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Saclay Lectures on Effective Field Theories

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Abstract

Effective field theories (EFTs) are widely used in particle physics and well beyond. The basic idea is to approximate a physical system by integrating out the degrees of freedom that are not relevant in a given experimental setting. These are traded for a set of effective interactions between the remaining degrees of freedom. In these lectures I review the concept, techniques, and applications of the EFT framework. In the 1st lecture I give an overview of the EFT landscape. I will review the basis techniques and usage of EFTs. A few prominent examples are presented in some detail, such as the Fermi theory, the Euler-Heisenberg Lagrangian, and the chiral perturbation theory. In the 2nd lecture I discuss quantitatively the procedure of integrating out heavy particles in a quantum field theory. An explicit example of the tree- and one-loop level matching between an EFT and its UV completion will be discussed in all gory details. The 3rd lecture is devoted to path integral methods of calculating effective Lagrangians. Finally, in the 4th lecture I discuss the EFT approach to constraining new physics beyond the Standard Model (SM). To this end, the so-called SM EFT is introduced, where the SM Lagrangian is extended by a set of non-renormalizable interactions representing the effects of new heavy particles.

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1 Illustrated philosophy of EFT

1.1 Introduction

Not so long ago the Holy Grail of theoretical particle physics was the fundamental theory underlying the SM and Einstein gravity. It was hoped that strong consistency constraints on quantum theories incorporating gravity would lead to an essentially unique solution. That *theory of everything*, valid at very high energies, should have the SM as its low-energy approximation, possibly with some intermediate field theories between the weak and the Planck scale. Such a top-down approach would allow one to predict the spectrum and interactions of the SM particles, and the existence of new particles, symmetries, and interactions. Alas, that program has not delivered what it promised, and the quest for the theory of everything is now largely abandoned. Instead the focus these days is on less ambitious but more practical *theories of something*, which describe particular physical systems in particular conditions. We think of them as *effective fields theories* because they are not meant to be valid at all energy scales, and often the degrees of freedom they describe are not fundamental but emergent.

Consider a physical system whose dynamics is described by the Lagrangian $\mathcal{L}_{\mathrm{UV}}(\phi, H)$. Here I divided the degrees of freedom of the system into two groups denoted as ϕ and H. In principle, $\mathcal{L}_{\mathrm{UV}}$ encodes the full information about the system, and allows one to predict the rate of any processes involving ϕ and H. However, quite often one is interested in amplitudes with only ϕ's in the external states, while those with H's are irrelevant in the particular experimental situation. This may be because H's are too heavy to be excited, given the characteristic energy scale of the experiment. Take e.g. the QED Lagrangian which contains the photon and electron fields. Both of these are crucial to describe light-by-light scattering, as photons interact with each other only via intermediate electrons. However, on-shell electrons cannot appear on-shell in scattering of very low energy photons well below the threshold for e^+e^- pair production. In undersatand to understand the photon behavior we do not need to calculate scattering amplitudes into electrons. Another example is the muon decay. This can be described by the SM Lagrangian, where the decay happens to due to an exchange of a virtual W boson at the leading order in perturbation theory (and also of other weak bosons at higher orders). However, in muon decay the weak bosons are not directly visible as resonances, simply because $m_\mu \ll m_W$.

In a situation as in the examples above it may be more efficient to use a simpler description of the system where H is absent from the beginning. In the particle jargon, we can integrate out H's from the UV theory, so as to obtain an effective theory for ϕ's only. This can be succinctly formulated using the path integrals. The UV theory is fully characterized by the partition function

$$Z_{\mathrm{UV}}[J_\phi, J_H] = \int [D\phi][DH] \exp\left[i \int d^4x \left(\mathcal{L}_{\mathrm{UV}}(\phi, H) + J_\phi\phi + J_H H \right) \right]. \qquad (1.1)$$

All n-point correlation functions of ϕ's and H's (and thus all S-matrix elements) can be obtained by differentiating Z_{UV} with respect to the currents J. In the effective theory

we only need the correlators of ϕ's, hence[1]

$$Z_{\text{EFT}}[J_\phi] = Z_{\text{UV}}[J_\phi, 0]. \tag{1.2}$$

One might well stop at that definition, however the partition function of a realistic quantum field theory is an extremely complicated and awkward object. For this reason we would like to encode the dynamics of ϕ in the effective Lagrangian $\mathcal{L}_{\text{EFT}}(\phi)$. This is defined by

$$Z_{\text{EFT}}[J_\phi] = \int [D\phi] \exp\left[i \int d^4x \left(\mathcal{L}_{\text{EFT}}(\phi) + J_\phi \phi \right) \right]. \tag{1.3}$$

The problem which I will discuss at length in these lectures is how to find $\mathcal{L}_{\text{EFT}}(\phi)$ given $\mathcal{L}_{\text{UV}}(\phi, H)$. It should be obvious that $\mathcal{L}_{\text{eff}}(\phi) \neq \mathcal{L}_{\text{UV}}(\phi, H)$, unless ϕ and H are totally decoupled. The difference $\mathcal{L}_{\text{eff}}(\phi) - \mathcal{L}_{\text{UV}}(\phi, 0)$ must be non-trivial in order to account for the effects of H exchange between the ϕ's.

In general, $\mathcal{L}_{\text{EFT}}(\phi)$ will be a complicated non-local object. In the particle physics jargon, *non-local* means non-polynomial in the fields and their derivative. For example, the interaction term $\mathcal{L}_{\text{EFT}}(\phi) \supset \phi^2 \Box \phi^2$ is local, whereas $\mathcal{L}_{\text{EFT}}(\phi) \supset \phi^2 (\Box + M^2)^{-1} \phi^2$ is non-local. The latter do arise in effective theories to describe effects of propagation of heavy H particles, and may be unfamiliar to regular QFT users. However, there is one important situation where $\mathcal{L}_{\text{EFT}}(\phi)$ can be adequately approximated by a local Lagrangian. This occurs when the integrated-out H particles are much heavier than ϕ's masses and energies, which is also the most common case where EFT techniques are applied. Heuristically this is easy to understand. When we operate at energies well below the mass M we are blind to physics at distance scales $L \ll M^{-1}$. Thus, the exchange of H's with masses of order M between ϕ's looks to a low-energy observer as a contact interaction between ϕ's only. Mathematically speaking, the H propagator has a local expansion when its 4-momentum is much smaller than M: $(\Box + M^2)^{-1} \approx M^{-2} - M^{-4}\Box + \dots$. In these lectures I will focus on the case when the EFT can be described by a local Lagrangian.

There are several possible motivations to work with $\mathcal{L}_{\text{EFT}}(\phi)$, rather than with the more fundamental $\mathcal{L}_{\text{UV}}(\phi, H)$:

- **Simplicity**. Calculations within the effective theory may be far more efficient, especially for multi-loop integrals. Moreover, it is often the case that non-trivial cancellations in the UV theory can be easily understood via power counting in the EFT.

- **Calculability**. Loop calculations in a theory with disparate mass scales suffer from the problem of large logarithms, which may invalidate the perturbative expansion even when the coupling constants are well within the perturbative regime. Effective theory techniques allow one to resum the large logarithms into a renormalization group flow of the EFT parameters.

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[1] We usually demand the weaker condition that the on-shell S-matrix elements of ϕ's and not their full off-shell correlators match between the UV theory and the EFT. The matching of the S-matrix elements follows from Eq. (1.2), but in fact $Z_{\text{EFT}}[J_\phi]$ is defined up to transformations that leave the S-matrix invariant.

- **Agnosticity**. The fundamental UV theory may be unknown, as is the case for the fundamental theory underlying the SM. Alternatively, the low-energy consequence of the UV theory may be difficult to calculate, as for QCD below the confinement scale. In such circumstances, the effective theory approach allows one to proceed in a systematic way, sweeping the ignorance under the carpet of free parameters in $\mathcal{L}_{\mathrm{EFT}}$.

In the following I will first discuss the general rules that enable systematic construction of effective Lagrangians. Then I will walk through a series of simple examples that illustrate some salient feature of effective theories. This review is by no means exhaustive. For further reading, I warmly recommend the general EFT reviews of David Kaplan [1], Ira Rothstein [2], and Aneesh Manohar [3]. For reviews focused on specific applications, see e.g. Ref. [4] for the effective theory of heavy mesons, Ref. [5] for the effective theory of classical gravitational phenomena, or Ref. [6] for the effective theory of excitations in conductors.

1.2   Scaling and power counting

Even when the EFT is described by a local Lagrangian, $\mathcal{L}_{\mathrm{EFT}}(\phi)$ in general contains an infinite number of interaction terms. Therefore any useful EFT comes with a set of *power counting* rules which allow one to organize the calculations in a consistent expansion and single out the most relevant contributions. For relativistic theories obtained by integrating out heavy particles H, the inverse of the mass scale M_H of H's provides a natural expansion parameter. Observables can then be expanded in powers of E/M_H, where E is the typical energy scale of the experiment.

To be more quantitative, let us discuss how different terms in a local relativistic EFT Lagrangian scale under the rescaling of the space-time coordinates:

$$x_\mu \to \xi x'_\mu. \tag{1.4}$$

The point here is that the limit $\xi \to 0$ corresponds to zooming in on small distance scales (large energies), while the limit $\xi \to \infty$ turns the focus on large distance scales (small energies). By studying how $\mathcal{L}_{\mathrm{EFT}}(\phi)$ changes as ξ increases we can understand the relative importance of various terms as we move towards lower energies away from the UV theory underlying the EFT.

As an example, consider the following effective action for a single scalar field ϕ:

$$S_{\mathrm{EFT}}(\phi) = \int d^4x \left[(\partial_\mu \phi)^2 - m^2 \phi^2 - \kappa \mu \phi^3 - \lambda \phi^4 - \sum_{n+d>4} \frac{c_{n,d}}{\Lambda^{n+d-4}} \phi^{n-1} \partial^d \phi \right], \tag{1.5}$$

where $n(d)$ in the sum are positive (non-negative) integers. The convention here is that the scalar has mass dimension $[\phi] = \mathrm{mass}^1$. The scales Λ and μ have been inserted to make sure the action is dimensionless. Typically, Λ is identified with the mass scale M_H of the integrated-out particles. Changing the coordinates as in Eq. (1.4):

$$S_{\mathrm{EFT}}(\phi) = \int d^4x' \left[\xi^2 (\partial_\mu \phi)^2 - m^2 \xi^4 \phi^2 - \kappa \mu \xi^4 \phi^3 - \lambda \xi^4 \phi^4 - \sum_{n+d>4} \frac{c_{n,d} \xi^{4-d}}{\Lambda^{n+d-4}} \phi^{n-1} \partial^d \phi \right]. \tag{1.6}$$

Next we rescale $\phi \to \phi' \xi^{-1}$ so that the kinetic term Eq. (1.6) has the same normalization as the one in Eq. (1.5):

$$S_{\text{EFT}}(\phi) = \int d^4 x' \left[(\partial_\mu \phi')^2 - m^2 \xi^2 (\phi')^2 - \kappa(\xi\mu)(\phi')^3 - \lambda(\phi')^4 - \sum_{n+d>4} \frac{c_{n,d}}{(\xi\Lambda)^{n+d-4}} (\phi')^{n-1} \partial^d \phi' \right].$$

$$(1.7)$$

The rationale of this last operation is that the path integral is dominated by the kinetic term. Once we normalize the leading contribution canonically, we can more easily isolate the scaling behavior of the interactions terms. Comparing Eq. (1.5) and Eq. (1.7) we observe that all the terms in the sum are suppressed as we send $\xi \to \infty$. These are called *irrelevant* interactions because their effects become more and more suppressed as we descent towards lower energies. Note that the canonical dimension of the interaction term, $D = n + d - 4$, uniquely determines the scaling properties. The larger the D, the more irrelevant the interaction term becomes far below the scale Λ. Power counting in a relativistic EFT in most cases can be organized based just on the canonical dimensions of the interaction terms. In a practical situation, one retains in the effective Lagrangian only the terms up to some $D_{\max}$, and neglect the higher-dimensional interactions. The calculations of observables are performed order by order in the $1/\Lambda$ expansion.

The coefficient of the ϕ^2 term in $\mathcal{L}_{\text{EFT}}(\phi)$ increases with $\xi \to \infty$. This is called a *relevant* term: understandably, the particle's mass becomes more and more important in the IR. Similarly, the ϕ^3 interaction is relevant. Note that for the power counting sake one needs to specify the order of magnitude of the scale μ multiplying ϕ^3: the possible choices $\mu \sim \Lambda$ or $\mu \ll \Lambda$ will lead to a different expansion of the observables. On the other hand, the coefficient of the ϕ^4 term in $\mathcal{L}_{\text{EFT}}(\phi)$ is dimensionless, which implies it scales trivially with ξ. We call this a *marginal interaction*. In fact, loop corrections modify the scaling dimensions, tipping the balance either in the relevant or in the irrelevant direction (unless the effective theory is conformal).

I should mention here that the simple power counting rules above are not applicable in all situations. Sometimes a large parameter $\Lambda \gg E$ is not available because the scale of the UV theory is not parametrically larger than the typical energy scale of the EFT. In such a case, the expansion of the effective Lagrangian may be organized according to the number of derivatives. Different power counting rules apply to non-relativistic theories, such as the non-relativistic approximation of QED, in which case space and time have different scaling dimensions due to the lack of Lorentz symmetry in the effective theory. Still different counting rules apply to relativistic systems with one heavy component (such as atoms or B-mesons).

Before moving to specific examples, I would like to briefly discuss a general selection rule that goes under the name of $\hbar$ *counting*. To derive it, it is convenient to temporarily retrieve the Planck constant $\hbar$ in the action (usually set to $\hbar = 1$ in the particle physics literature). Then the action must have the dimension $\hbar^1$, because the path integrand contains $e^{iS/\hbar}$. Choosing the convention that no $\hbar$ factors multiply kinetic terms, each field with a quadratic kinetic term needs to have dimension $\hbar^{1/2}$. Then the coefficient of an interaction term with n fields needs to have dimension $\hbar^{1-n/2}$, independently of the number of derivatives. It follows that the parameters in the UV Lagrangian of Eq. (1.5) have the following $\hbar$ dimensions:

$$[m^2] = \hbar^0, \quad [\kappa] = \hbar^{-1/2}, \quad [\lambda] = \hbar^{-1}, \quad [c_{n,d}] = \hbar^{1-n/2}. \tag{1.8}$$

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The point of this exercise is that when $\mathcal{L}_{\text{EFT}}$ is derived from the underlying UV theory, its parameters are functions of the masses and couplings in $\mathcal{L}_{\text{UV}}$. If the typical mass scale of the heavy degrees of freedom in $\mathcal{L}_{\text{UV}}$ is M_*, the usual dimensional analysis leads to identifying $\Lambda \sim M_*$. The $\hbar$ counting allows one in addition to estimate the magnitude of *dimensionless* EFT parameters. Suppose the UV theory contains only one gauge coupling denoted as g_*. By the same rules, the gauge coupling has $[g_*] = \hbar^{-1/2}$. This leads to the estimate

$$(\text{tree}): \qquad c_{n,d} \sim g_*^{n-2}, \tag{1.9}$$

which predicts a pattern of all EFT couplings depending on just one fundamental parameter. This is not the whole story though. The loop expansion is also associated with powers of $\hbar$. One can prove that each loop correspond to another factor of $\hbar^1$. Thus, if the EFT couplings are effectively generated by one loop diagrams in the UV theory, the estimate should be modified as

$$(\text{one loop}): \qquad c_{n,d} \sim \frac{g_*^n}{(4\pi)^2}. \tag{1.10}$$

These simple heuristic rules are very useful, yet a warning is in order. Evidently, the UV theory may contain more than one coupling. Moreover, the symmetries in the UV theory may lead to additional selection rules. For example, the naive dimensional analysis and $\hbar$ counting lead to the estimate $m \sim g_*^0 \Lambda$ if the mass term is generated by tree level processes in the UV theory. In practice we always assume the hierarchy $m \ll \Lambda$, because otherwise there would not be any energy range where the EFT description is useful. The hierarchy can arise via fine-tuning, or because the mass term is generated at a loop level. But if the hierarchy is due to some symmetries in the UV theory, there will always be additional selection rules that need to be taken into account when estimating the magnitude of the EFT parameters. All in all, the dimensional analysis and the $\hbar$ counting in the EFT are always used in conjunction with additional assumptions about the dynamics of the UV theory.

1.3 Euler-Heisenberg Lagrangian

One illustrative example of an EFT is the Euler-Heisenberg Lagrangian [7].[2] Consider the low-energy effective theory for massless photons where all charged degrees of freedom have been integrated out. The lightest know charged particle is the electron, thus we expect such an EFT to be valid for photon energies $E \ll m_e \sim 0.5$ MeV. The effective Lagrangian should be Lorentz- and gauge-invariant, thus it should take the form

$$\mathcal{L}_{\text{EH}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{\Lambda^4} \mathcal{L}_{\text{EH}}^{d=8} + \frac{1}{\Lambda^8} \mathcal{L}_{\text{EH}}^{d=12} + \dots \tag{1.11}$$

Gauge invariance dictates that the photon field A_μ can enter only via the field strength tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ or its dual $\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma} \partial_\rho A_\sigma$. The first term in Eq. (1.11) is the usual kinetic term, which does not contain any interactions. The remaining terms describe higher-dimensional photon self-interactions, which arise after integrating out

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[2]Here, Euler is of course not the famous 18th century mathematician, but a PhD student and Luftwaffe pilot in Nazi Germany. Similarly, Heisenberg is a 20th century German physicist, and not the famous kingpin from Breaking Bad.

charged particles. As usual in a relativistic EFT, the interactions are organized according to their canonical dimensions, each consecutive term being suppressed by appropriate power of the cut-off scale $\Lambda \sim m_e$. For $E \ll \Lambda$ the power counting suggests that the interactions with the lowest canonical dimensions are the most relevant ones. At the dimension-6 level there is no possible gauge-invariant interaction due to the identity $F_{\mu\nu}F_{\nu\rho}F_{\rho\mu} = 0$ (more generally, all invariants with an odd number of $F_{\mu\nu}$ vanish). Thus, the leading interactions arise at the level of dimension-8 operators. For simplicity, I assume that parity is conserved in the EFT, so that the interactions do not depend $\tilde{F}$ (in principle, the Lagrangian may contain terms with an even number of $\tilde{F}$, but in that case they can always be traded for F). There are 2 independent parity-conserving dimension-8 operators , which I parametrize as[3]

$$\mathcal{L}_{\text{EH}}^{d=8} = a_1 \frac{\alpha^2}{16} (F_{\mu\nu}F_{\mu\nu})(F_{\rho\sigma}F_{\rho\sigma}) + a_2 \frac{\alpha^2}{16} (F_{\mu\rho}F_{\nu\rho})(F_{\mu\sigma}F_{\nu\sigma}), \qquad (1.12)$$

where $\alpha = \frac{e^2}{4\pi}$ and e is the electromagnetic coupling. When these effective interactions arise from integrating out charged particles, any of the 4 external photon legs in the vertex has to pick up at least one power of e. Since the coefficient of an F^4 operators has the $\hbar$ dimension -1, while $[e] = \hbar^{-1/2}$, the mismatch has to be balanced by a loop factor. For these reasons I factored out $\alpha^2 = e^4/16\pi^2$ from the Wilson coefficients, such that a_i are simply order one numerical coefficients. Any other dimension-8 terms can be expressed by the ones in Eq. (1.12). The Wilson coefficients a_1 and a_2 are arbitrary at this point. They can be fixed when the UV completion of this EFT is specified.

