Geometric methods in Theoretical Physics

---

1. Compute the coset representatives, vielbeins, connection, metric and curvature for the homogeneous manifolds
$$\frac{SO(3)}{SO(2)} \qquad \text{and} \qquad \frac{SU(1,1)}{U(1)}.$$

2. Consider the three-sphere embedded in $\mathbb{R}^4$
$$(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = 1$$
and the map $\Pi : S^3 \to S^2$
$$\begin{aligned} \xi^1 &= 2(x_1 x_3 + x_2 x_4), \\ \xi^2 &= 2(x_2 x_3 - x_1 x_4), \\ \xi^3 &= (x_1)^2 + (x_2)^2 - (x_3)^2 - (x_4)^2. \end{aligned}$$
Use two charts on S^2, called U_N and U_S, covering the northern and southern hemispheres and use stereographic projection coordinates, i.e.
$$X + i\,Y = \frac{\xi^1 + i\,\xi^2}{1 - \xi^3} \tag{1}$$
for the southern hemisphere. The sphere S^3 is then a fibre bundle, where S^2 is the base space and $U(1)$ is the fibre. Construct the local trivializations for such bundle and the transition functions at the equator.

3. Write in differential form language the action
$$S = -\frac{1}{4} \int d^4x \,\mathrm{tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$
and the equations of motion following from this action, using the definition
$$F = \frac{1}{2}\, dx^\mu \wedge dx^\nu F_{\mu\nu} = dA + \frac{1}{2}\, A \wedge A,$$
with $A = A^I t_I$, for $t_I \in \mathfrak{g}$, the algebra of a non-abelian, semisimple gauge group.