1. Start from the stress-energy tensor for dust in its rest frame

$$T_{00} = \rho\,, \quad T_{ij} = p\,\delta_{ij}\,.$$

Derive its form in a general frame.

(Apply a Lorentz boost)

2. Consider the gravity wave metric

$$ds^2 = -dt^2 + (1 + A\cos k(z+t))dx^2 + (1 - A\cos k(z-t))dy^2 + dz^2,$$

where $A << 1$.

(a) Show that $x^\mu(\tau) = (\tau, x_0, y_0, z_0)$, for x_0, y_0, z_0 constants, is a geodesic;

(b) Find (at first order in A) the time difference between two light signals emitted at the same time from the origin of the reference frame given above and traveling to mirrors located at $(L, 0, 0)$ and $(0, L, 0)$ and back.

3. For a null geodesic γ on a given metric $g_{\mu\nu}$ we can find adapted coordinates so that the line element becomes

$$ds^2 = 2dUdV + a(U,V,x^k)dV^2 + 2b_i(U,V,x^k)dx^i dV + g_{ij}(U,V,x^k)dx^i dx^j.$$

To approach the massless particles moving along the null geodesic, we can perform a boost

$$(U,V,x^k) \mapsto \left(\frac{U}{\lambda}, \lambda\, V, x^k\right)$$

and try to take the infinite boost limit $\lambda \to 0$. We can avoid singularities if we uniformly rescale the coordinates by

$$(U,V,x^k) \mapsto \lambda\,\left(U,V,x^k\right).$$

(a) Show that the net effect is an asymmetric rescaling of the coordinates;

(b) Show that by also rescaling the line element and taking the limit

$$d\bar{s}^2 = \lim_{\lambda\to 0} \lambda^{-2} ds^2_\lambda$$

one obtains the metric of a plane wave $d\bar{s}^2 = 2du dv + h_{ij}(u)dx^i dx^j$.