One practical effect of the interactions in Eq. (1.12) is to modify Maxwell's equations of electrodynamics. Due to the non-linearities in the field equations, a superposition of two solutions is not a solution anymore. In practice, however, the non-linear effects are extremely small in the low-energy regime, such that they have not been unambiguously observed to date.[4]

The effective Lagrangian in Eq. (1.11) can be used to calculate any scattering processes with low-energy photons. In particular, the dimension-8 interactions in Eq. (1.12) provide the tree-level contributions to the 2-to-2 scattering. It is straightforward (even if a bit tedious) to derive the Feynman rules and calculate the scattering amplitudes. I will write down the results as a function of the helicities $(+/-)$ of the incoming and outgoing photons. One can express the helicity amplitudes in terms of the kinematical invariants $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ as

$$\begin{aligned}
\mathcal{M}(\pm\pm; \pm\pm) &= \frac{\alpha^2}{8\Lambda^4}(4a_1 + 3a_2)\,s^2, \\
\mathcal{M}(\pm\mp; \pm\mp) &= \frac{\alpha^2}{8\Lambda^4}(4a_1 + 3a_2)\,u^2, \\
\mathcal{M}(\pm\mp; \mp\pm) &= \frac{\alpha^2}{8\Lambda^4}(4a_1 + 3a_2)\,t^2, \\
\mathcal{M}(\pm\pm; \mp\mp) &= \frac{\alpha^2}{8\Lambda^4}(4a_1 + a_2)\left(s^2 + t^2 + u^2\right).
\end{aligned} \qquad (1.13)$$

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[3]In the literature another parametrization is typically used: $\mathcal{L}_{\text{EH}}^{d=8} = c_1(F_{\mu\nu}F_{\mu\nu})(F_{\rho\sigma}F_{\rho\sigma}) + c_2(F_{\mu\nu}\tilde{F}_{\mu\nu})(F_{\rho\sigma}\tilde{F}_{\rho\sigma})$. This is equivalent to the one in Eq. (1.12) upon relating the Wilson coefficients as $a_1\alpha^2 = 16c_1 - 32c_2$, $a_2\alpha^2 = 64c_2$.

[4]Light-by-light scattering is routinely observed in colliders, however for photon energies $E \gg m_e$ where the EFT in Eq. (1.11) is no longer valid.

The remaining helicity amplitudes, not displayed above, are zero. The first 3 amplitudes depend on the same linear combination of the Wilson coefficients. That is because they are not independent, as they are related by the *crossing symmetry*. t-channel crossing corresponds to exchanging $s \leftrightarrow u$, and replacing the 1st helicity entry with the opposite of the 3rd and vice-versa. u-channel crossing corresponds to exchanging $s \leftrightarrow t$, and replacing the 1st helicity entry with the opposite of the 4th and vice-versa. The crossing symmetry must hold in any sensible quantum field theory, and one can easily see that the amplitudes in Eq. (1.13) are consistent with the t- and u-channel crossing. The last amplitude in Eq. (1.13) transform into itself under crossing symmetry.

The Euler-Heisenberg Lagrangian is an effective theory valid at energies up to the cut-off Λ. At that scale it is replaced by another theory containing some other particle of mass $\sim \Lambda$ coupled to the photon. The dimension-8 interactions in Eq. (1.12) approximate the effects of exchanging the new particle between photons at energy scales $E \ll \Lambda$. The coefficients a_1 and a_2 can then be *matched* by comparing the EFT results for the 2-to-2 scattering amplitude with that calculated in the UV theory with the new particle.

One example of the UV completion is the QED Lagrangian:

$$\mathcal{L}_{\text{UV}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + i\psi\gamma_\mu\partial_\mu\psi + eQ_e\psi\gamma_\mu\psi A_\mu, \tag{1.14}$$

where ψ is the electron field and $Q_e = -1$. In QED, 2-to-2 photon scattering proceeds through box diagrams with the electron in the loop, like the one in Fig. **??**. Calculating the helicity amplitudes from these diagrams, expanding the results in m_e^2, and keeping the leading $1/m_e^4$ term in this expansion, the QED amplitudes take the form of Eq. (**??**) with the following identification:

$$\frac{a_1^{\text{QED}}}{\Lambda^4} = -\frac{4Q_e^2}{9m_e^4}, \qquad \frac{a_2^{\text{QED}}}{\Lambda^4} = \frac{56Q_e^2}{45m_e^4}. \tag{1.15}$$

For other SM fermions the result is the same upon replacing $Q_e \to Q_f$, $m_e \to m_f$, and including the color factor where appropriate. Integrating out a scalar with the mass m_s and the electric charge Q_e leads to a different pattern of the Wilson coefficients (see e.g. [8]):

$$\frac{a_1^{\text{SQED}}}{\Lambda^4} = \frac{Q_s^2}{18m_s^4}, \qquad \frac{a_2^{\text{SQED}}}{\Lambda^4} = \frac{2Q_s^2}{45m_s^4}. \tag{1.16}$$

Finally, consider a UV completion which is itself an effective theory. The Lagrangian contains a real scalar h of mass m_h coupled to photons via dimension-5 interactions

$$\mathcal{L}_{\text{UV}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu h)^2 - \frac{m_h^2}{2}h^2 + b_h\frac{\alpha}{4\pi f}hF_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{1.17}$$

The coupling b_h is characteristic for axions, or more generally for Goldstone bosons of a global symmetry spontaneously broken at the scale f where the global symmetry current has a mixed triangle anomaly with the electromagnetic $U(1)$ currents. Integrating out h at tree level leads to the effective dimension-8 interactions in Eq. (1.12) with the Wilson coefficients

$$\frac{a_1^{\text{axion}}}{\Lambda^4} = -\frac{b_h^2}{\pi^2 f^2 m_h^2}, \qquad \frac{a_2^{\text{axion}}}{\Lambda^4} = \frac{2b_h^2}{\pi^2 f^2 m_h^2}. \tag{1.18}$$

Let us go back to the EFT. In theory, by measuring elastic scattering of low-energy polarized photons we could determine both Wilson coefficients a_1 and a_2. In practice, however, this does not seem feasible at present. Instead, the hope is to observe one day the so-called *vacuum birefrigence*, that is the rotation of the polarization vector of light traveling in vacuum in a strong magnetic field. Note that the measurement would me much more than an academic exercise. It is certain that the Wilson coefficients receive contributions in Eq. (1.15) from integrating out the electron, and subleading contributions from the remaining SM particles. However, for all we know other unknown particles could give larger or comparable contributions, especially if they are light. For example, light milli-charged particles or axions remain an interesting possibility. Therefore, a precision measurement of a_1 and a_2 would provide relevant constraints on hypothetical hidden sectors.

Up this point we treated the Wilson coefficients in Eq. (1.12) as free parameters. This is broadly correct, as in principle there are many consistent theories with charged particles that could serve as a UV completion and which would yield various patterns of a_1 and a_2. However, they are not completely unconstrained, even if we remain agnostic about the UV completion. There are two kinds of theoretical constraints:

- **Positivity**. Certain linear combinations of Wilson coefficients are bound to be positive under mild assumptions that the UV completion does not violate unitarity, causality, and the crossing symmetry.

- **Perturbativity**. The magnitude of Wilson coefficients is limited if the effective theory is to remain unitary and perturbative all the way to the cut-off.

Let us discuss these two in turn.

We start with positivity.[5] For this purpose, it is convenient to rewrite the amplitudes in Eq. (1.13) in the basis of linearly polarized photons: $|x\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$, $|y\rangle = (|+\rangle - |-\rangle)/\sqrt{2}i$,

$$
\begin{aligned}
\mathcal{M}(x/y, x/y; x/y, x/y) &= \frac{\alpha^2}{4\Lambda^4} \left(2a_1 + a_2\right) \left(s^2 + t^2 + u^2\right), \\
\mathcal{M}(x/y, x/y; y/x, y/x) &= \frac{\alpha^2}{8\Lambda^4} \left(4a_1 s^2 + a_2(2s^2 - u^2 - t^2)\right), \\
\mathcal{M}(x/y, y/x; x/y, y/x) &= -\frac{\alpha^2}{8\Lambda^4} \left(4a_1 t^2 + a_2(2t^2 - s^2 - u^2)\right), \\
\mathcal{M}(x/y, y/x; y/x, x/y) &= \frac{\alpha^2}{8\Lambda^4} \left(4a_1 u^2 + a_2(2u^2 - s^2 - t^2)\right),
\end{aligned}
\tag{1.19}
$$

I denote the forward amplitudes as $f_{\sigma_1\sigma_2}(s) = \mathcal{M}(\sigma_1\sigma_2 \to \sigma_1\sigma_2)|_{t=0,s=-u}$, where $\sigma_i = x, y$. One can derive the following dispersion relation for the forward amplitude [9, 10]

$$
\frac{1}{2!} \frac{df_{\sigma_1\sigma_2}}{ds^2}|_{s=0} = \int_0^\infty ds \frac{\text{Im} f_{\sigma_1\sigma_2}(s) + \text{Im} f_{\sigma_1\sigma_2}(u)}{\pi s^3}.
\tag{1.20}
$$

This equality holds in any theory with photons where amplitudes satisfy the postulates of causality (analyticity in the s plane), crossing symmetry, as well as the Froissart

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[5]Thanks to Brando Bellazzini for the help to work out these constraints.

bound [11] ($f(s) < c_\infty s \log^2 s$ for $s \to \infty$ and some constant c_∞). The last postulate does *not* hold the Euler-Heisenberg effective theory, as $f(s) \sim s^2$. Although Eq. (1.20) cannot be applied in this EFT, it should be valid in its UV completion provided the latter is well-behaved. On the other side, the left-hand side can be reliably calculated within the EFT, as it refers to the $s \to 0$ limit of the amplitudes.

We choose the spin state $\sigma_1 \sigma_2$ such that that spins are the same after t-channel crossing: $f_{\sigma_1 \sigma_2}(u) = f_{\sigma_1 \sigma_2}(s)$. Then unitarity (the optical theorem) allows us to trade the imaginary forward amplitude for the total elastic and inelastic scattering cross section from the given initial spin state: $\text{Im} f_{\sigma_1 \sigma_2}(s) = s \sigma_{\sigma_1 \sigma_2 \to \text{all}}(s)$. Thus, we rewrite Eq. (1.20) as

$$\frac{1}{2!}\frac{df_{\sigma_1 \sigma_2}}{ds^2}\Big|_{s=0} = 2 \int_0^\infty ds \frac{\sigma_{\sigma_1 \sigma_2 \to \text{all}}(s)}{\pi s^2}. \tag{1.21}$$

The right-hand side is unknown and depends on the UV completion of the EFT, however one thing we are sure is that it is manifestly positive. This in turn implies the positivity of the left-hand side, which can be adequately approximated in the EFT and expressed in terms of the Wilson coefficients a_1, a_2.

To obtain concrete bounds, we use the amplitudes in the linear polarization basis in Eq. (1.19). Under crossing the linear polarizations remain the same. Hence, the forward amplitudes for the spin states $|x/y, x/y\rangle$ and $|x/y, y/x\rangle$ are transformed into itself under t-channel crossing, as required in the derivation of Eq. (1.21). The dispersion relations for these states read

$$\begin{aligned}
(2a_1 + a_2)\frac{\alpha^2}{2\Lambda^4} &= 2\int_0^\infty ds \frac{\sigma_{x/y,x/y \to \text{all}}(s)}{\pi s^2}, \\
a_2 \frac{\alpha^2}{4\Lambda^4} &= 2\int_0^\infty ds \frac{\sigma_{x/y,y/x \to \text{all}}(s)}{\pi s^2}.
\end{aligned} \tag{1.22}$$

Since the right-hand sides are manifestly positive, we obtain the *positivity conditions* on the Wilson coefficients in the Euler-Heisenberg Lagrangian:

$$2a_1 + a_2 > 0, \qquad a_2 > 0. \tag{1.23}$$

These conditions have to be satisfied for any sensible UV completion of the Euler-Heisenberg Lagrangian. It is easy to verify it that the positivity conditions are satisfied in (scalar) QED examples in Eq. (1.15) and Eq. (1.16). For the axion in Eq. (1.18) one has $2a_1 + a_2 = 0$ at tree level, which means that another contribution must arise at a loop level to make this combination strictly positive.

The dispersion relations in Eq. (1.22) also provide a neat path to derive perturbativity bounds on the Wilson coefficients. Let us assume that the $2 \to 2$ cross section calculated in the EFT approximates well the one in the full UV complete theory all the way to the cut-off Λ, that is for $\sqrt{s} \lesssim \Lambda$. Cutting off the integral on the right-hand side we obtain

$$\begin{aligned}
2a_1 + a_2 &\gtrsim \frac{\alpha^2}{16\pi^2}\left(\frac{19a_1^2}{20} + \frac{13a_1 a_2}{15} + \frac{13a_2^2}{64}\right) \\
a_2 &\gtrsim \frac{\alpha^2}{16\pi^2}\left(\frac{a_1^2}{5} + \frac{a_1 a_2}{60} + \frac{3a_2^2}{40}\right)
\end{aligned} \tag{1.24}$$

These constrain a_1 and a_2 to a be inside a circle of finite radius, and they cannot be arbitrarily large in a consistent EFT (although they can be rather huge due to the smallness
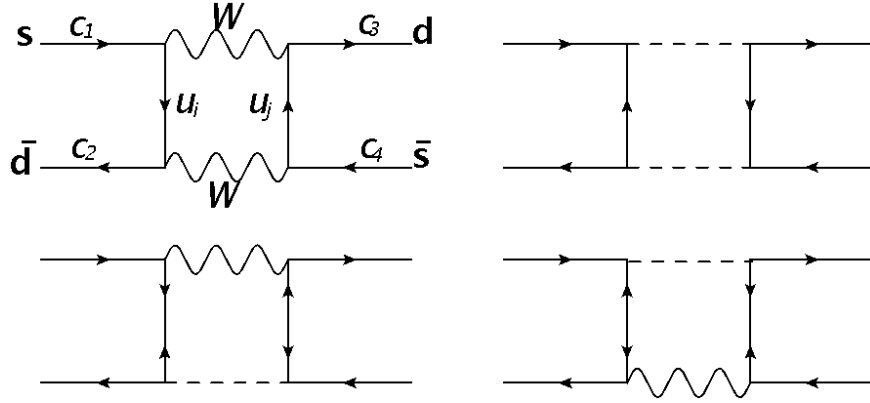
Figure 1: One-loop diagrams with W bosons and unphysical Goldstone bosons $G_\pm$ contributing to the process $s\bar{d} \to d\bar{s}$ in the SM. Only the diagram corresponding to the color structure $\delta_{c_1 c_2}\delta_{c_3 c_4}$ are displayed. Similar diagrams with horizontal quark lines give the other color structure $\delta_{c_1 c_3}\delta_{c_2 c_4}$.

of α). Turning this around, if the EFT is generated from a particular UV completion (such as QED), by power counting one expects $a_i \sim 1$. Then the perturbativity condition becomes a bound on α in the UV completion:

$$\alpha \lesssim 4\pi, \tag{1.25}$$

or equivalently $e \lesssim 4\pi$. This is the familiar perturbativity bound on the coupling strength in a gauge theory.

1.4 Effective Lagrangians and naturalness

In this example, I will illustrate how naturalness or lack thereof may manifest itself in the EFT beyond tree level. To this end, I will study the low-energy effective description of $\Delta S = 2$ transitions in the SM. I am more interested here in theoretical aspects rather than in the flavor phenomenology. Therefore, I will work in a simplified framework of the SM with only 2 generations of quarks. Quantitatively, this is a very bad approximation of reality, as the top quark contribution to these transitions is non-negligible. However, the clear hierarchy of scales, $m_q \ll m_W$ for the first two generations, makes the EFT discussion more straightforward and transparent.

Let us first consider the process $s\bar{d} \to d\bar{s}$ in the SM. It proceeds via the box diagrams in Fig. 2 with the W and physical Goldstone bosons exchanged between the fermion lines. The relevant SM interactions are

$$\mathcal{L}_{\mathrm{SM}} \supset \sum_{i,k=1,2} \left[\frac{g_L}{\sqrt{2}} V_{ik} \bar{u}_i \bar{\sigma}_\mu d_k W_\mu^+ + i \frac{\sqrt{2} m_{u_i}}{v} V_{ik} u_i^c d_k G_+ + \mathrm{h.c.} \right]. \tag{1.26}$$

I denote $u_1 \equiv u$, $u_2 \equiv c$, $d_1 \equiv d$, $d_2 \equiv s$. The unitary CKM matrix V in the 2-generation case becomes just a rotation by the Cabibbo angle θ_c:

$$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}. \tag{1.27}$$

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I will use the notation $V_{11} \equiv V_{ud}$, $V_{12} \equiv V_{us}$, etc. In the limit of vanishing external momenta, the amplitude takes the form

$$\mathcal{M}(s_A^{c_1}\bar{d}_{\dot{A}}^{c_2} \to d_B^{c_3}\bar{s}_{\dot{B}}^{c_4}) = \left(\delta^{c_1c_2}\delta^{c_3c_4}\bar{\sigma}_\mu^{\dot{A}A}\bar{\sigma}_\mu^{\dot{B}B} - \delta^{c_1c_3}\delta^{c_2c_4}\bar{\sigma}_\mu^{\dot{A}B}\bar{\sigma}_\mu^{\dot{B}A}\right)F, \qquad (1.28)$$

where $A, B, \dot{A}, \dot{B}$ are spinor indices of the external fermions, and $c_1 \ldots c_4$ are their color indices. The form factor F calculated in the SM can be written as

$$F_{\mathrm{SM}} = \sum_{ij\in uc} \xi_i \xi_j F_{ij}, \quad \xi_i \equiv V_{is}V_{id}^*. \qquad (1.29)$$

In the 2-generation case $\xi_u = -\xi_c = \sin\theta_c \cos\theta_c$. One finds the following expression for the function F_{ij} in the limit $m_{i,j} \ll m_W$:

$$F_{ij} = \frac{g_L^4}{64\pi^2} \times \begin{cases} -\frac{1}{m_W^2} + \frac{m_i^2}{m_W^4}\left[2\log\left(\frac{m_W^2}{m_i^2}\right) - 3\right] + \mathcal{O}(m_W^{-6}), & i = j, \\ -\frac{1}{m_W^2} + \frac{1}{m_W^4}\frac{m_i^4\log\left(\frac{m_W^2}{m_i^2}\right) - m_j^4\log\left(\frac{m_W^2}{m_j^2}\right) - m_i^4 + m_j^4}{m_i^2 - m_j^2} + \mathcal{O}(m_W^{-6}), & i \neq j. \end{cases}$$
$$(1.30)$$

Note that the first term of F does not contribute to the amplitude because $\sum_i \xi_i = 0$ due to the unitarity of the CKM matrix. Hence, the $\Delta S = 2$ amplitude is $\mathcal{O}(m_W^{-4})$ rather than $\mathcal{O}(m_W^{-2})$ as one might naively estimate via dimensional analysis. This is the Glashow-Iliopoulos-Maiani (GIM) mechanism [12]. In fact, for two generations in the limit $m_u \to 0$, $m_c > 0$ the GIM suppression is even stronger, as also the log-enhanced terms cancel. In that limit i find

$$F_{\mathrm{SM}} \to -\sin^2\theta_c \cos^2\theta_c \frac{g_L^4 m_c^2}{64\pi^2 m_W^4} = -\sin^2\theta_c \cos^2\theta_c \frac{m_c^2}{4\pi^2 v^4} \qquad (1.31)$$

I descend now to the EFT. I consider an effective theory with two generations of light quarks where the massive SM bosons (W, Z, and h) are integrated out. At the scales far below the weak scale the effective Lagrangian can be organized as an expansion in $1/m_W$. I will match this EFT to the SM (more precisely, to its simplified version without the top and bottom quarks), working up to $\mathcal{O}(m_W^{-4})$ and up to one-loop precision.

