi_1\rangle = \langle\phi_2\rangle = 0 \ , \quad \langle x\rangle = \text{const} \ , \tag{8.16}$$

so we have infinitely many non-supersymmetric degenerate vacua, in which the scalar potential takes the value $\langle\mathcal{V}\rangle = |\mu^2|^2$.

Let us look at the classical mass spectrum around the susy breaking vacua. The full chiral superfield X remains massless. The massless fermion mode ψ_X plays the role of the Goldstino (indeed the only non-vanishing F-term in the vacuum is F_X, so $\psi^G \propto \langle F_X\rangle\psi_X$). The real scalar $|x|$ is the modulus parameterizing the classical moduli space, while the phase $x = e^{i\alpha}|x|$ can be seen as the Goldstone boson associated with the spontaneous breaking of the R-symmetry in the vacuum (indeed the model has an R-symmetry under which X has R-charge 2; since this takes a VEV, it breaks the R-symmetry spontaneously). All other fields have a non-vanishing mass in the vacuum, and one can check that the fermion and boson masses are different functions of the parameters h, μ, m and of the VEV x, so the spectrum is manifestly non-supersymmetric.

Classically we have a moduli space of vacua, as x can take any constant VEV. However we should keep in mind that these are susy breaking vacua, so generically they will not be protected against quantum corrections. One can indeed check by computing the Coleman-Weinberg potential that the quantum corrections lift the flat direction and leave just the vacuum in $x = 0$. [For details see Bertolini's lectures, pp. 137–139.]

♦ **Exercise.** Compute the full mass spectrum for the field fluctuations around the non-supersymmetric vacua above. Check that the boson and fermion masses are different, as expected since supersymmetry is broken. Notice that however one has the relation

$$\text{STr}\mathcal{M}^2 \equiv \sum m_0^2 - 2\sum m_{1/2}^2 = 0 \ , \tag{8.17}$$

where m_0 denotes the scalar field masses while $m_{1/2}$ denotes the spin 1/2 field masses. This identity is called the *supertrace formula* and (after adding the contribution of vector fields) holds when supersymmetry is preserved (obviously) but also when supersymmetry is spontaneously broken at tree level as in the present case. Also note that for $\mu = 0$ supersymmetry is restored and the masses become equal. [For the answer see Bertolini's lectures, pp. 134-135].

D-term breaking: Fayet-Iliopoulos model

A different mechanism for supersymmetry breaking uses the Fayet-Iliopoulos parameters in the D-terms. Recall that Fayet-Iliopoulos terms can be introduced whenever the gauge group has U(1) factors.

As an example, let us consider the case where the gauge group is just U(1) and there are two massive chiral superfields with opposite charge $+e$ and $-e$:

$$\mathcal{L} = \frac{1}{32\pi}\mathrm{Im}\left(\tau \int \mathrm{d}^2\theta\, W^\alpha W_\alpha\right) + \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\left(\bar{\Phi}_+\, e^{2eV}\,\Phi_+ + \bar{\Phi}_-\, e^{-2eV}\,\Phi_- + \xi V\right) + \left(m \int \mathrm{d}^2\theta\, \Phi_+\Phi_- + h.c.\right) \quad . \tag{8.18}$$

A supersymmetric gauge transformation acts on the chiral superfields as $\Phi_\pm \to e^{\pm ie\Lambda}\Phi_\pm$, where Λ is the chiral superfield of gauge parameters. The equations of motion for the auxiliary fields are:

$$\begin{aligned} \bar{F}_\pm &= m\phi_\mp \ , \\ D &= -\frac{1}{2}\left[2e\left(|\phi_+|^2 - |\phi_-|^2\right) + \xi\right] \ . \end{aligned} \tag{8.19}$$

Because of the shift in the D-term due to the Fayet-Iliopoulos parameter ξ, the first and the second line cannot vanish separately, hence any extremum of the scalar potential must break supersymmetry. Notice that although the presence of the Fayet-Iliopoulos parameter in the D-term is crucial, this mechanism for supersymmetry breaking also requires the equation from the F-term.

The scalar potential can be written as:

$$\begin{aligned} \mathcal{V} &= \frac{1}{8}\left[2e\left(|\phi_+|^2 - |\phi_-|^2\right) + \xi\right]^2 + m^2\left(|\phi_+|^2 + |\phi_-|^2\right) \\ &= \tfrac{1}{8}\xi^2 + \left(m^2 - \tfrac{1}{2}e\xi^2\right)|\phi_-|^2 + \left(m^2 + \tfrac{1}{2}e\xi^2\right)|\phi_+|^2 + \tfrac{1}{2}e^2\left(|\phi_+|^2 - |\phi_-|^2\right)^2 \ . \end{aligned} \tag{8.20}$$

For $m^2 > \frac{1}{2}e\xi$, all terms in the potential are non-negative and $\langle\phi_\pm\rangle = 0$ is a minimum. In this point, the value of the potential is $\langle\mathcal{V}\rangle = \frac{1}{8}\xi^2$; as long as $\xi \neq 0$, this is positive, confirming that supersymmetry is broken. On the other hand, since $\langle\phi_\pm\rangle = 0$, there is no Higgs mechanism taking place and the gauge symmetry is preserved. Notice that in this case the F-term in (8.19) vanishes and the source of supersymmetry breaking is entirely in the D-term $D = -\frac{1}{2}\xi$. One then speaks of *pure* D-term breaking. Correspondingly, the Goldstino mode is identified with the photino λ (namely, the fermion in the U(1) vector supermultiplet), because $\langle F_\pm\rangle = 0$ and $\psi^G \propto \langle D\rangle\lambda$.

♦ **Exercise.** Consider the case $m^2 < \frac{1}{2}e\xi$. Show that the potential is extremized for non-zero $\langle\phi_-\rangle$. This implies that the Higgs mechanism takes place and both supersymmetry and gauge symmetry are broken in the vacuum. Check that both the F-term and the D-term get a VEV; in this case one speaks of *mixed* D-term and F-term breaking. By computing the fermion mass matrix explicitly, you can check that the Goldstino mode is $\psi^G \propto \langle D\rangle\lambda + \langle F_+\rangle\psi_+$. For details see Bertolini's lectures, section 7.5.