In the EFT, the $s\bar{d} \to d\bar{s}$ transition can be mediated at tree level by the $\Delta S = 2$ flavor-violating dimension-6 four-fermion operator:

$$\mathcal{L}_{\mathrm{EFT}_4} \supset \frac{C_{ss}}{m_W^2}(\bar{d}\bar{\sigma}_\mu s)(\bar{d}\bar{\sigma}_\mu s) + \mathrm{h.c.}, \qquad (1.32)$$

where C_{ss} is a dimensionless Wilson coefficient. We know that $\Delta S = 2$ processes arise only at 1-loop level in the UV theory and, due to the GIM mechanism, they vanish at $\mathcal{O}(m_W^{-2})$. Hence $C_{ss} \sim \frac{m_q^2}{16\pi^2 m_W^2}$. At this order, to properly match C_{ss} to the UV theory, one also needs to take into account the 1-loop contribution from the $\Delta S = 1$ flavor-violating dimension-6 four-fermion operators in the EFT:

$$\mathcal{L}_{\mathrm{EFT}_4} \supset \sum_{i,j=1,2} \frac{C_s V_{is}V_{jd}^*}{m_W^2}(\bar{u}_i\bar{\sigma}_\mu s)(\bar{d}\bar{\sigma}_\mu u_j) + \mathrm{h.c.}. \qquad (1.33)$$
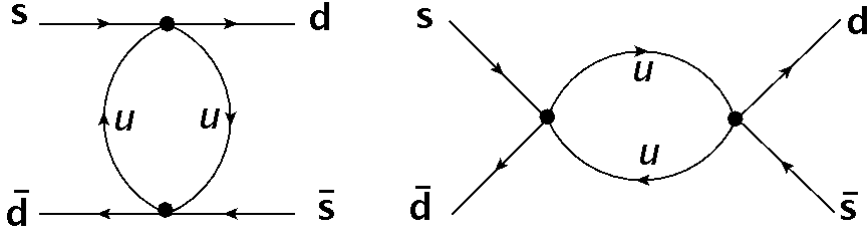
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Figure 2: One-loop diagrams with contributing to the process $s\bar{d} \to d\bar{s}$ in the EFT where the W boson is integrated out. The first diagram corresponds to the color structure $\delta_{c_1c_2}\delta_{c_3c_4}$, and the other to $\delta_{c_1c_3}\delta_{c_2c_4}$.

Integrating out the W-boson at tree level fixes the $\Delta S = 1$ Wilson coefficients as

$$C_s = -\frac{g_L^2}{2} + \mathcal{O}(m_W^{-2}) + \mathcal{O}(1/16\pi^2). \tag{1.34}$$

For our purpose of matching C_{ss} at the next-to-leading order, the leading order expression for C_s is enough because the latter's contribution to $\Delta S = 2$ processes scales as $C_s^2/m_W^4/16\pi^2$.

As in the SM case, we can factor out the color and spin indices and parametrize the amplitude as in Eq. (1.28). The form factor F calculated in the EFT takes the form

$$F_{\text{EFT}} = \frac{C_{ss}}{m_W^2} + \frac{C_s^2}{m_W^4} \sum_{ij \in u,c} \xi_i \xi_j I_{ij} \quad, \tag{1.35}$$

where the loop integral I_{ij} is given by

$$I_{ij} = -i\frac{4}{d} \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 - m_{u_i}^2)(k^2 - m_{u_j}^2)}. \tag{1.36}$$

This integral is UV divergent. Evaluating it in the dimensional regularization (where the quadratic divergence is not visible) I get

$$I_{ij} = \frac{1}{8\pi^2} \times \begin{cases} m_i^2 \left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m_i^2}\right) + 1\right], & i = j, \\ (m_i^2 + m_j^2)\left(\frac{1}{2\epsilon} + \frac{3}{4}\right) + \frac{m_i^4 \log\left(\frac{\mu^2}{m_i^2}\right) - m_j^4 \log\left(\frac{\mu^2}{m_j^2}\right)}{2(m_i^2 - m_j^2)}, & i \neq j. \end{cases} \tag{1.37}$$

Moreover, evaluating the diagram with the cut-off one finds $I_{ij} = -\frac{\Lambda^2}{16\pi^2} + \dots$. However, once we sum over the up-quarks in the loop, the GIM mechanism is again at play, and both quadratic and logarithmic divergences cancel thanks to $\sum \xi_i = 0$. In the limit $m_u = 0$, $m_c > 0$ I get

$$\sum_{ij \in u,c} \xi_i \xi_j I_{ij} = -\frac{m_c^2}{16\pi^2} \sin^2 \theta_c \cos^2 \theta_c. \tag{1.38}$$

Matching to the SM, we require that the amplitudes $s\bar{d} \to d\bar{s}$ calculated in the EFT and in the SM are the same up to order $\frac{m_c^2}{16\pi^2 m_W^4}$. Comparing Eq. (1.35) and Eq. (1.29), we

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require

$$\sum_{ij \in u,c} \xi_i \xi_j F_{ij} = \frac{C_{ss}}{m_W^2} + \frac{C_s^2}{m_W^4} \sum_{ij} \xi_i \xi_j I_{ij} + \mathcal{O}\left(\frac{1}{16\pi^2 m_W^6}\right). \tag{1.39}$$

Given Eq. (1.31) and Eq. (1.38), this is satisfied for $C_{ss} = 0$. Thus, the $\Delta S = 2$ four-fermion operator is absent in the EFT Lagrangian also at order m_W^{-4}, while the proper description of $\Delta S = 2$ processes at that order is assured by loops involving a pair $\mathcal{O}(m_W^{-2})$ $\Delta S = 1$ vertices present in that EFT.

We can move one more step down the EFT ladder. What happens when the charm quark is also integrated out? Consider an effective theory with 3 massless quarks s, d, u and the interactions

$$\mathcal{L}_{\text{EFT}_3} \supset \frac{C'_{ss}}{m_W^2}(\bar{d}\bar{\sigma}_\mu s)(\bar{d}\bar{\sigma}_\mu s) + \frac{C_s}{m_W^2}(\bar{u}\bar{\sigma}_\mu s)(\bar{d}\bar{\sigma}_\mu u) + \text{h.c.}. \tag{1.40}$$

In such a setup, the form factor for the $s\bar{d} \to d\bar{s}$ amplitude is given by

$$F_{\text{EFT}'} = \frac{C'_{ss}}{m_W^2} + \frac{C_s^2}{m_W^4}\xi_u^2 \frac{(-4i)}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} = \frac{C'_{ss}}{m_W^2} - \frac{g_L^4 \sin^2\theta_c \cos^2\Lambda^2}{64\pi^2 m_W^4}. \tag{1.41}$$

The integral depends quadratically on the cut-off Λ of the 3-quark EFT, and now there is no GIM mechanism to come to rescue. Therefore the Wilson coefficient C'_{ss} cannot vanish in the 3-quark EFT, as it is needed as a counterterm to cancel divergences. Matching to the SM (or the 4-quark EFT) result we have

$$C'_{ss} = \frac{g_L^4 \sin^2\theta_c \cos^2\theta_c}{64\pi^2 m_W^2}\left(\Lambda^2 - m_c^2\right). \tag{1.42}$$

We could have used the dimensional regularization instead, in which case no infinite counterterm would be needed. However, doing so in this particular case we would miss an interesting hint. In fact, the leading order result in the 4-quark EFT can be obtained by replacing $\Lambda \to m_c$ in the 3-quark EFT calculation. Quadratic divergences do have a physical meaning, sometimes.

Back in the 60s, before the charm quark discovery, the 3-quark EFT was the default description of flavor transitions. In this context, the quadratic divergence suggested that $\Delta S = 2$ transitions receive dominant contributions from UV physics that scale at least as the cutoff squared. At the same time, an upper bound on the sum of all existing and new contributions was known from experimental observations of the neutral kaon mass difference. *Naturalness* then suggested that Λ of that EFT should not be large, and that above Λ the EFT is replaced by UV completion where the quadratic sensitivity is cured. This expectation was qualitatively borne out with the discovery of the charm quark with $m_c \sim 1$ GeV and interactions consistent the GIM mechanism. At the quantitative level, the full story is a bit more complicated as the top quark contributions to $\Delta S = 2$ transitions cannot be neglected in this discussion (they are of the same order as those of the charm quark). Nevertheless, predicting the charm quark in Ref. [12] can be considered a successful application of the naturalness principle in high-energy physics. Until today it remains the only successful prediction (as opposed to several postdictions) based on the naturalness principle...

2 Effective Toy Story

In this lecture I will work out the low-energy EFT for a theory of two reals scalars, one light and one heavy, interacting via cubic and quartic couplings in the Lagrangian. The goal is to illustrate, in a simple and calculable example, some central concepts pervading EFT calculations, such as the tree- and loop-level matching, the on-shell vs off-shell matching, the basis independence, and the resummation of large logarithms. The discussion is inspired by the one in the TASI lectures of Ira Rothstein [2], although my approach is different in several important details. In particular, I match the EFT and the full theory on-shell rather than off-shell, and I get rid of redundant operators before matching.

2.1 Settings

Consider a light real scalar field ϕ with mass m_L and a heavy one H with mass M. The Lagrangian of the UV theory is

$$
\begin{aligned}
\mathcal{L}_{\text{UV}} &= \frac{1}{2}\left[(\partial_\mu \phi)^2 - m_L^2 \phi^2 + (\partial_\mu H)^2 - M^2 H^2\right] \\
&\quad - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H - \frac{\lambda_2}{4}\phi^2 H^2.
\end{aligned} \tag{2.1}
$$

Note that I factored out the heavy mass scale M in the dimensionful coefficient of the trilinear term, which will affect the power counting below. I'm imposing the $\mathbb{Z}_2$ symmetry $\phi \to -\phi$, thus odd powers of ϕ do not appear in the Lagrangian. The H^3 and H^4 interactions are irrelevant for this discussion, and for simplicity I'm assuming they are absent in the Lagrangian (even if this assumption is not stable, as loop corrections will generate H^3 and H^4 counterterms).

We want to derive the EFT valid at $E \ll M$ where H is integrated out. The EFT Lagrangian has to be of the form

$$
\mathcal{L}_{\text{EFT}} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2 \phi^2\right] - C_4 \frac{\phi^4}{4!} - \frac{C_6}{M^2}\frac{\phi^6}{6!} + \mathcal{O}(M^{-4}). \tag{2.2}
$$

The interaction terms are organized as the expansion in inverse powers of M, $\sum_d \frac{C_d}{M^{d-4}}O_d$, where each operator O_d has the canonical dimension d. In this simple EFT there is only one non-redundant operator at the dimension-4 level, and only one at the dimension-6 level. Operators with an odd number of ϕ's do not appear because of the $\mathbb{Z}_2$ symmetry $\phi \to -\phi$ of the UV Lagrangian in Eq. (2.1), which is inherited by the low-energy theory. I will not trace here the operators with $d > 6$, although it is easy to complicate the analysis and truncate the Lagrangian at some higher d.

Note that one could write other possible operators at $\mathcal{O}(M^{-2})$, e.g.

$$
\hat{O}_6 \equiv (\Box\phi)^2, \quad \tilde{O}_6 \equiv \phi\Box\phi^3, \quad \tilde{O}_6' \equiv \phi^2\Box\phi^2, \quad \tilde{O}_6'' \equiv \phi^2\partial_\mu\phi\partial_\mu\phi, \quad \dots \tag{2.3}
$$

It turns out that the operators in Eq. (2.3) are *redundant*, that is to say, adding them to the Lagrangian in Eq. (2.2) does not change the physical content of the theory. First, $\tilde{O}_6''$ and $\tilde{O}_6'$ can be traded for $\tilde{O}_6$ via integration by parts: $\phi^2(\partial_\mu\phi)^2 = -\frac{1}{3}\phi^3\Box\phi$, $\phi^2\Box\phi^2 = \frac{4}{3}\phi^3\Box\phi$. On the other hand, $\hat{O}_6$ and $\tilde{O}_6$ can be eliminated in favor of the

interaction term present in Eq. (2.2) by using the classical equations of motion. It was proven in Ref. [13] that shifting the higher-dimensional operators by a term proportional to the classical equations of motion does not change the S-matrix elements, even at the loop level. The point is that trading one interaction term for another using the equations of motion is the same as redefining the fields in the Lagrangian in a non-linear way. For example, going from the unbox basis to the box basis corresponds to the redefinition $\phi \to \phi \left(1 - \frac{C_6}{120C_4M^2}\phi^2\right)$. It is rather intuitive that the manner in which you define your fields should not affect the physical content of the theory. Independence of the S-matrix on field redefinitions is the consequence of the *equivalence theorem* [14, 15].

In our case, the equation of motion for ϕ read

$$\Box\phi + m^2\phi + \frac{C_4}{6}\phi^3 = \mathcal{O}(M^{-2}). \tag{2.4}$$

For our purpose, we don't need to write down the $\mathcal{O}(M^{-2})$ piece explicitly as it is relevant only for manipulating $\mathcal{O}(M^{-4})$ terms in the Lagrangian. Using the equation of motion we find, for example, the following operator equation:

$$\frac{1}{M^2}\phi^3\Box\phi = -\frac{m^2}{M^2}\phi^4 - \frac{C_4}{6M^2}\phi^6 + \mathcal{O}(M^{-4}). \tag{2.5}$$

This means that $\tilde{O}_6$ has the same effect on on-shell amplitudes as a particular linear combination of the terms already present in Eq. (2.2). Since the coefficients in Eq. (2.2) are free parameters at this point, $\tilde{O}_6$ can be left out without any loss of generality. Conversely, one can use Eq. (2.5) to trade O_6 for $\tilde{O}_6$, leading to the Lagrangian

$$\mathcal{L}_{\text{EFT}} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - \tilde{C}_4\frac{\phi^4}{4!} - \frac{\tilde{C}_6}{4!M^2}\phi^3\Box\phi + \mathcal{O}(M^{-4}). \tag{2.6}$$

In my jargon, the Lagrangian in Eq. (2.2) is written in the *unbox basis*, and the one in Eq. (2.6) is written in the *box basis*.[6] At any order in perturbation theory, both Lagrangians give equivalent predictions for all on-shell scattering amplitudes up to $\mathcal{O}(M^{-2})$ terms. The two sets of predictions are related by the map

$$\tilde{C}_4 = C_4 - \frac{m^2}{5M^2}\frac{C_6}{C_4},$$
$$\tilde{C}_6 = -\frac{C_6}{5C_4}. \tag{2.7}$$

Exercise: Express the operator $\hat{O}_6$ by the ones present in the Lagrangian of Eq. (2.2). Write down the map between the double-box basis, and the unbox basis.

Before moving on let us write the $\hbar$ dimensions of the fields and couplings. Demanding that the action has $\hbar^1$ dimension, and choosing the convention that no $\hbar$ factors multiply kinetic terms, each field needs to have dimension $\hbar^{1/2}$. Then the coefficient of

---

[6]In this toy model the basis of dimension-6 operators is one-dimensional, but that is just because I picked a particularly simple example to introduce the concept. The simplicity should be appreciated, especially when compared to the 2499-dimensional basis of dimension-6 operators in the SM EFT.
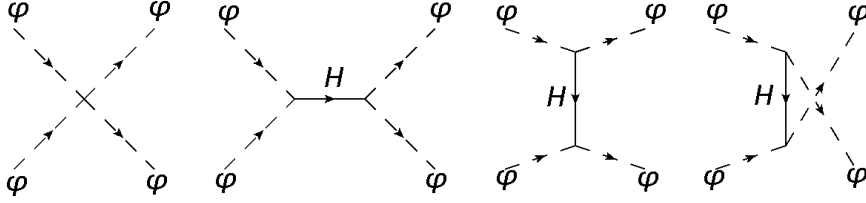
Figure 3: Tree-level Feynman diagrams contributing to the $\phi\phi \to \phi\phi$ scattering amplitude in the toy model described by the Lagrangian Eq. (2.1).

an interaction term with n fields needs to have dimension $\hbar^{1-n/2}$. It follows that the couplings in the UV Lagrangian have the following $\hbar$ dimensions:

$$[\lambda_0] = \hbar^{-1}, \qquad [\lambda_1] = \hbar^{-1/2}, \qquad [\lambda_2] = \hbar^{-1}, \tag{2.8}$$

while the EFT couplings have the $\hbar$ dimensions:

$$[\hat{C}_6] = \hbar^0, \qquad [C_4] = \hbar^{-1}, \qquad [\tilde{C}_6] = \hbar^{-1}, \qquad [C_6] = \hbar^{-2}. \tag{2.9}$$

Matching the UV theory and the EFT consists fixing the EFT Wilson coefficients such that all on-shell scattering amplitudes of ϕ are the same when calculated in either framework. In practice, the above condition is imposed up to a fixed order in the $1/M$ expansion and the in the loop expansion. In the following I will perform the matching between the EFT Wilson coefficients C_i and the UV parameters, working up to $1/M^2$ order and up to the 1-loop precision.

2.2 Tree-level matching

I start with the tree-level matching. At this order, the condition for the ϕ propagator to be the same in the UV theory and the EFT is trivial:

$$m^2 = m_L^2. \tag{2.10}$$

Let us move to the on-shell scattering amplitudes of ϕ, starting with the 2-to-2 scattering. In the UV theory this process receives a contribution from the contact ϕ^4 interaction, and also from the s-, t-, and u-channel exchange of H, see Fig. 3. The resulting amplitude is given by

$$\begin{aligned}
\mathcal{M}_4^{\text{UV}} &= -\lambda_0 - \lambda_1^2 M^2 \left[\frac{1}{s - M^2} + \frac{1}{t - M^2} + \frac{1}{u - M^2} \right] \\
&\approx -\lambda_0 + 3\lambda_1^2 + \frac{\lambda_1^2}{M^2}(s + t + u) + \mathcal{O}(M^{-4}) \\
&\approx -\lambda_0 + 3\lambda_1^2 + \frac{4m_L^2\lambda_1^2}{M^2} + \mathcal{O}(M^{-4})
\end{aligned} \tag{2.11}$$

where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ are the Mandelstam variables, and in the last step we used $s + t + u = 4m_L^2$.

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The analogous calculation in the EFT is simpler, as only the leftmost diagram in Fig. 3 contributes. In the unbox basis this yields

$$\mathcal{M}_4^{\text{EFT}} = -C_4. \tag{2.12}$$

The matching between the EFT and the UV theory consists in imposing the requirement $\mathcal{M}_4^{\text{EFT}} = \mathcal{M}_4^{\text{UV}} + \mathcal{O}(M^{-2})$. The matching condition thus reads

$$C_4 = \lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}. \tag{2.13}$$

We can equally well perform the matching in the box basis. Here the EFT calculation is a tad less trivial due to the presence of the derivative 4-point vertex. With the external momenta on-shell I get

$$\mathcal{M}_4^{\text{EFT}} = -\tilde{C}_4 + \frac{m^2}{M^2}\tilde{C}_6, \tag{2.14}$$

and the matching condition becomes

$$\tilde{C}_4 - \frac{m^2}{M^2}\tilde{C}_6 = \lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}. \tag{2.15}$$

This is consistent with Eq. (2.13) after using the map in Eq. (2.7).

In order to match the Wilson coefficient C_6 in the unbox basis one needs to calculate the 6-point function on the UV and EFT sides. This is a lengthy and complicated exercise, so I will only quote the final result here. In Section 3 I will derive this result using much more powerful methods of path integrals. All in all, the tree-level matching conditions in the unbox basis up to $\mathcal{O}(M^{-2})$ read

$$\begin{aligned} m^2 &= m_L^2, \\ C_4 &= \lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}, \\ C_6 &= 45\lambda_1^2\lambda_2 - 20\lambda_0\lambda_1^2 + 60\lambda_1^4. \end{aligned} \tag{2.16}$$

Using Eq. (2.7) one also finds the matching condition in the box basis:

$$\begin{aligned} m^2 &= m_L^2, \\ \tilde{C}_4 &= \lambda_0 - 3\lambda_1^2 - \frac{9m_L^2}{M^2}\frac{\lambda_1^2\lambda_2}{\lambda_0 - 3\lambda_1^2}, \\ \tilde{C}_6 &= 4\lambda_1^2 - 9\frac{\lambda_1^2\lambda_2}{\lambda_0 - 3\lambda_1^2}. \end{aligned} \tag{2.17}$$

2.3   One-loop matching of 2-point function

We start with matching the 1PI 2-point function $\Pi(p^2)$. At tree level we have $\Pi_0^{\text{EFT}} = p^2 - m^2$, and $\Pi_0^{\text{UV}} = p^2 - m_L^2$. This will be corrected by 1-loop diagrams.[7] On the EFT

---

[7] Loop integrals have been calculated with the help of the `Package-X` code [16].
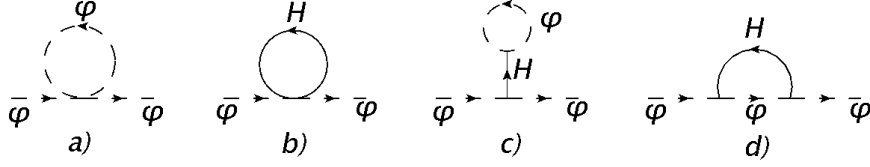
Figure 4: One-loop Feynman diagrams contributing the two-point function of ϕ in the toy model described by the Lagrangian Eq. (2.1).

side, there is just one diagram marked as a) in Fig. 4. Working in the unbox basis and evaluating the diagram in dimensional regularization I get

$$
\begin{aligned}
\delta\Pi^{\mathrm{EFT}} &= (-i)\frac{-iC_4}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} \\
&= C_4\frac{m^2}{32\pi^2}\left[\frac{1}{\bar{\epsilon}}+\log\left(\frac{\mu^2}{m^2}\right)+1\right].
\end{aligned}
\tag{2.18}
$$

where $1/\bar{\epsilon}=1/\epsilon+\gamma_E+\log(4\pi)$, and μ is a dimensionful parameter introduced through the dimensional regularization prescription. The physical mass m^2_{phys} is defined as the pole of $\Pi(p^2)$. For renormalization, I will always use the $\overline{\mathrm{MS}}$ scheme, which consists in simply dropping the $1/\bar{\epsilon}$ pole in all amplitudes. The parameter μ can be identified with the renormalization scale, as explained in detailin the following. Then the physical mass at one loop is related to the Lagrangian parameters by

$$
m^2_{\mathrm{phys}} = m^2 - C_4\frac{m^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right)+1\right].
\tag{2.19}
$$

The physical mass on the left-hand side is an observable, thus it should not depend on the arbitrary scale μ. The answer is that, at loop level, the mass parameter in the Lagrangian should be promoted to a scale-dependent object, $m^2(\mu)$. To ensure it, the parameter m^2 at one loop order must satisfy the RG equation

$$
\frac{dm^2}{d\log\mu} = C_4\frac{m^2}{16\pi^2},
\tag{2.20}
$$

such that the right-hand side is also μ-independent at one loop.

In the box basis of the EFT the 2-point function takes the form:

$$
\begin{aligned}
\delta\tilde{\Pi}^{\mathrm{EFT}} &= -\frac{i\tilde{C}_4}{2!}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}+(-i)\frac{i\tilde{C}_6}{2!4M^2}\int\frac{d^dk}{(2\pi)^d}\frac{2i(k^2+m^2)}{k^2-m^2} \\
&= \tilde{C}_4\frac{m^2}{32\pi^2}\left[\frac{1}{\bar{\epsilon}}+\log\left(\frac{\mu^2}{m^2}\right)+1\right]-\tilde{C}_6\frac{p^2+m^2}{64\pi^2}\frac{m^2}{M^2}\left[\frac{1}{\bar{\epsilon}}+\log\left(\frac{\mu^2}{m^2}\right)+1\right].
\end{aligned}
\tag{2.21}
$$

It is important to notice that $\delta\Pi^{\mathrm{EFT}}$ and $\delta\tilde{\Pi}^{\mathrm{EFT}}$ coincide on-shell but are *not* the same off-shell! In particular, the dependence on p^2 is completely different. It follows that the

anomalous dimensions are basis-dependent: at one loop the wave-function renormalization in the box basis is non-trivial,

$$\tilde{\delta}_\phi \equiv \frac{d\delta\tilde{\Pi}^{\text{EFT}}}{dp^2}\Big|_{p^2=m_{\text{phys}}^2} = -\frac{\tilde{C}_6}{64\pi^2}\frac{m^2}{M^2}\left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 1\right], \quad (2.22)$$

whereas $\delta_\phi = 0$ in the unbox basis. In general, the equivalence theory only ensure the equality of on-shell S-matrix elements, while off-shell correlation functions are basis-dependent. However, physical quantities must be the same, irrespectively of the basis in which they are calculated. The physical mass (the solution to $p^2 - m^2 + \delta\tilde{\Pi}^{\text{EFT}} = 0$) in the box basis at one loop is given by

$$m_{\text{phys}}^2 = m^2 - \frac{m^2}{32\pi^2}\left(\tilde{C}_4 - \frac{m^2}{M^2}\tilde{C}_6\right)\left[\log\left(\frac{\mu^2}{m^2}\right) + 1\right], \quad (2.23)$$

which is the same as in Eq. (2.19) after mapping the parameters as in Eq. (2.7). By the same token, the RG equation in the box basis matches that in Eq. (2.20).

We move to the UV side. First, there is again the diagram a) in Fig. 4, but now the mass parameter and the quartic coupling of ϕ are different:

$$\mathcal{M}_2^{\text{UV,a)}} = \lambda_0\frac{m_L^2}{32\pi^2}\left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m_L^2}\right) + 1\right]. \quad (2.24)$$

Next, there's the analogous diagram b) with the heavy scalar in the loop.

$$\mathcal{M}_2^{\text{UV,b)}} = \lambda_2\frac{M^2}{32\pi^2}\left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right) + 1\right]. \quad (2.25)$$

Next, there's the diagram c) containing a tadpole:

$$\begin{aligned}
\mathcal{M}_2^{\text{UV,c)}} &= (-i)(-i\lambda_1 M)^2\frac{1}{0^2 - M^2}\int\frac{d^d k}{(2\pi)^d}\frac{i}{k^2 - m^2} \\
&= -\lambda_1^2\frac{m_L^2}{32\pi^2}\left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m_L^2}\right) + 1\right].
\end{aligned} \quad (2.26)$$

Finally, the mixed heavy-light loop expanded to $\mathcal{O}(M^{-2})$ and evaluated at $p^2 = m_L^2$ reads

$$\begin{aligned}
\mathcal{M}_2^{\text{UV,d)}} &= (-i)(-i\lambda_1 M)^2\int\frac{d^d k}{(2\pi)^d}\frac{i^2}{(k^2 - M^2)((k+p)^2 - m_L^2)} \\
&\rightarrow \lambda_1^2\frac{M^2}{16\pi^2}\left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right) + 1\right] + \lambda_1^2\frac{m_L^2}{32\pi^2}\left[-2\log\left(\frac{M^2}{m_L^2}\right) + 1\right] \\
&+ \lambda_1^2\frac{m_L^4}{48\pi^2 M^2}\left[-6\log\left(\frac{M^2}{m_L^2}\right) + 5\right].
\end{aligned} \quad (2.27)$$

All in all, the physical mass of the light scalar in the UV theory at one loop in the $\overline{\text{MS}}$ scheme is given by

$$\begin{aligned}
m_{\text{phys}}^2 &= m_L^2 - \left(\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2\frac{m_L^2}{M^2}\right)\frac{m_L^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m_L^2}\right) + 1\right] \\
&- \frac{1}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 2\lambda_1^2 m_L^2 + 4\lambda_1^2\frac{m_L^4}{M^2}\right] \\
&- \frac{1}{32\pi^2}\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right].
\end{aligned} \quad (2.28)$$

Equating Eq. (2.19) and Eq. (2.28), one finds the matching equation between the $\overline{\text{MS}}$ mass parameters in the UV and EFT Lagrangians:

$$
\begin{aligned}
m^2(\mu) &= m_L^2(\mu) - \frac{1}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 2\lambda_1^2 m_L^2 + 4\lambda_1^2\frac{m_L^4}{M^2}\right] \\
&- \frac{1}{32\pi^2}\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right].
\end{aligned} \tag{2.29}
$$

Note that $\hbar$ counting still works at the loop-level. To see this, one should take into account that, when $\hbar$ is retrieved in the action, the loop expansion parameter is really $\frac{\hbar}{16\pi^2}$.

One can see that the $\log(\mu^2/m_L^2)$ cancel out in the matching equation. Choosing $\mu = M$, the matching condition further simplifies to

$$
m^2(M) = m_L^2(M) - \frac{1}{32\pi^2}\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]. \tag{2.30}
$$

Thanks to the absence of $\log(\mu^2/m_L^2)$ in the matching equation, choosing the high matching scale $\mu \sim M$ ensures that there are no large logarithms in the matching equation. Otherwise, these large logs could invalidate the perturbative expansion if $\lambda_1^2 \log(\mu^2/M^2) \gg 1$, or $\lambda_2 \log(\mu^2/M^2) \gg 1$.

Furthermore, Eq. (2.30) shows the modern way to formulate the UV sensitivity of scalar masses. In the low-energy theory for the scalar ϕ, the loop corrections to m^2 are proportional to m^2 as long as we use the dimensional regularization. On the other hand, one would find large quadratically divergent corrections to m^2 in the EFT, $\delta m^2 \sim \Lambda^2/(16\pi^2)$, if a mass-dependent regulator (e.g. the momentum cut-off) were used. This is often confusing: the question whether of there is the fine-tuning problem seems to depend on the regularization procedure. Viewing the low-energy theory as an EFT that is matched to some up completion at the high scale M allows one to avoid this conundrum and formulate the fine-tuning problem in the regularization-independent manner. What Eq. (2.30) tells us is that the natural value of the parameter m^2 in the EFT is $m^2 \sim M^2$ (more precisely $m^2 \sim M^2/(16\pi^2)$), which is also what one would expect from the simple scaling arguments. In order to arrive at a small $m \ll M/(4\pi)$ (and thus $m_{\text{phys}} \ll M$) in the EFT one needs to choose $m_L^2 \sim M^2/16\pi^2 \gg m$ in the UV theory, and carefully fine-tune the cancellation.

2.4 One-loop matching of 4-point function

For the 4-point function, the number of 1-loop diagrams in the UV model explodes, so I will restrict the discussion to the case $\lambda_1 = 0$, which greatly simplifies the calculation. Note that in this limit the tree-level matching conditions becomes simply $C_4 = \lambda_0$, $C_6 = 0$.

For the EFT calculation, the relevant diagrams are drawn in the top row of Fig. 5. In the unbox basis I find the following result for the 2-to-2 scattering up to 1-loop precision:

$$
\begin{aligned}
\mathcal{M}_4^{\text{EFT}} &= -C_4 + \frac{C_4^2}{32\pi^2}\left[f(s,m) + f(t,m) + f(u,m)\right] \\
&+ \frac{3C_4^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right) + \frac{C_6 m^2}{32\pi^2 M^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 1\right), \tag{2.31}
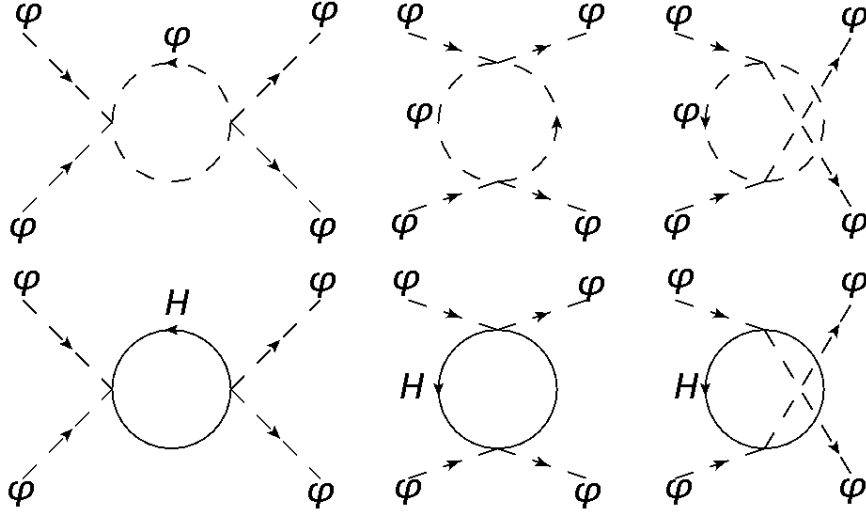\end{aligned}
$$

Figure 5: One-loop Feynman diagrams contributing to the $\phi\phi \to \phi\phi$ scattering amplitude in the toy model described by the Lagrangian Eq. (2.1) in the limit $\lambda_1 = 0$.

where

$$f(s,m) \equiv \sqrt{1 - \frac{4m^2}{s}} \log\left(\frac{2m^2 - s + \sqrt{s(s-m^2)}}{2m^2}\right). \tag{2.32}$$

The S-matrix element is

$$S_4^{\text{EFT}} \equiv \frac{\mathcal{M}_4^{\text{EFT}}(\overline{\text{MS}})}{(1+\delta_\phi)^2}\Big|_{p_i^2 = m_{\text{phys}}^2} \tag{2.33}$$

from which the observable differential cross-section can be calculated. Here δ_ϕ is the wave-function renormalization which however vanishes at one loop in the unbox basis, and $\overline{\text{MS}}$ prescription consists in dropping the $1/\bar{\epsilon}$ poles in the amplitude. Demanding that S_4^{EFT} is renormalization-scale independent, one obtains the RG equation for the Wilson coefficient C_4,

$$\frac{dC_4}{d\log\mu} = \frac{3}{16\pi^2}C_4^2 + \frac{m^2}{16\pi^2 M^2}C_6. \tag{2.34}$$

It is instructive to repeat the same calculation in the box basis. Off-shell, the 2-to-2 amplitude reads

$$\begin{aligned}
\tilde{\mathcal{M}}_4^{\text{EFT}} &= -\tilde{C}_4 + \frac{\sum_{i=1}^4 p_i^2}{4M^2}\tilde{C}_6 + \frac{\tilde{C}_4^2 - \frac{2m^2}{M^2}\tilde{C}_4\tilde{C}_6}{32\pi^2}[f(s,m) + f(t,m) + f(u,m)] \\
&+ \frac{3\tilde{C}_4^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right) - \frac{3\tilde{C}_4\tilde{C}_6 m^2}{16\pi^2 M^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + \frac{3}{2}\right) \\
&- \frac{3\tilde{C}_4\tilde{C}_6(\sum_{i=1}^4 p_i^2)}{128\pi^2 M^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right).
\end{aligned} \tag{2.35}$$

Much as for the two point function, the on-shell matrix elements $\tilde{\mathcal{M}}_4^{\text{EFT}}$ and $\mathcal{M}_4^{\text{EFT}}$ are different, and the two have a different momentum dependence. However, the S-matrix

element calculated in the box basis,

$$S_4^{\text{EFT}} = \frac{\tilde{\mathcal{M}}_4^{\text{EFT}}(\overline{\text{MS}})}{(1 + \tilde{\delta}_\phi)^2}\Big|_{p_i^2 = m_{\text{phys}}^2}, \tag{2.36}$$

is *exactly* the same as in the one in the unbox basis Eq. (2.33), once the Wilson coefficients are related by the map in Eq. (2.7).

We move to the UV theory computation, where we also include the bottom row diagrams in Fig. 5 with the heavy scalar in the loop. In the limit $\lambda_2 = 0$ the result is

$$
\begin{aligned}
\mathcal{M}_4^{\text{UV}} &= -\lambda_0 + \frac{3\lambda_0^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right) + \frac{3\lambda_2^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right) + 2\right) \\
&+ \frac{\lambda_0^2}{32\pi^2}\left[f(s,m) + f(t,m) + f(u,m)\right] + \frac{\lambda_2^2}{32\pi^2}\left[f(s,M) + f(t,M) + f(u,M)\right].
\end{aligned}
\tag{2.37}
$$

Expanding this in $1/M$,

$$
\begin{aligned}
\mathcal{M}_4^{\text{UV}} &\approx -\lambda_0 + \frac{3\lambda_0^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right) + \frac{3\lambda_2^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right)\right) + \frac{m^2\lambda_2^2}{48\pi^2 M^2} \\
&+ \frac{\lambda_0^2}{32\pi^2}\left[f(s,m) + f(t,m) + f(u,m)\right].
\end{aligned}
\tag{2.38}
$$

In the $\overline{\text{MS}}$ scheme, the renormalized amplitudes are those of Eq. (2.31) and Eq. (2.38) with $1/\bar{\epsilon}$ poles dropped and μ is interpreted as the renormalization scale. We then read off the matching condition for C_4 at 1-loop level:

$$C_4 = \lambda_0 - \frac{3\lambda_2^2}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right) - \frac{\lambda_2^2 m^2}{48\pi^2 M^2}. \tag{2.39}$$

As in the previous case of matching m^2, logs of the small mass scale m do not show up in the matching equation. Again, we are free to choose $\mu = M$ (or another $\mu \sim M$, e.g. $\mu_R = 11M/17$ if you wish) to avoid large logs in the matching equation. Then the matching between the EFT and UV parameters simplifies to

$$C_4(M) = \lambda_0(M) - \frac{\lambda_2^2 m^2}{48\pi^2 M^2}. \tag{2.40}$$

2.5 RG equations in the EFT

We have seen that the matching between the UV theory and the EFT simplifies if the matching is performed at the scale $\mu \sim M$. Then, to calculate the EFT couplings at $\mu \ll M$ we need to evolve them using the RG equations. The latter are obtained by demanding the observables, such as the physical mass and S-matrix elements, do not depend on the renormalization scale. In the unbox basis this led us to the following RG equations (c.f. Eq. (2.20) and Eq. (2.34))

$$
\begin{aligned}
\frac{dm^2}{d\log\mu} &= \frac{m^2 C_4}{16\pi^2}, \\
\frac{dC_4}{d\log\mu} &= \frac{1}{16\pi^2}\left[3C_4^2 + \frac{m^2}{M^2}C_6\right].
\end{aligned}
\tag{2.41}
$$

The $\mathcal{O}(M^0)$ terms on the r.h.s. are the standard result in the ϕ^4 theory. There is also an $\mathcal{O}(M^{-2})$ terms in the running equation for C_4 that is proportional to the Wilson coefficient of the dimension-6 operator. In general, the EFT at one loop, Wilson coefficients of higher dimensional operators may affect RG running of lower-dimensional ones (never the other way around) if there are explicit mass parameters in the EFT.

Let us pause for a moment to brood on the physical meaning of Eq. (2.41). To one-loop accuracy the first of these equations is solved by

$$m^2(\mu) = m^2(M) \left(\frac{\mu}{M}\right)^{\frac{C_4}{16\pi^2}} . \qquad (2.42)$$

Given that $m^2 \sim M^2$, c.f. Eq. (2.30), Eq. (2.42) tells us that the naive scaling is modified by 1-loop effects, leading to $m^2 \sim M^{2+\frac{C_4}{16\pi^2}}$. For these reason the coefficients on the right-hand side of the RG equations are referred to as *anomalous dimensions*. Furthermore, using $a^\epsilon = e^{\epsilon \log a} \approx 1 + \epsilon \log a$ one can re-write Eq. (2.42) as

$$m^2(\mu) \approx m^2(M) \left[1 + \frac{C_4}{16\pi^2} \log \left(\frac{\mu}{M}\right)\right], \qquad (2.43)$$

which is a valid approximation as long as $C_4 \log(\mu/M) \ll 16\pi^2$. This should be compared to the expression for the physical mass in the full UV theory in Eq. (2.28). Up to one loop accuracy this is the same expression once the map between the EFT and the UV parameters is taken into account (c.f. Eq. (2.30) and Eq. (2.7)). What the RG evolution accomplishes is re-summing the possibly large logarithms in Eq. (2.28) into the expression in Eq. (2.42), which is valid also $C_4 \log(\mu/M) \gtrsim 16\pi^2$ (assuming $C_4 \ll 16\pi^2$, that is to say, the breakdown of the perturbation theory is due to large logarithms, and not due large couplings).

.

2.6 Summary and lessons learned

To summarize, given the heavy-light scalar system described by Eq. (2.1), in order to calculate scattering amplitudes of the light scalar ϕ at energies E far below the heavy mass scale M one should apply the following algorithm:

1. Express the parameters of the EFT Lagrangian Eq. (2.2) at the scale M by those of the full theory at that scale using the matching equations such as the ones in Eq. (2.30) and Eq. (2.40).

2. Evolve the Wilson coefficients down to the scale $\mu \sim E$ using the RG equations in the EFT, such as the ones in Eq. (2.41).

3. Calculate the amplitudes using the EFT Lagrangian with the parameters at the scale $\mu \sim E$.

4. If working at $\mathcal{O}(M^{-2})$ is insufficient for the given E/M and the experimental precision, write down $d = 8$ operators (or higher) in the EFT Lagrangian and generalize the matching and RG equations to the corresponding order in $1/M$. Similarly, if the 1-loop precision is insufficient, generalize the matching and RG equations to higher loops.

Alternatively, one could perform the same calculations within the full theory with two scalars. This would be of course more involved computationally. Moreover, when $E \ll M$, one may run into the problem of large logs that invalidate the perturbative expansion. Thus, calculations in the EFT are most often superior, both from the practical and from the precision point of view.

One could also go the opposite way, from the EFT to the UV theory. Imagine that some hypothetical experimental data are well described by a theory with a single scalar field and the effective Lagrangian Eq. (2.2). Due to the presence of higher-dimensional operators, this theory cannot be valid up to an arbitrary high scale. Indeed, one finds that the amplitude calculated in the effective theory lose perturbative unitarity at the high-energy scale $\Lambda \sim \frac{4\pi M}{\sqrt{|C_6|}}$ which we assume to be beyond the reach of the hypothetical experiment. In such circumstances, a hypothetical theorist would speculate that at the single scalar theory is embedded in a two scalar theory where the new scalar has mass $M \leq \Lambda$. Then, running the experimentally measured Wilson coefficients C_4 and C_6 up to a high scale, and matching them to the parameters to the heavy-light scalar system, one could gain some insight about certain combinations of masses and couplings of the heavy scalar.

3 Path integral methods for EFT

The path integral formulation of quantum field theories offers powerful methods to match the low-energy effective Lagrangian to any UV f relativistic theory. These methods go under the name of the covariant derivative expansion (CDE) and have been developed recently in Refs. [17, 18, 19, 20, 21].

3.1 Flash review of path integrals

In a quantum field theory the correlation functions can be computed from a path integral according to

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{\int [D\phi] \phi(x_1) \dots \phi(x_n) \exp\left[i \int d^4x \mathcal{L}(\phi)\right]}{\int [D\phi] \exp\left[i \int d^4x \mathcal{L}(\phi)\right]}. \tag{3.1}$$

All correlation functions can be encoded in the generating functional $Z[J]$ called the *partition function*:

$$Z[J] = \int [D\phi] \exp\left[i \int d^4x \left(\mathcal{L}(\phi) + J\phi\right)\right]. \tag{3.2}$$

Given $Z[J]$, the correlation function can be recovered by differentiating with respect to the source term:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \left(-i\frac{\delta}{\delta J(x_1)}\right) \dots \left(-i\frac{\delta}{\delta J(x_n)}\right) (\log Z[J])_{J=0}. \tag{3.3}$$

Actually, the full information about the theory is stored in the 1-particle irreducible (1PI) diagrams, that is to say, in connected diagrams which cannot be disconnected by cutting a single internal propagator. There exists a generating functional for the 1PI irreducible diagrams, which I call the *quantum 1PI action*[8], which is defined as the Legendre transform of $\log Z[J]$:

$$\Gamma[\phi_b] = -i \log Z[J] - \int d^4x J(x)\phi_b(x), \qquad \phi_b(x) = \frac{\delta \log Z[J]}{\delta J(x)}, \tag{3.4}$$

where ϕ_b, the so-called classical field, is a solution to the classical equations of motion, $\delta\mathcal{L}|_{\phi=\phi_b} + J\delta\phi = 0$. Given the quantum 1PI action, the 1PI diagrams can be recovered by differentiating $\Gamma[\phi]$:

$$\langle \phi(x_1) \dots \phi(x_n) \rangle_{1\text{PI}} = i\frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1) \dots \delta\phi(x_n)}. \tag{3.5}$$

The central formula in most application is the Gaussian path integral. For a real scalar ϕ, a complex scalar Φ, and a Dirac fermion Ψ we have

$$\int [D\phi] \exp\left(\frac{i}{2}\phi\hat{A}\phi\right) = C_1[\det \hat{A}]^{-1/2},$$

$$\int [D\Phi^\dagger][D\Phi] \exp\left(i\Phi^\dagger\hat{A}\Phi\right) = C_2[\det \hat{A}]^{-1},$$

$$\int [D\bar{\Psi}][D\Psi] \exp\left(i\bar{\Psi}\hat{A}\Psi\right) = C_3[\det \hat{A}], \tag{3.6}$$

---

[8]In the literature it is usually called the effective action, but I prefer not to use this term because of possible confusion with the low-energy effective action $\int d^4x \mathcal{L}_{\text{EFT}}$.

where $\hat{A}$ is a differential operator acting in the space of functions, and C_i's are some renormalization constant which is irrelevant in practical applications.

3.2   Matching - general discussion

I follow closely the presentation of Ref. [21]. Consider a theory with a collection of light fields ϕ and heavy fields H with the Lagrangian $\mathcal{L}_{\text{UV}}(\phi, H)$. We are interested in the low energy EFT where H are integrated out. The EFT is valid at energies below the heavy fields mass scale M and the interactions of the light fields ϕ at these scales are described by the local Lagrangian $\mathcal{L}_{\text{EFT}}(\phi)$. The problem is to determine $\mathcal{L}_{\text{EFT}}(\phi)$ given $\mathcal{L}_{\text{UV}}(\phi, H)$.

The matching between a UV theory and its low-energy EFT can be concisely characterized at the level of path integrals. The UV theory is defined by the generating functional and the quantum action as follows:

$$
\begin{aligned}
Z_{\text{UV}}[J_\phi, J_H] &= \int [D\phi][DH] \exp\left[i \int d^4x \left(\mathcal{L}_{\text{UV}}(\phi, H) + J_\phi\phi + J_H H\right)\right] \\
\Gamma_{\text{UV}}[\phi_b, H_b] &= -i \log Z_{\text{UV}}[J_\phi, J_H] - \int d^4x J_\phi(x)\phi_b(x) - \int d^4x J_H(x) H_b(x) \\
\phi_b(x) &= \frac{\delta \log Z_{\text{UV}}[J_\phi, J_H]}{\delta J_\phi(x)}, \quad H_b(x) = \frac{\delta \log Z_{\text{UV}}[J_\phi, J_H]}{\delta J_H(x)}
\end{aligned}
\tag{3.7}
$$

On the other hand, the EFT is defined by the generating functional and the quantum action as follows:

$$
\begin{aligned}
Z_{\text{EFT}}[J_\phi] &= \int [D\phi] \exp\left[i \int d^4x \left(\mathcal{L}_{\text{EFT}}(\phi) + J_\phi\phi\right)\right] \\
\Gamma_{\text{EFT}}[\phi_b] &= -i \log Z_{\text{EFT}}[J_\phi] - \int d^4x J_\phi(x)\phi_b(x) \\
\phi_b(x) &= \frac{\delta \log Z_{\text{EFT}}[J_\phi]}{\delta J_\phi(x)}.
\end{aligned}
\tag{3.8}
$$

Since $Z_{\text{UV}}[J_\phi, 0]$ generates all correlation functions for the light fields in the UV theory, while $Z_{\text{EFT}}[J_\phi]$ generates the same correlation functions in the EFT, we require that

$$
Z_{\text{EFT}}[J_\phi] = Z_{\text{UV}}[J_\phi, 0].
\tag{3.9}
$$

This condition is somewhat stronger than we need, as it ensures that the equality of on-shell as well as off-shell correlation functions. What we really need is the equality of S-matrix elements in the UV and the EFT theories. Thus, once we find $\mathcal{L}_{\text{EFT}}(\phi)$ that solves Eq. (3.9), we will allow for field redefinitions of ϕ, which changes the correlation function but preserves the S-matrix elements. Furthermore, one never demands that Eq. (3.9) is exact. Instead, one typically assumes that it holds up to some prescribed order in the loop expansion, and/or up to some order in $1/M$ Taylor expansion of both sides.

Thus, the original problem is transformed into the one of constructing $\mathcal{L}_{\text{EFT}}(\phi)$ such that Eq. (3.9) is satisfied. However, $Z[\phi]$ is in general a highly complicated object. For

practical reasons, it is simpler more convenient to manipulate the 1PI quantum action. We can equivalently formulate the condition in Eq. (3.9) as

$$\Gamma_{\text{UV}}[\phi, 0] = \Gamma_{\text{EFT}}[\phi] \tag{3.10}$$

In the following I review the construction that satisfies Eq. (3.10) up to the one loop order. I will work under the assumption that ϕ and H are a collection of real scalar fields, but the discussion can be easily generalized to other spins.

Let me first calculate the EFT side. In the path integral for Z_{EFT} I change variables as $\phi = \phi_b + \phi'$, where the background value ϕ_b satisfies $\delta\mathcal{L}_{\text{EFT}}|_{\phi=\phi_b} + J_\phi \delta\phi = 0$. I also expand the Lagrangian around ϕ_b. This gives

$$Z_{\text{EFT}} = e^{i \int d^4x \left(\mathcal{L}_{\text{EFT}}|_{\phi=\phi_b} + J_\phi \phi_b \right)} \int [D\phi'] \exp\left[-\frac{i}{2} \int d^4x \, \phi'^T Q_{\text{EFT}} \phi' \right] + \dots \tag{3.11}$$

where $Q_{\text{EFT}} = -\frac{\delta^2 \mathcal{L}_{\text{EFT}}}{\delta\phi^2}|_{\phi=\phi_b}$. The linear term in ϕ' vanish as a consequence of the definition of ϕ_b, while the cubic and higher terms, represented by the dots, correspond to two- and higher-loop contributions which are neglected here. Note that, since ϕ_b solves equations of motion with some arbitrary source term J_ϕ, it is effectively arbitrary. In the following we skip the index b for the light fields ϕ.

Using Eq. (3.6), Eq. (3.11) becomes:

$$Z_{\text{EFT}} = e^{i \int d^4x \left(\mathcal{L}_{\text{EFT}} + J_\phi \phi \right)} \left[\det Q_{\text{EFT}} \right]^{-1/2} + \dots \tag{3.12}$$

It follows

$$\Gamma_{\text{EFT}}[\phi] = \int d^4x \mathcal{L}_{\text{EFT}} + \frac{i}{2} \log \det Q_{\text{EFT}} + \dots \tag{3.13}$$

The calculation on the UV side is similar with one important difference. Since we set $J_H = 0$, H_b solves $\frac{\delta\mathcal{L}_{\text{UV}}}{\delta H}|_{H=H_b} = 0$, and thus it becomes a function of ϕ only. I denote it as $H_b = H_c(\phi)$, where H_c is a solution to the classical equations of motion in the background of ϕ. All in all,

$$\Gamma_{\text{UV}}[\phi] = \int d^4x \mathcal{L}_{\text{UV}}|_{H=H_c(\phi)} + \frac{i}{2} \log \det Q_{\text{UV}} + \dots \tag{3.14}$$

where

$$Q_{\text{UV}} \equiv \begin{pmatrix} \Delta_H & X_{LH} \\ X_{LH} & \Delta_L \end{pmatrix} = \begin{pmatrix} -\frac{\delta^2 \mathcal{L}_{\text{UV}}}{\delta H^2} & -\frac{\delta^2 \mathcal{L}_{\text{UV}}}{\delta\phi\delta H} \\ -\frac{\delta^2 \mathcal{L}_{\text{UV}}}{\delta\phi\delta H} & -\frac{\delta^2 \mathcal{L}_{\text{UV}}}{\delta\phi^2} \end{pmatrix}\Bigg|_{H=H_c(\phi)}. \tag{3.15}$$

We write the EFT Lagrangian as $\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{EFT}}^{(0)} + \mathcal{L}_{\text{EFT}}^{(1)} + \dots$, where for $\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{EFT}}^{(0)}$ Eq. (3.10) is satisfied at tree level, while $\mathcal{L}_{\text{EFT}}^{(1)}$ is the correction required for Eq. (3.10) to be satisfied at one loop. Comparing Eq. (3.14) and Eq. (3.13) we immediately obtain a simple formula for the tree-level effective Lagrangian:

$$\mathcal{L}_{\text{EFT}}^{(0)}(\phi) = \hat{\mathcal{L}}_{\text{UV}}(\phi, H_c(\phi)). \tag{3.16}$$

The hat above signifies the expansion in $1/M$ up to the order in which we perform the matching. The point is that $\mathcal{L}_{\text{UV}}(\phi, H_c(\phi))$ may contain non-local interactions due to $H_c(\phi)$ containing derivatives in denominators. Only after expanding in $1/M$ and

truncating at some finite order we obtain a local Lagrangian, as required by our matching procedure.

More work is needed to determine $\mathcal{L}_{\text{EFT}}^{(1)}$. To this end we need to massage $\Gamma[\phi]$ a bit more. Let's start with the UV side. Q_{UV} can be diagonalized as

$$Q_{\text{UV}} = V \begin{pmatrix} \Delta_H - X_{LH}\Delta_L^{-1}X_{LH} & 0 \\ 0 & \Delta_L \end{pmatrix} V^\dagger, \qquad V = \begin{pmatrix} 1 & X_{LH}\Delta_L^{-1} \\ 0 & 1 \end{pmatrix}. \qquad (3.17)$$

Since $\det V = 0$, we have

$$\log \det Q_{\text{UV}} = \log \det \Delta_L + \log \det \left(\Delta_H - X_{LH}\Delta_L^{-1}X_{LH} \right) \qquad (3.18)$$

The first term is interpreted as the loops of the light fields in the UV theory, while the second describes the heavy and mixed loops.

We go back to the EFT. When calculating $\log \det Q_{\text{EFT}}$ we need to take into account only $\mathcal{L}_{\text{EFT}}^{(0)}$, which we already determined, because $\mathcal{L}_{\text{EFT}}^{(1)}$ contributes to the determinant only at two loops. The functional derivatives of $\mathcal{L}_{\text{EFT}}^{(0)}$ can be simplified as

$$\begin{aligned} \frac{\delta^2 \mathcal{L}_{\text{EFT}}^{(0)}}{\delta\phi^2} &= \frac{\delta^2}{\delta\phi^2} \left(\hat{\mathcal{L}}_{\text{UV}}(\phi, H_c(\phi)) \right) = \frac{\delta}{\delta\phi} \left(\frac{\delta\hat{\mathcal{L}}_{\text{UV}}}{\delta\phi}\big|_{H=H_c(\phi)} \right) \\ &= \frac{\delta^2 \hat{\mathcal{L}}_{\text{UV}}}{\delta\phi^2}\big|_{H=H_c(\phi)} + \frac{\delta^2 \hat{\mathcal{L}}_{\text{UV}}}{\delta\phi\delta H}\big|_{H=H_c(\phi)} \frac{dH_c}{d\phi} \end{aligned} \qquad (3.19)$$

Now using

$$0 = -\frac{\delta}{\delta\phi} \left(\frac{\delta\mathcal{L}_{\text{UV}}}{\delta H}\big|_{H=H_c(\phi)} \right) = X_{LH} + \Delta_H \frac{dH_c}{d\phi} \qquad (3.20)$$

we get

$$Q_{\text{EFT}} = \hat{\Delta}_L - \hat{X}_{LH}\hat{\Delta}_H^{-1}\hat{X}_{LH} \qquad (3.21)$$

Again, the hat here denote the $1/M$ expansion. Thus,

$$\begin{aligned} \log \det Q_{\text{EFT}} &= \log \det \hat{\Delta}_L + \log \det \left(1 - \hat{\Delta}_L^{-1}\hat{X}_{LH}\hat{\Delta}_H^{-1}\hat{X}_{LH} \right) \\ &= \log \det \hat{\Delta}_L + \log \det \left(1 - \hat{X}_{LH}\hat{\Delta}_L^{-1}\hat{X}_{LH}\hat{\Delta}_H^{-1} \right) \\ &= \log \det \hat{\Delta}_L - \log \det \hat{\Delta}_H + \log \det \left(\hat{\Delta}_H - \hat{X}_{LH}\hat{\Delta}_L^{-1}\hat{X}_{LH} \right) \end{aligned} \qquad (3.22)$$

All in all, to ensure $\Gamma_{\text{UV}}[\phi] = \Gamma_{\text{EFT}}[\phi]$ up to one-loop order we need to choose $\mathcal{L}_{\text{EFT}}^{(1)}$ as

$$\begin{aligned} \int d^4x \mathcal{L}_{\text{EFT}}^{(1)} &= \frac{i}{2} \log \det \left(\Delta_H - X_{LH}\Delta_L^{-1}X_{LH} \right) - \frac{i}{2} \log \det \left(\hat{\Delta}_H - \hat{X}_{LH}\hat{\Delta}_L^{-1}\hat{X}_{LH} \right) \\ &+ \frac{i}{2} \log \det \hat{\Delta}_H \end{aligned} \qquad (3.23)$$

I used that $\det \hat{\Delta}_L = \det \Delta_L$; in other words, in the light loops it doesn't matter if the expansion in $1/M$ is done before or after taking the integral. That is in general not true for the heavy and mixed loops. A surprising theorem, known as the method of regions, allows one separate hard and soft contributions to the loop integrals. The

method of regions postulates that loop integrals with two disparate mass scales m and M are exactly given by the sum of their soft region obtained by expanding the integrand assuming the loop momenta $k \sim mi \ll M$ and the hard expansion region assuming $k \sim M \gg m$. See Appendix A for an explicit example. Note that isolating the soft region is equivalent to expanding the integrand in $1/M$, thus it is the same as putting the hat. Thus we have $\log \det X = \log \det X_{\text{hard}} + \log \det X_{\text{soft}}$ and moreover

$$
\begin{aligned}
\log \det \left(\Delta_H - X_{LH} \Delta_L^{-1} X_{LH} \right)_{\text{soft}} &= \log \det \left(\hat{\Delta}_H - \hat{X}_{LH} \hat{\Delta}_L^{-1} \hat{X}_{LH} \right) \\
\left(\log \det \Delta_H \right)_{\text{soft}} &= \log \det \hat{\Delta}_H = 0
\end{aligned}
\tag{3.24}
$$

The last equality hold in dimensional regularization because the expansion leads to scaleless integrals of the form $M^{-c} \int d^d k \, k^n = 0$. Finally, the compact expression for the 1-loop part of the effective Lagrangian reads

$$
\int d^4 x \mathcal{L}_{\text{EFT}}^{(1)} = \frac{i}{2} \log \det \left(\Delta_H - X_{LH} \Delta_L^{-1} X_{LH} \right)_{\text{hard}}
\tag{3.25}
$$

3.3 Calculating functional determinants

We turn to discussing how one actually calculate the functional determinant in Eq. (3.25). The central formula for this enterprise is

$$
\log \det \hat{A} = \operatorname{Tr} \log \hat{A}.
\tag{3.26}
$$

Here, Tr should be taken both in the space of functions where $\hat{A}$ is acting, and over all internal indices (color, spin, flavor, etc.) of the quantum field being integrated over. To differentiate the latter, the trace over internal indices only will be denoted as tr. One can derive the following useful expression for the functional trace of an operator which is a function of the momentum operator $\hat{p}$ ($\hat{p} = p$ in the momentum basis, $\hat{p} = i\partial_x$ in the coordinate basis). Representing the trace as the sum of the eigenstates in the momentum basis we have:

$$
\operatorname{Tr} O(\hat{p}) = \int \frac{d^d q}{(2\pi)^d} \langle q | \operatorname{tr} O(\hat{p}) | q \rangle = \int d^d x \int \frac{d^d q}{(2\pi)^d} \langle q | x \rangle \langle x | \operatorname{tr} O(\hat{p}) | q \rangle
\tag{3.27}
$$

Now we can insert the coordinate basis representation of the momentum eigenstate: $\langle x \| q \rangle = e^{-iqx}$.

$$
\operatorname{Tr} O(\hat{p}) = \int d^d x \int \frac{d^d q}{(2\pi)^d} e^{iqx} \operatorname{tr} O(i\partial_x) e^{-iqx} = \int d^d x \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} O(i\partial_x + q) = \int d^d x \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} O(\hat{p} - q)
\tag{3.28}
$$

The last step is just a trivial change of variable $q \to -q$. Hence we have

$$
\mathcal{L}_{\text{EFT}}^{(1)} = \frac{i}{2} \int \frac{d^d q}{(2\pi)^d} \operatorname{tr} \log \left(\Delta_H - X_{LH} \Delta_L^{-1} X_{LH} \right)_{\text{hard}} \big|_{\hat{P} = \hat{P} - q}
\tag{3.29}
$$

Now assume Δ_H is of the form

$$
\Delta_H = -\hat{P}^2 + M^2 + X_H(H, \phi, \hat{P})
\tag{3.30}
$$

31

Then

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}}^{(1)} &= \frac{i}{2}\int \frac{d^d q}{(2\pi)^d}\,\mathrm{tr}\log\left(-q^2 + 2q\hat{P} - \hat{P}^2 + M^2 + X_H|_{\hat{P}=\hat{P}-q} - X_{LH}\Delta_L^{-1}X_{LH}|_{\hat{P}=\hat{P}-q}\right)_{\text{hard}} \\
&= \frac{i}{2}\int \frac{d^d q}{(2\pi)^d}\,\mathrm{tr}\log\left[1 - (q^2 - M^2)^{-1}\left(2q\hat{P} - \hat{P}^2 + X_H|_{\hat{P}=\hat{P}-q} - X_{LH}\Delta_L^{-1}X_{LH}|_{\hat{P}=\hat{P}-q}\right)\right]_{\text{hard}} \quad (3.31)
\end{aligned}
$$

Above we dropped a constant ϕ-independent term. Finally

$$
\mathcal{L}_{\text{EFT}}^{(1)} = -\frac{i}{2}\sum_{n=1}^{\infty} n^{-1}\int \frac{d^d q}{(2\pi)^d}\,\mathrm{tr}\left(\frac{2q\hat{P} - \hat{P}^2 + X_H|_{\hat{P}=\hat{P}-q} - X_{LH}\Delta_L^{-1}X_{LH}|_{\hat{P}=\hat{P}-q}}{q^2 - M^2}\right)^n_{\text{hard}}
$$
$$(3.32)$$

This way, calculation of the effective Lagrangian at one loop is reduced to calculating a number of momentum integrals.

3.4 Scalar toy model example

I go back to the scalar toy model discussed in Section 2. I will rederive the matching conditions between the parameters of the UV theory and the EFT using the path integral techniques discussed in this section. We will see that path integrals offer a much more efficient way to determine the effective Lagrangian.

3.4.1 Tree level

At tree-level the matching equation is given by Eq. (3.16). To find the EFT Lagrangian we first need $H_c(\phi)$ which solves the equation of motion for the heavy field H in the background of ϕ. I get

$$
H_c(\phi) = -\frac{\lambda_1 M}{2}\left[M^2 + \Box + \frac{\lambda_2}{2}\phi^2\right]^{-1}\phi^2. \qquad (3.33)
$$

To obtain the tree-level effective Lagrangian, this solution should be inserted in the UV Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}}^{(0)}(\phi) &= \mathcal{L}_{\text{UV}}(\phi, H_c(\phi)) \\
&= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H_c(\phi) - \frac{1}{2}H_c(\phi)\left[\Box + M^2 + \frac{\lambda_2}{2}\phi^2\right]H_c(\phi) \\
&= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 + \frac{\lambda_1^2 M^2}{8}\phi^2\left[M^2 + \Box + \frac{\lambda_2}{2}\phi^2\right]^{-1}\phi^2. \qquad (3.34)
\end{aligned}
$$

Expanding this up to order $1/M^2$:

$$
\mathcal{L}_{\text{EFT}}^{(0)} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \left(\lambda_0 - 3\lambda_1^2\right)\frac{\phi^4}{4!} - 45\lambda_1^2\lambda_2\frac{\phi^6}{6!M^2} - 4\lambda_1^2\frac{\phi^3\Box\phi}{4!M^2} + \mathcal{O}(M^{-4}). \quad (3.35)
$$

Note that both the ϕ^6 and $\phi^3\Box\phi$ dimension-6 terms appear as a result of this procedure, however one of the pair is redundant. One can simplify the effective theory by projecting it into one of the bases introduces in Section 2. If we remove $\phi^3\Box\phi$ using Eq. (2.5) we get

in the unbox basis defined by Eq. (2.2). At the tree level and at $\mathcal{O}(M^{-2})$ the matching conditions read

$$
\begin{aligned}
m^2 &= m_L^2, \\
C_4 &= \lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}, \\
C_6 &= 45\lambda_1^2\lambda_2 - 20\lambda_0\lambda_1^2 + 60\lambda_1^4.
\end{aligned}
\tag{3.36}
$$

On the other hand, eliminating ϕ^6 we get the matching conditions in the box basis defined by Eq. (2.6):

$$
\begin{aligned}
m^2 &= m_L^2, \\
\tilde{C}_4 &= \lambda_0 - 3\lambda_1^2 - \frac{9m_L^2}{M^2}\frac{\lambda_1^2\lambda_2}{\lambda_0 - 3\lambda_1^2}, \\
\tilde{C}_6 &= 4\lambda_1^2 - 9\frac{\lambda_1^2\lambda_2}{\lambda_0 - 3\lambda_1^2}.
\end{aligned}
\tag{3.37}
$$

These match the results in Eq. (2.16) and Eq. (2.17) previously obtained by diagrammatic methods. However, the path integral way is much more straightforward, given that one can avoid calculations of the 6-point function.

3.4.2 One loop

We move to the 1-loop level. For simplicity, I consider the Lagrangian of Eq. (2.1) in the limit $\lambda_1 = 0$. In that case there is no linear coupling of the heavy field to the light one, which implies that $H_c(\phi) = 0$ is a solution to the equation of motion for H. Furthermore,

$$
X_H = \frac{\lambda_2}{2}\phi^2, \qquad X_{LH} = 0.
\tag{3.38}
$$

Because $H_c(\phi)$ vanishes, the tree-level EFT Lagrangian is just the UV Lagrangian with the heavy fields set to zero:

$$
\mathcal{L}_{\text{EFT}}^{(0)} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4.
\tag{3.39}
$$

We now compute the 1-loop EFT Lagrangian using Eq. (3.32):

$$
\mathcal{L}_{\text{EFT}}^{(1)} = -\frac{i}{2}\sum_{n=1}^{\infty} n^{-1} \int \frac{d^dq}{(2\pi)^d} \left(\frac{2q\hat{P} - \hat{P}^2 + \frac{\lambda_2}{2}\phi^2}{q^2 - M^2}\right)^n.
\tag{3.40}
$$

In this case the "hard" qualifier can be dropped because there is no tree-level Lagrangian to expand in local operators.

The $n = 1$ term yields

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}}^{(1)} &\supset -\frac{i}{2}\frac{\lambda_2\phi^2}{2} \int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2 - M^2} \\
&= \frac{\lambda_2 M^2}{32\pi^2}\left[\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 1\right]\frac{\phi^2}{2}
\end{aligned}
\tag{3.41}
$$

The $n = 2$ term yields

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}}^{(1)} &\supset -\frac{i}{2}\frac{\lambda_2^2\phi^4}{4}\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-M^2)^2} \\
&= \frac{3\lambda_2^2}{32\pi^2}\left[\frac{1}{\epsilon}+\log\left(\frac{\mu^2}{M^2}\right)\right]\frac{\phi^4}{4!}
\end{aligned}
\tag{3.42}
$$

where I dropped total derivatives (such as e.g. $\partial^2(\phi^2)$).

The $n = 3$ term yields

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}}^{(1)} &\supset -\frac{i}{2}\frac{1}{3}\left(\frac{\lambda_2^3\phi^6}{8}+\frac{7}{2}\phi^2(\partial_\mu\phi)^2+\frac{3}{2}\phi^3\Box\phi\right)\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-M^2)^3} \\
&= -\frac{i}{2}\frac{1}{3}\left(\frac{\lambda_2^3\phi^6}{8}+\frac{1}{3}\phi^3\Box\phi\right)\int\frac{d^dq}{(2\pi)^d}\frac{1}{(q^2-M^2)^3} \\
&= -\frac{15\lambda_2^3}{32\pi^2}\frac{\phi^6}{6!M^2}-\frac{\lambda_2^2}{24\pi^2}\frac{\phi^3\Box\phi}{24M^2}
\end{aligned}
\tag{3.43}
$$

where I integrated by parts to trade $\phi^2(\partial_\mu\phi)^2 = -\frac{1}{3}\phi^3\Box\phi$.

There is also an $\mathcal{O}(1/M^2)$ contribution from the $n = 4$ term:

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}}^{(1)} &\supset -\frac{i}{2}\frac{1}{4}\left(-\frac{4\lambda^2}{3}\phi^3\partial_\mu\partial_\nu\phi\right)\int\frac{d^dq}{(2\pi)^d}\frac{q_\mu q_\nu}{(q^2-M^2)^4} \\
&= \frac{\lambda_2^2}{48\pi^2M^2}\frac{\phi^3\Box\phi}{24}
\end{aligned}
\tag{3.44}
$$

The $n > 4$ terms in the expansion of the logarithm only contribute at $\mathcal{O}(M^{-4})$.

Putting this together, the effective Lagrangian up to one loop in the MSbar scheme is given by

$$
\begin{aligned}
\mathcal{L}_{\text{EFT}}^{(0)} + \mathcal{L}_{\text{EFT}}^{(1)} &\supset \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\phi^2}{2}\left(m_L^2 - \frac{\lambda_2 M^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{M^2}\right)+1\right]\right) \\
&\quad - \frac{\phi^4}{4!}\left[\lambda_0 - \frac{3\lambda_2^2}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\right] \\
&\quad - \frac{\phi^6}{6!M^2}\frac{15\lambda_2^3}{32\pi^2} - \frac{\phi^3\Box\phi}{24M^2}\frac{\lambda_2^2}{48\pi^2}.
\end{aligned}
\tag{3.45}
$$

One can see that the path integral approach leads to dimension-6 operators from both the box and the unbox bases. To project onto one of these bases we need to trade the operators using the identity

$$
\frac{1}{4!M^2}\phi^3\Box\phi = -\frac{m^2}{4!M^2}\phi^4 - \frac{5C_4}{6!M^2}\phi^6 + \mathcal{O}(M^{-4}).
\tag{3.46}
$$

Eliminating $\phi^3\Box\phi$ in Eq. (3.45), in the unbox basis the matching between the Wilson coefficients and the UV parameters is thus

$$
\begin{aligned}
m^2 &= m_L^2 - \frac{\lambda_2 M^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{M^2}\right)+1\right] \\
C_4 &= \lambda_0 - \frac{3\lambda_2^2}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right) - \frac{\lambda_2^2 m_L^2}{48\pi^2M^2} \\
C_6 &= \frac{15\lambda_2^3}{32\pi^2} - \frac{5\lambda_0\lambda_2^2}{48\pi^2}
\end{aligned}
\tag{3.47}
$$

One can see this correctly reproduces the results for the 1-loop matching of m^2 and C_4 calculated diagramatically, c.f. Eq. (2.29) and Eq. (2.39). The 1-loop matching for C_6 is a new result, which would be time-consuming to calculate diagramatically.

On the other hand, eliminating ϕ^6 in Eq. (3.45) one obtains the matching conditions in the box basis:

$$
\begin{aligned}
\tilde{C}_4 &= \lambda_0 - \frac{3\lambda_2^2}{32\pi^2} \log\left(\frac{\mu^2}{M^2}\right) - \frac{3\lambda_2^3 m_L^2}{32\pi^2 \lambda_0 M^2} \\
\tilde{C}_6 &= \frac{\lambda_2^2}{48\pi^2} - \frac{3\lambda_2^3}{32\pi^2 \lambda_0}
\end{aligned}
\tag{3.48}
$$

4   SM EFT: effective theory for BSM

The Standard Model (SM) of particle physics is a theory of quarks and leptons interacting via strong, weak, and electromagnetic forces. It is built on the following principles:

#1 The basic framework is that of a relativistic quantum field theory, with interactions between particles described by a local Lagrangian.

#2 The Lagrangian is invariant under the linearly realized local $SU(3) \times SU(2) \times U(1)$ symmetry.

#3 The vacuum state of the theory preserves only $SU(3) \times U(1)$ local symmetry, as a result of the Brout-Englert-Higgs mechanism [22, 23, 24]. The spontaneous breaking of the $SU(2) \times U(1)$ symmetry down to $U(1)$ arises due to a vacuum expectation value (VEV) of a scalar field transforming as $(1,2)_{1/2}$ under the local symmetry.

#4 Interactions are *renormalizable*, which means that only interactions up to the canonical mass dimension 4 are allowed in the Lagrangian.

Given the quantum numbers of the 3 families of quarks and leptons under the $SU(3) \times SU(2) \times U(1)$, these rules completely specify the theory up to 19 free parameters. The local symmetry implies the presence of spin-1 vector bosons which mediate the strong and electroweak forces. The breaking pattern of the local symmetry ensures that the carriers of the strong and electromagnetic force are massless, whereas the carriers of the weak force are massive. Finally, the particular realization of the Brout-Englert-Higgs mechanism in the SM leads to the emergence of exactly one spin-0 scalar boson - the famous Higgs boson [25, 26, 27].

The SM passed an incredible number of experimental tests. It correctly describes the rates and differential distributions of particles produced in high-energy collisions; a robust deviation from the SM predictions has never been observed. It allows one to accurately calculate many properties of elementary particles, such as e.g. magnetic dipole moments, as well as certain properties of simple enough composite particles, such as e.g. atomic energy levels. The discovery of a 125 GeV boson at the Large Hadron Collider (LHC) [28, 29] nails down the last propagating degree of freedom predicted by the SM. Measurements of its production and decay rates vindicate the simplest realization of the Brout-Englert-Higgs mechanism, in which a VEV of a *single* SU(2) doublet field spontaneously breaks the electroweak symmetry. Last not least, the SM is a consistent quantum theory (as long as the gravitational interactions can be neglected). In particular, for the measured value of the Higgs boson mass the vacuum of the theory is metastable, with a lifetime many orders of magnitude longer than the age of the universe. Therefore the validity range of the SM can be extended all the way up to the Planck scale (at which point the gravitational interactions become strong and can no longer be neglected) without encountering any theoretical inconsistency.

Yet we know that the SM is not the ultimate theory. It cannot account for dark matter, neutrino masses, matter/anti-matter asymmetry, and cosmic inflation, which are all experimental facts. In addition, some theoretical or esthetic arguments (the strong

CP problem, flavor hierarchies, unification) suggest that the SM should be extended.[9] This justifies the ongoing searches for *new physics*, that is particles or interactions not predicted by the SM.

In spite of good arguments for the existence of new physics, a growing body of evidence suggests that, at least up to energies of a few hundred GeV, the fundamental degrees of freedom are those of the SM. Given the absence of any direct or indirect collider signal of new physics, it is reasonable to assume that new particles from beyond the SM are much heavier than the SM particles. If that is correct, physics at the weak scale can be adequately described using *effective field theory* (EFT) methods.

The framework adopted here is referred to as the *SM EFT*. In this framework the assumptions #1...#3 above continue to be valid.[10] Thus, much as in the SM, the Lagrangian is constructed from gauge invariant operators involving the SM fermion, gauge, and Higgs fields. The difference is that the assumption #4 is dropped and interactions with arbitrary large mass dimension D are allowed. These interactions can be organized in a systematic expansion in the operator dimensions:

$$\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{\Lambda_L} \sum_i c_i^{(5)} \mathcal{O}_i^{D=5} + \frac{1}{\Lambda^2} \sum_i c_i^{(6)} \mathcal{O}^{D=6} + \frac{1}{\Lambda_L^3} \sum_i c_i^{(7)} \mathcal{O}^{D=7} + \frac{1}{\Lambda^4} \sum_i c_i^{(8)} \mathcal{O}^{D=8}$$
$$+ \dots, \tag{4.1}$$

where each $\mathcal{O}_i^D$ is a gauge invariant operator of canonical dimension D constructed from the SM fields, and the parameters $c_i^{(D)}$ are called the *Wilson coefficients*. The leading order term in this expansion is the SM Lagrangian with operators up to $D=4$. All possible effects of heavy new physics are encoded in operators with $D > 4$, which are suppressed in the Lagrangian by appropriate powers of the mass scales Λ and Λ_L (I will justify later why two different scales are introduced in Eq. (4.1)). In other words, the SM EFT is defined as a double expansion in $1/\Lambda$ and $1/\Lambda_L$. The expansion is useful assuming $v \ll \Lambda$ and $v \ll \Lambda_L$.

The EFT defined by Eq. (4.1) is intended to parametrize observable effects of a large class of BSM theories with new particles much heavier than the SM ones and much heavier than the energy scale at which the experiment is performed. Constraints on the EFT parameters can be recast into constraints on particular BSM models as soon as the matching between its parameters and the EFT Wilson coefficients is known. In the rest of this section I discuss the effective Lagrangian in Eq. (4.1) and the relationship between its parameters, BSM physics, and collider observables.

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[9]Historically, a strong esthetic motivation to extend the SM was offered by its *naturalness problem*. Due to the large sensitivity of the Higgs mass to physics at high energy scales many researchers believed that the SM has to be replaced by another theory at energies very close the Z boson mass. Given the failure to observe any non-SM degrees of freedom at the LHC, one concludes that the naturalness problem of the SM is probably a red herring, much as the cosmological constant problem.

[10]One could consider a more general effective theory where the assumptions #2 and #3 are also relaxed and the electroweak symmetry is realized *non-linearly*, see Section II.2.4 of [30] for a review. In that framework, the Higgs boson h is a perfect singlet under the gauge transformations, while the 3 Goldstone bosons π^a eaten by W and Z transform non-linearly under $SU(2)_L \times U(1)_Y$, but linearly under the electromagnetic $U(1)$. Such a framework can be viewed as a generalization of the SM EFT where the Lagrangian contains terms that are non-analytic in $H^\dagger H = 0$. Consequently, it is relevant to describe the low-energy effects of SM extensions containing new particles whose masses vanish in the limit of unbroken electroweak symmetry.

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
$q = \begin{pmatrix} u \\ d \end{pmatrix}$	**3**	**2**	$1/6$
u^c	$\mathbf{\bar{3}}$	**1**	$-2/3$
d^c	$\mathbf{\bar{3}}$	**1**	$1/3$
$\ell = \begin{pmatrix} \nu \\ e \end{pmatrix}$	**1**	**2**	$-1/2$
e^c	**1**	**1**	1
H	**1**	**2**	$1/2$

Table 1:   Transformation properties of the SM fields under the SM gauge group.

4.1   Standard Model

For the sake of defining my notation and conventions, I start by reviewing the SM field content and Lagrangian. The fields entering the Lagrangian in Eq. (4.1) are:

- Vector fields G_μ^a, W_μ^i, B_μ, where $i = 1\ldots3$ and $a = 1\ldots8$. They transform as four-vectors under the Lorentz symmetry and are the gauge fields of the G_{SM} group.

- 3 generations of fermionic fields $q_I = (u_I, d_I)$, u_I^c, d_I^c, $\ell_I = (\nu_I, e_I)$, e_I^c. They transform as 2-component spinors under the Lorentz symmetry.[11] The transformation properties under G_{SM} are listed in Table 1. The generation index, $I, J = 1\ldots3$ will be sometimes suppressed to reduce the clutter.

- Scalar field $H = (H^+, H^0)$ transforming as $(1,2)_{1/2}$ under G_{SM}. I also define $\tilde{H}_i = \epsilon_{ij} H_j^*$ with $\epsilon_{12} = -\epsilon_{21} = 1$, which transforms as $(1,2)_{-1/2}$.

The Lagrangian can be split as

$$\mathcal{L}^{\text{SM}} = \mathcal{L}_{\text{V}}^{\text{SM}} + \mathcal{L}_{\text{F}}^{\text{SM}} + \mathcal{L}_{\text{H}}^{\text{SM}} + \mathcal{L}_{\text{Y}}^{\text{SM}}. \tag{4.2}$$

The first term above contains gauge invariant kinetic terms for the vector fields:[12]

$$\mathcal{L}_{\text{V}}^{\text{SM}} = -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a - \frac{1}{4} W_{\mu\nu}^i W_{\mu\nu}^i - \frac{1}{4} B_{\mu\nu} B_{\mu\nu}. \tag{4.3}$$

The field strength tensors are expressed by the corresponding gauge fields as

$$\begin{aligned} B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ W_{\mu\nu}^i &= \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g_L \epsilon^{ijk} W_\mu^j W_\nu^k, \\ G_{\mu\nu}^a &= \partial_\mu G_\nu^a - \partial_\nu G_\mu^a + g_s f^{abc} G_\mu^b G_\nu^c. \end{aligned} \tag{4.4}$$

---

[11]Throughout this review I use the 2-component spinor notation for fermions. I follow the conventions of Ref. [31] unless otherwise noted. One difference is that my definition of the $\sigma_{\mu\nu}$ matrices differs by a factor of 2 from that in Ref. [31]: $\sigma_{\mu\nu} \equiv \frac{i}{2} (\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)$, $\bar{\sigma}_{\mu\nu} \equiv \frac{i}{2} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu)$.

[12]Here and everywhere, repeating Lorentz indices $\mu, \nu, \ldots$ are implicitly contracted using the Lorentz tensor $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. I do not adhere to the convention of writing upper and lower Lorentz indices.

where ϵ^{ijk} and f^{abc} are the totally anti-symmetric structure tensors of $SU(2)$ and $SU(3)$, and g_s, g_L, g_Y are gauge couplings of $SU(3)_C \times SU(2)_L \times U(1)_Y$. For the future use, I also define the electromagnetic coupling $e = g_L g_Y / \sqrt{g_L^2 + g_Y^2}$, and the weak mixing angle $s_\theta = g_Y / \sqrt{g_L^2 + g_Y^2}$.

The second term in Eq. (4.2) contains covariant kinetic terms of the fermion fields:

$$\mathcal{L}_F^{\rm SM} = i\bar{q}\bar{\sigma}_\mu D_\mu q + iu^c\sigma_\mu D_\mu \bar{u}^c + id^c\sigma_\mu D_\mu \bar{d}^c + i\bar{\ell}\bar{\sigma}_\mu D_\mu \ell + ie^c\sigma_\mu D_\mu \bar{e}^c. \tag{4.5}$$

Each fermion field is a 3-component vector in the generation space. The covariant derivatives are defined as

$$D_\mu f = \left(\partial_\mu - ig_s G_\mu^a T_f^a - ig_L W_\mu^i T_f^i - ig_Y Y_f B_\mu\right) f. \tag{4.6}$$

Here $T_f^a = (\lambda^a, -\lambda^a, 0)$ for f in the triplet/anti-triplet/singlet representation of SU(3), where λ^a are Gell-Mann matrices; $T_f^i = (\sigma^i/2, 0)$ for f in the doublet/singlet representation of SU(2); Y_f is the U(1) hypercharge. The electric charge is given by $Q_f = T_f^3 + Y_f$.

The third term in Eq. (4.2) contains Yukawa interactions between the Higgs field and the fermions:

$$\mathcal{L}_Y^{\rm SM} = -\tilde{H}^\dagger u^c y_u q - H^\dagger d^c y_d q - H^\dagger e^c y_e \ell + {\rm h.c.}, \tag{4.7}$$

where y_f are 3×3 matrices in the generation space.

The last term in Eq. (4.2) are the Higgs kinetic and potential terms:

$$\mathcal{L}_H^{\rm SM} = D_\mu H^\dagger D_\mu H + \mu_H^2 H^\dagger H - \lambda (H^\dagger H)^2, \tag{4.8}$$

where the covariant derivative acting on the Higgs field is

$$D_\mu H = \left(\partial_\mu - \frac{i}{2}g_L W_\mu^i \sigma^i - \frac{i}{2}g_Y B_\mu\right) H. \tag{4.9}$$

For a future use, I write here the equations of motion for the SM gauge bosons:

$$\partial_\nu B_{\nu\mu} = -\frac{ig_Y}{2}H^\dagger \overleftrightarrow{D_\mu} H - g_Y j_\mu^Y,$$
$$\left(\partial_\nu W_{\nu\mu}^i + \epsilon^{ijk}g_L W_\nu^j W_{\nu\mu}^k\right) = D_\nu W_{\nu\mu}^i = -\frac{i}{2}g_L H^\dagger \sigma^i \overleftrightarrow{D_\mu} H - g_L j_\mu^i,$$
$$\left(\partial_\nu G_{\nu\mu}^a + f^{abc}g_s G_\nu^b G_{\nu\mu}^c\right) = D_\nu G_{\nu\mu}^a = -g_s j_\mu^a,$$
$$\Box H = \mu_H^2 H - 2\lambda (H^\dagger H) - j_H, \tag{4.10}$$

where

$$j_\mu^Y = \sum_{f \in \nu,e,u,d} Y_f \bar{f}\bar{\sigma}_\mu f + \sum_{f \in e,u,d} Y_{\bar{f}^c} f^c \sigma_\mu \bar{f}^c,$$
$$j_\mu^i = \bar{q}\bar{\sigma}_\mu \frac{\sigma^i}{2} q + \bar{\ell}\bar{\sigma}_\mu \frac{\sigma^i}{2}\ell,$$
$$j_\mu^a = \bar{q}\bar{\sigma}_\mu T^a q + u^c \sigma_\mu T^a \bar{u}^c + d^c \sigma_\mu T^a \bar{d}^c, \tag{4.11}$$

are the fermionic currents corresponding to the $U(1)$, $SU(2)$, and $SU(3)$ factors of the SM gauge group, and

$$
\begin{aligned}
H^\dagger \overleftrightarrow{D_\mu} H &\equiv H^\dagger D_\mu H - D_\mu H^\dagger H, \\
j_H &\equiv -\bar{u}^c y_u^\dagger \tilde{q} + d^c y_d q + e^c y_e \ell, \qquad \tilde{q}_i \equiv \epsilon_{ij} \bar{q}_j.
\end{aligned} \tag{4.12}
$$

Now I discuss the mass eigenstates in the SM after electroweak symmetry breaking. Because of the negative mass squared term in the Higgs potential the Higgs field gets a VEV,

$$
\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad \mu_H^2 = \lambda v^2. \tag{4.13}
$$

This generates mass terms for W_μ^i and B_μ and a field rotation is needed to diagonalize the mass matrix. The mass eigenstates are defined to the electroweak vector fields by

$$
\begin{aligned}
W_\mu^1 &= \frac{1}{\sqrt{2}} \left(W_\mu^+ + W_\mu^- \right), & W_\mu^3 &= \frac{1}{\sqrt{g_L^2 + g_Y^2}} \left(g_L Z_\mu + g_Y A_\mu \right), \\
W_\mu^2 &= \frac{i}{\sqrt{2}} \left(W_\mu^+ - W_\mu^- \right), & B_\mu &= \frac{1}{\sqrt{g_L^2 + g_Y^2}} \left(-g_Y Z_\mu + g_L A_\mu \right).
\end{aligned} \tag{4.14}
$$

The mass terms W and Z boson in the SM Lagrangian are expressed by the electroweak parameters as

$$
m_W = \frac{g_L v}{2}, \qquad m_Z = \frac{\sqrt{g_L^2 + g_Y^2}\, v}{2}. \tag{4.15}
$$

In the presence of $D=6$ operators there may be new contributions to W and Z mass terms, as discussed later in this chapter.

The SM fermions (except for the neutrinos) also acquire mass terms after electroweak symmetry breaking

$$
\mathcal{L}_{\text{fermion mass}} = -u_I^c [M_u]_{IJ} u_J - d_I^c [M_d]_{IJ} d_J - e_I^c [M_e]_{IJ} e_J + \text{h.c.} \tag{4.16}
$$

In the SM, these mass terms are produced by the Yukawa interactions in Eq. (4.7): $M_f = \frac{v}{\sqrt{2}} y_f$, and can be non-diagonal in general. If that is a case, a set of rotations is needed to bring the mass terms to the diagonal form:

$$
u \to L_u u, \quad u^c \to u^c R_u^\dagger, \quad d \to L_d d, \quad d^c \to d^c R_d^\dagger, \quad e \to L_e e, \quad e^c \to e^c R_e^\dagger, \tag{4.17}
$$

where each L_i and R_i is a unitary 3×3 matrix such that $R_f^\dagger M_f L_f = \text{diag}(m_{f_I})$. In the SM, the observable effects of these rotations are described by the unitary Cabbibo-Kobayashi-Maskawa (CKM) matrix $V_{\text{CKM}} = L_u^\dagger L_d$, which controls flavor changing charged currents in the quark sector. The remaining effects of the rotations cancel out in the SM mass eigenstate Lagrangian. Again, $D=6$ operators may contribute to the fermion mass terms in Eq. (4.16), which also affects the unitary rotations to the mass eigenstate basis. From now on I will assume the fermion fields are written in the mass eigenstate basis *after* all contribution from $D=6$ operators are taken into account. In that basis, the effects of the rotations are absorbed in the Lagrangian parameters and in the definition of the quark doublet fields: $q_I = (u_I, [V_{\text{CKM}}]_{IJ} d_J)$.

Finally, I discuss the degrees of freedom in the Higgs sector. Once the Higgs field gets a VEV as in Eq. (4.13), it can be conveniently parametrized as

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} i\sqrt{2}G_+ \\ v + h + iG_z \end{pmatrix}. \tag{4.18}$$

The fields G_z and $G_\pm$ do not correspond to new physical degrees of freedom (they kinetically mix with the massive gauge bosons and can be gauged away). From now on I will work in the unitary gauge and set $G_\pm = 0 = G_z$. The scalar field h corresponds to a scalar particle called *the Higgs boson*. Its mass can be expressed by the parameters of the Higgs potential as

$$m_h^2 = 2\mu_H^2 = 2\lambda v^2. \tag{4.19}$$

4.2 Higher-dimensional operators

I start with discussing the dimension-5 operators in the SM EFT Lagrangian Eq. (4.1). They are of the form

$$[O_5]_{IJ} = (\epsilon_{ij} H^i L_I^j)(\epsilon_{kl} H^i L_J^j), \tag{20}$$

where $I, J = 1, 2, 3$ are the generation (flavor) indices, Clearly, these operators violate the lepton number (and B-L). The most important effect of the operators in Eq. (20) is the appearance of Majorana-type neutrino masses after electroweak symmetry breaking:

$$\frac{1}{\Lambda_L}[c_5]_{IJ}[O_5]_{IJ} \to +\frac{v^2}{2\Lambda_L}[c_5]_{IJ}\nu_I\nu_J. \tag{21}$$

The SM neutrino masses are bound to be below eV, and from the oscillation experiments tell us that at least two masses are non-zero, and at least one must be equal or larger than 0.06 eV. From that we deduce that $\Lambda_L/c_5 \gtrsim 10^{15}$ GeV for all eigenvalues of the matrix c. The large scale suppressing the dimension-5 operators in the SM EFT Lagrangian makes them practically unobservable in collider experiments.

Because there is an approximate symmetry protecting the dimension-5 operators, it makes sense to assume that the scales Λ_L and Λ in Eq. (4.1) are vastly different, and loop corrections will not destabilize that. From now on I assume the following hierarchy of scales:

$$v \ll \Lambda, \qquad \Lambda^2 \ll v\Lambda_L. \tag{22}$$

As a consequence, the contributions of the odd-D operators to low-energy observables that conserve B or L will be subdominant with respect to that of the even-D operators.

Given Eq. (22), the leading contributions to collider observables are expected from $D = 6$ operators suppressed by Λ^2. The scale Λ is identified with the mass scale of heavy particles in the B and L conserving UV theory that completes the SM. The importance of dimension-6 operators for characterizing low-energy effects of heavy particles has been recognized long ago, see e.g. [32, 33]. More recently, advantages of using a complete and non-redundant set of operators have been emphasized. The point is that seemingly different higher-dimensional operators can lead to the same S-matrix elements describing scattering of the SM particles. This is the case if the operators can be related by using equations of motion, integration by parts, field redefinitions, or Fierz transformations. This leads to non-trivial and often counter-intuitive relations between operators. For

	Bosonic CP-even		Bosonic CP-odd
O_H	$(H^\dagger H)^3$		
$O_{H\square}$	$(H^\dagger H)\square(H^\dagger H)$		
O_{HD}	$\left|H^\dagger D_\mu H\right|^2$		
O_{HG}	$H^\dagger H\, G^a_{\mu\nu}G^a_{\mu\nu}$	$O_{H\widetilde{G}}$	$H^\dagger H\, \widetilde{G}^a_{\mu\nu}G^a_{\mu\nu}$
O_{HW}	$H^\dagger H\, W^i_{\mu\nu}W^i_{\mu\nu}$	$O_{H\widetilde{W}}$	$H^\dagger H\, \widetilde{W}^i_{\mu\nu}W^i_{\mu\nu}$
O_{HB}	$H^\dagger H\, B_{\mu\nu}B_{\mu\nu}$	$O_{H\widetilde{B}}$	$H^\dagger H\, \widetilde{B}_{\mu\nu}B_{\mu\nu}$
O_{HWB}	$H^\dagger \sigma^i H\, W^i_{\mu\nu}B_{\mu\nu}$	$O_{H\widetilde{W}B}$	$H^\dagger \sigma^i H\, \widetilde{W}^i_{\mu\nu}B_{\mu\nu}$
O_W	$\epsilon^{ijk}W^i_{\mu\nu}W^j_{\nu\rho}W^k_{\rho\mu}$	$O_{\widetilde{W}}$	$\epsilon^{ijk}\widetilde{W}^i_{\mu\nu}W^j_{\nu\rho}W^k_{\rho\mu}$
O_G	$f^{abc}G^a_{\mu\nu}G^b_{\nu\rho}G^c_{\rho\mu}$	$O_{\widetilde{G}}$	$f^{abc}\widetilde{G}^a_{\mu\nu}G^b_{\nu\rho}G^c_{\rho\mu}$

Table 2: Bosonic D=6 operators in the Warsaw basis.

example, by using equations of motion one can establish equivalence between certain purely bosonic operators and a linear combination of 2- and 4-fermion operators! Thus, starting from the set of all distinct D=6 operators that can be constructed from the SM fields, a number of these operators will be redundant as they are equivalent to linear combinations of other operators. The redundant operators can be removed to simplify the EFT description and to establish an unambiguous map from observables to the EFT Wilson coefficients. A minimal, non-redundant set of operators is called a *basis*. There is infinitely many equivalent bases, and for dimension-6 operators several distinct constructions are frequently used in the literature. Furthermore, bases of operators have been constructed for D=7 [34, 35], and for D=8 [36, 35] (for 1 generation of fermions), but I will not discuss them here.

Because of a humungous number of D=6 operators, and because establishing equivalence between operators may be time consuming, identifying a basis is not a trivial task. Indeed, a complete construction for one generation of SM fermions was accomplished only a few years ago in Ref. [37], and than extended to 3 generations in Ref. [38]. This construction is referred to as the *Warsaw basis*, and is often used in the literature. One important conclusion from this exercise is that a complete basis of D=6 operators is characterized by 2499 independent parameters [38]. Note that construction of one complete basis is a qualitative tour de force, as now any other basis can be systematically derived by transforming operators from the Warsaw basis. Another popular choice of operators is the so-called *SILH basis*[13] which was proposed in Ref. [39] and completed in Ref. [40]. Finally, Ref. [41] proposed a slightly different (but fully equivalent) way to parametrize the space of D=6 operators using a subset of couplings characterizing the interactions of mass eigenstates in the effective Lagrangian. It should be stressed that any complete basis leads to equivalent predictions concerning possible new contri-

---

[13]SILH stands for Strongly Interacting Light Higgs, because this operator basis is more convenient to describe low energy effects of strongly interacting BSM sectors from which the Higgs doublet emerges as a light composite state.

	Yukawa
$[O_{eH}^\dagger]_{IJ}$	$H^\dagger H e_I^c H^\dagger \ell_J$
$[O_{uH}^\dagger]_{IJ}$	$H^\dagger H u_I^c \widetilde{H}^\dagger q_J$
$[O_{dH}^\dagger]_{IJ}$	$H^\dagger H d_I^c H^\dagger q_J$

	Vertex		Dipole
$[O_{H\ell}]_{IJ}$	$i\bar\ell_I \bar\sigma_\mu \ell_J H^\dagger \overleftrightarrow{D}_\mu H$	$[O_{eW}^\dagger]_{IJ}$	$e_I^c \sigma_{\mu\nu} H^\dagger \sigma^i \ell_J W_{\mu\nu}^i$
$[O_{H\ell}^{(3)}]_{IJ}$	$i\bar\ell_I \sigma^i \bar\sigma_\mu \ell_J H^\dagger \sigma^i \overleftrightarrow{D}_\mu H$	$[O_{eB}^\dagger]_{IJ}$	$e_I^c \sigma_{\mu\nu} H^\dagger \ell_J B_{\mu\nu}$
$[O_{He}]_{IJ}$	$ie_I^c \sigma_\mu \bar e_J^c H^\dagger \overleftrightarrow{D}_\mu H$	$[O_{uG}^\dagger]_{IJ}$	$u_I^c \sigma_{\mu\nu} T^a \widetilde{H}^\dagger q_J G_{\mu\nu}^a$
$[O_{Hq}]_{IJ}$	$i\bar q_I \bar\sigma_\mu q_J H^\dagger \overleftrightarrow{D}_\mu H$	$[O_{uW}^\dagger]_{IJ}$	$u_I^c \sigma_{\mu\nu} \widetilde{H}^\dagger \sigma^i q_J W_{\mu\nu}^i$
$[O_{Hq}^{(3)}]_{IJ}$	$i\bar q_I \sigma^i \bar\sigma_\mu q_J H^\dagger \sigma^i \overleftrightarrow{D}_\mu H$	$[O_{uB}^\dagger]_{IJ}$	$u_I^c \sigma_{\mu\nu} \widetilde{H}^\dagger q_J B_{\mu\nu}$
$[O_{Hu}]_{IJ}$	$iu_I^c \sigma_\mu \bar u_J^c H^\dagger \overleftrightarrow{D}_\mu H$	$[O_{dG}^\dagger]_{IJ}$	$d_I^c \sigma_{\mu\nu} T^a H^\dagger q_J G_{\mu\nu}^a$
$[O_{Hd}]_{IJ}$	$id_I^c \sigma_\mu \bar d_J^c H^\dagger \overleftrightarrow{D}_\mu H$	$[O_{dW}^\dagger]_{IJ}$	$d_I^c \sigma_{\mu\nu} \bar H^\dagger \sigma^i q_J W_{\mu\nu}^i$
$[O_{Hud}]_{IJ}$	$iu_I^c \sigma_\mu \bar d_J^c \tilde H^\dagger D_\mu H$	$[O_{dB}^\dagger]_{IJ}$	$d_I^c \sigma_{\mu\nu} H^\dagger q_J B_{\mu\nu}$

Table 3: Two-fermion $D=6$ operators in the Warsaw basis. The flavor indices are denoted by I, J. For complex operators (O_{Hud} and all Yukawa and dipole operators) the corresponding complex conjugate operator is implicitly included.

butions to physical observables. Nevertheless, working with a particular basis may be more convenient for specific applications.

The full list of dimension-6 operators in the Warsaw basis is given in Tables 2, 3, and 4. It takes a moment to realize that not all gauge invariant operators that one can construct from the SM fields are present there. This is because the redundant operators have already been removed. Let us see how this works in practice, taking the example of the dimension-6 operator $O'_{HD} \equiv (H^\dagger H) D_\mu H^\dagger D_\mu H$. First, integrating by parts we obtain

$$O'_{HD} = \frac{1}{2} H^\dagger H \left(\Box(H^\dagger H) - H^\dagger D_\mu D_\mu H - D_\mu D_\mu H^\dagger H \right) \tag{23}$$

Now we can use the Higgs equations of motion in Eq. (4.10) to get rid of the last two terms in the bracket

$$O'_{HD} = -\mu_H^2 (H^\dagger H)^2 + \frac{1}{2}(H^\dagger H)\Box(H^\dagger H) + 2\lambda(H^\dagger H)^3 + \frac{1}{2} H^\dagger H \left[-\bar u^c y_u^\dagger \tilde q + d^c y_d q + e^c y_e \ell + \text{h.c.} \right]. \tag{24}$$

At this point all the operators on the right-hand side above belong to the Warsaw basis, which proves that O'_{HD} is redundant. Note that many different Warsaw basis operators must be simultaneously present in a specific linear combination to exactly reproduce the effect of O'_{HD}. Repeating similar calculations one can prove that any gauge invariant operator can expressed by the ones in the Warsaw basis [37]. A more systematic method of constructing a basis is possible using the Hilbert series techniques [36, 35].

	$(\bar{R}R)(\bar{R}R)$		$(\bar{L}L)(\bar{R}R)$
O_{ee}	$\eta(e^c\sigma_\mu\bar{e}^c)(e^c\sigma_\mu\bar{e}^c)$	$O_{\ell e}$	$(\bar{\ell}\bar{\sigma}_\mu\ell)(e^c\sigma_\mu\bar{e}^c)$
O_{uu}	$\eta(u^c\sigma_\mu\bar{u}^c)(u^c\sigma_\mu\bar{u}^c)$	$O_{\ell u}$	$(\bar{\ell}\bar{\sigma}_\mu\ell)(u^c\sigma_\mu\bar{u}^c)$
O_{dd}	$\eta(d^c\sigma_\mu\bar{d}^c)(d^c\sigma_\mu\bar{d}^c)$	$O_{\ell d}$	$(\bar{\ell}\bar{\sigma}_\mu\ell)(d^c\sigma_\mu\bar{d}^c)$
O_{eu}	$(e^c\sigma_\mu\bar{e}^c)(u^c\sigma_\mu\bar{u}^c)$	O_{eq}	$(e^c\sigma_\mu\bar{e}^c)(\bar{q}\bar{\sigma}_\mu q)$
O_{ed}	$(e^c\sigma_\mu\bar{e}^c)(d^c\sigma_\mu\bar{d}^c)$	O_{qu}	$(\bar{q}\bar{\sigma}_\mu q)(u^c\sigma_\mu\bar{u}^c)$
O_{ud}	$(u^c\sigma_\mu\bar{u}^c)(d^c\sigma_\mu\bar{d}^c)$	$O_{qu}^{(8)}$	$(\bar{q}\bar{\sigma}_\mu T^a q)(u^c\sigma_\mu T^a\bar{u}^c)$
$O_{ud}^{(8)}$	$(u^c\sigma_\mu T^a\bar{u}^c)(d^c\sigma_\mu T^a\bar{d}^c)$	O_{qd}	$(\bar{q}\bar{\sigma}_\mu q)(d^c\sigma_\mu\bar{d}^c)$
		$O_{qd}^{(8)}$	$(\bar{q}\bar{\sigma}_\mu T^a q)(d^c\sigma_\mu T^a\bar{d}^c)$

	$(\bar{L}L)(\bar{L}L)$		$(\bar{L}R)(\bar{L}R)$
$O_{\ell\ell}$	$\eta(\bar{\ell}\bar{\sigma}_\mu\ell)(\bar{\ell}\bar{\sigma}_\mu\ell)$	O_{quqd}	$(u^c q^j)\epsilon_{jk}(d^c q^k)$
O_{qq}	$\eta(\bar{q}\bar{\sigma}_\mu q)(\bar{q}\bar{\sigma}_\mu q)$	$O_{quqd}^{(8)}$	$(u^c T^a q^j)\epsilon_{jk}(d^c T^a q^k)$
O_{qq}'	$\eta(\bar{q}\bar{\sigma}_\mu\sigma^i q)(\bar{q}\bar{\sigma}_\mu\sigma^i q)$	$O_{\ell equ}$	$(\bar{\ell}^j\bar{e}^c)\epsilon_{jk}(\bar{q}^k\bar{u}^c)$
$O_{\ell q}$	$(\bar{\ell}\bar{\sigma}_\mu\ell)(\bar{q}\bar{\sigma}_\mu q)$	$O_{\ell equ}^{(3)}$	$(\bar{\ell}^j\bar{\sigma}_{\mu\nu}\bar{e}^c)\epsilon_{jk}(\bar{q}^k\bar{\sigma}^{\mu\nu}u^c)$
$O_{\ell q}'$	$(\bar{\ell}\bar{\sigma}_\mu\sigma^i\ell)(\bar{q}\bar{\sigma}_\mu\sigma^i q)$	$O_{\ell edq}$	$(\bar{\ell}\bar{e}^c)(d^c q)$

Table 4: Four-fermion $D{=}6$ operators in the Warsaw basis. Flavor indices are suppressed here to reduce the clutter. The factor η is equal to $1/2$ when all flavor indices are equal (e.g. in $[O_{ee}]_{1111}$), and $\eta = 1$ otherwise. For each complex operator the complex conjugate is included in the Lagrangian and the corresponding Wilson coefficient is complex.

Exercise: Express the following dimension-6 operators

$$H^\dagger\sigma^i D_\mu H D_\nu W_{\mu\nu}^i,$$
$$B_{\mu\nu}D_\mu H^\dagger D_\nu H,$$
$$(\bar{\ell}\bar{\sigma}_\mu\sigma^i\ell)(\bar{\ell}\bar{\sigma}_\mu\sigma^i\ell),$$

by the operators in the Warsaw basis.

In general, the Wilson coefficients of the higher-dimensional operators are free parameters which are only subject to experimental constraints. However, given some broad assumptions about the UV physics, it is possible to make useful estimates. First of all, we can use the $\hbar$ counting to determine how the Wilson coefficients depend on the couplings strength g_* of new physics to the SM. Assuming there is only one relevant g_* and

one mass scale Λ in the UV theory, we can write down the following estimates:

$$
\begin{aligned}
O_H = |H|^6 : &\qquad c_H \sim g_*^4, \\
O_{eH} = |H|^2 \bar{\ell} H e_c : &\qquad c_{eH} \sim g_*^3, \\
O_{H\Box} = |H|^2 \Box |H|^2 : &\qquad c_H \sim g_*^2, \\
O_W = \epsilon_{ijk} W_{\mu\nu}^i W_{\nu\rho}^j W_{\rho\mu}^k : &\qquad c_W \sim g_*, \\
&\cdots
\end{aligned}
\tag{25}
$$

These estimates may be very useful to guess which operators are the most important ones in some scenarios, but as it stands they can be misleading too. Let us assume a strongly coupled UV completion of the SM, where the Higgs is coupled to new resonances with the strength $g_* \gg 1$. Then Eq. (25) suggests that O_H should come with a huge numerical coefficient, which could even be $\mathcal{O}(10^4)$! However, the same UV physics would also contribute to the SM quartic Higgs interaction $\lambda |H|^4$ such that $\lambda \sim g_*^2$. But we know that $\lambda \sim 0.1$ from the measurement of the Higgs boson mass. Barring fine-tuning, the UV theory should have some protection mechanism (e.g. an approximate shift symmetry) to ensure that, the contributions to the Higgs potential in the EFT are proportional to λ. This would lead to an additional selection rule in the EFT, and we conclude

$$
O_H = |H|^6 : \qquad c_H \sim \lambda g_*^2.
\tag{26}
$$

Now, the strongly coupled scenario could lead only to $c_H \lesssim 10$ (which would still be wonderful). By the same token, we expect the new physics to have an order parameter to protect the SM flavor structure. This would lead to a selection rule that chirality violating operators should be accompanied by the corresponding Yukawa coupling:

$$
O_{eH} = |H|^2 \bar{\ell} H e_c : \qquad c_{eH} \sim y_e g_*^2.
\tag{27}
$$

The scaling for c_W is naive for another reason. If the $SU(2)_L$ gauge bosons are fundamental, any amplitude with n external W bosons should be accompanied by n powers of the $SU(2)_L$ gauge coupling g_L. From the $\hbar$ coupling it then immediately follows that O_W cannot be generated at tree level. These consideration change the scaling as

$$
O_W = \epsilon_{ijk} W_{\mu\nu}^i W_{\nu\rho}^j W_{\rho\mu}^k : \qquad c_W \sim \frac{g_L^3}{16\pi^2},
\tag{28}
$$

which makes O_W difficult to observe in practice, even for $\Lambda \sim 1$ TeV, as its Wilson coefficient is $c_W \lesssim 10^{-3}$.

4.3 From BSM to operators

We use the dimension-6 operators as a prop to parametrize the effects of heavy BSM particles on weak-scale observables. Suppose that one day it will be demonstrated that some linear combination of the higher-dimensional operators must indeed be present in the SM EFT Lagrangian to correctly account for all experimental results. What will this tell us about the new physics? In order to address this question the best way is to go through a number of examples that relate dimension-6 operators to the coupling and masses in BSM models.

Example no. 1 is akin to the Fermi theory. Consider a heavy neutral vector boson V_μ with mass M_V coupled to the SM fermionic currents:

$$\mathcal{L}_{\text{UV}} \supset V_\mu \left(g_{Xf,L} \bar{f} \bar{\sigma}_\mu f + g_{Xf,R} f_c \sigma_\mu \bar{f}_c \right). \tag{29}$$

In this theory the fermions interact by exchanging V in addition to the SM gauge bosons. For energies well below m_V one can ignore the momentum dependence in the V propagator, which effectively leads to contact interaction between the fermion currents. In the effective theory below m_V this corresponds to the dimension-6 operators in the Lagrangian:

$$\mathcal{L}_{\text{EFT}} \supset -\frac{1}{2M_V^2} \left(g_{Vf,L} \bar{f} \bar{\sigma}_\mu f + g_{Vf,R} f_c \sigma_\mu \bar{f}_c \right)^2 \tag{30}$$

Thus, heavy neutral vectors in the UV theory lead to the appearance of a linear combination of 4-fermion operators in the SM EFT. All the operators generated in our example can be straightforwardly matched to the Warsaw basis 4-fermion operators displayed in Table 4. The Wilson coefficients are matched as $\frac{c_{f_1 f_2}}{\Lambda^2} = -\frac{g_{Vf_1} g_{Xf_2}}{M_V^2}$. Note that low-energy experiments probe only c/Λ^2, thus they can only determine the ratio of the BSM couplings and masses, and not separately about the two. Still, pinpointing a dimension-6 operators set the upper limit on the mass scale of new physics, as the couplings cannot be larger than 4π.

Vector bosons can also couple to the Higgs current. As *example no. 2* consider a complex vector boson X_μ with mass M_X coupled to the Higgs as

$$\mathcal{L}_{\text{UV}} \supset g_X X_\mu H^\dagger D_\mu H + \text{h.c.} \tag{31}$$

This sort of coupling is characteristic of composite Higgs models, where the Higgs is a composite particle made of quark-like partons charged under new strong interactions, and X_μ is a ρ-meson-like vector resonance in the strong sector. Below the scale of the resonance the X exchange can be approximated by the contact interaction between 4 Higgs fields:

$$\mathcal{L}_{\text{EFT}} \supset -\frac{g_X^2}{M_X^2} |H^\dagger D_\mu H|^2 \tag{32}$$

Exercise: Consider a triplet of heavy real vector bosons V_μ^i coupled to the $SU(2)_L$ Higgs current as

$$\mathcal{L} \supset g_V V_\mu^i H^\dagger \sigma^i D_\mu H \tag{33}$$

Which dimension-6 operator mimics the effect of V_μ^i exchange at energies below M_V? Express this operator by the ones present in the Warsaw basis.

In both examples above, the SM EFT operators approximate tree-level exchange of heavy particles. But loop corrections in the UV theory may also be mimicked by dimension-6 operators. Some of the Warsaw basis operators can be generated *only* at loop level if the UV theory is renormalizable. For the example, this is the case of the O_{HG} operators in Table 2. Consider *example no. 3* where we consider a colored scalar particle $\tilde{t}_c$ with mass M_T and the quantum numbers of the SM right-handed top. The interactions in the UV theory include

$$\mathcal{L}_{\text{UV}} \supset -y_T H^\dagger H \tilde{t}_c^\dagger \tilde{t}_c. \tag{34}$$

46

This sort of structure is characteristic for supersymmetric models where the scalar play a central role in restoring the naturalness of electroweak symmetry breaking. At tree-level the new scalar does not affect the scattering amplitudes of the SM fields in any way. However it does contribute at the 1-loop level to certain processes. One of them is the $gg \to h$ production where it enters via the triangle and bubble diagrams. That contribution can be reproduced in the SM EFT *at tree level* by the operator O_{HG} with the Wilson coefficient

$$\frac{c_{HG}}{\Lambda^2} = \frac{y_T g_s^2}{256\pi^2 M_T^2}. \tag{35}$$

4.4 From operators to observables

In order to understand the phenomenological effects of higher-dimensional operators SM EFT it is convenient to leave the $SU(3) \times SU(2) \times U(1)$ invariant formulation and instead study the interaction terms of the mass eigenstates after electroweak symmetry breaking. There are 2 ways in which the higher-dimensional operators can lead to deviations from the SM interactions:

1. **Modified couplings:** corrections to the coupling strength for the SM-like interactions.

2. **New vertices:** New interaction terms appear that are not present in the SM Lagrangian.

One example leading to the modified couplings is the operator $O_{He} = ie_c\sigma_\mu \bar{e}_c(H^\dagger D_\mu H - D_\mu H^\dagger H)$ in Table 3. Inserting the Higgs VEV, it leads to a coupling of the Z-boson to right-handed electrons:

$$\frac{c_{He}}{\Lambda^2} O_{He} \to -\frac{c_{He}\sqrt{g_L^2 + g_Y^2}v^2}{2\Lambda^2} Z_\mu e_c\sigma_\mu \bar{e}_c. \tag{36}$$

In the SM, the couplings strengths of the fermions to the Z boson are fully specified by the electroweak gauge couplings g_L, g_Y and the fermion's quantum numbers: $g_{Zf} = \sqrt{g_L^2 + g_Y^2}(T_f^3 - s_\theta^2 Q_f)$. In particular, for the right-handed electron $g_{Ze} = \sqrt{g_L^2 + g_Y^2}s_\theta^2$. The effect of O_{He} is to shift the interaction strength by the amount $\Delta g_{Ze} = -\frac{c_{He}\sqrt{g_L^2+g_Y^2}v^2}{2\Lambda^2}$ which is a-priori independent of the fermion's quantum numbers. The effect can be measured (or constrained) e.g. by precision studies of the process $e^+e^- \to Z \to e^+e^-$ process, as was done in the previous century by the LEP collider at CERN.

The same operator O_{He} provides also an example for the new vertices. Namely, it also leads to the Higgs boson couplings to two right-handed electrons and one Z boson:

$$\frac{c_{He}}{\Lambda^2} O_{He} \to -\frac{c_{He}\sqrt{g_L^2 + g_Y^2}v}{2\Lambda^2} hZ_\mu e_c\sigma_\mu \bar{e}_c. \tag{37}$$

Such coupling is not present in the SM Lagrangian (in the SM, h always couples exactly 2 other fields). Thus, the above terms leads to a new vertex that should be included when calculating the amplitudes for Higgs processes. Physically, the new vertex affects the decay width and differential distributions for the Higgs boson decay to 4 leptons, which is routinely studied by the LHC collider.

It is important to mention that sometimes there is no invariant way to separate the modified couplings and new vertices. Much as for the dimension-6 operators, we can always redefine the couplings and fields in the mass eigenstate Lagrangian so as to reshuffle the interaction terms and rewrite the Lagrangian in a different albeit equivalent form. There is even more freedom than before as, once electroweak symmetry is broken, there is nothing unnatural in redefining separately then Z, W, and photon fields (while in the unbroken phase one usually redefines $SU(2) \times U(1)$ gauge fields only in order to maintain manifest gauge invariance). Let me give one example. We consider the following interactions of the Higgs boson h:

$$\mathcal{L} \supset \frac{1}{2}(\partial_\mu h)^2 - \frac{m_h^2}{2}h^2 - \frac{m_h^2}{2v}\left(1 + \delta_1 \frac{v^2}{\Lambda^2}\right)h^3 - \delta_2 \frac{v}{\Lambda^2}h\partial_\mu h\partial_\mu h + \dots \tag{38}$$

where the dots denote terms with 4 and more Higgs bosons, which are not of interest in this discussion. Two effects of dimension-6 operators appear in the Lagrangian above. The one proportional to δ_1 is the modification of the triple Higgs self-coupling, which is already present in the SM Lagrangian but with the magnitude strictly fixed by the Higgs boson mass. The other effect proportional to δ_2 is a new Higgs self-interaction terms which does not appear in the SM Lagrangian in its canonical form. Both of these corrections can be generated by the dimension-6 operators in the Warsaw basis, e.g. by O_H and $O_{H\square}$ in Table 2. Both contribute in a non-trivial way to the Higgs scattering amplitudes, for example to $hh \to hh$, or to double Higgs production at the LHC, $pp \to hh$, once interactions of h with the rest of the SM are taken into account.

Nevertheless, we can equivalently work with an effective Lagrangian where the 2-derivative $h(\partial_\mu h)^2$ interaction is completely eliminated via field redefinitions. To this end we redefine the Higgs boson field as

$$h \to h + \delta_2 \frac{v}{2\Lambda^2}h^2. \tag{39}$$

After this redefinition the effective Lagrangian of Eq. (38) takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 - \frac{m_h^2}{2}h^2 - \frac{m_h^2}{2v}\left(1 + (\delta_1 + \delta_2)\frac{v^2}{\Lambda^2}\right)h^3 + \dots, \tag{40}$$

where I ignored $\mathcal{O}(\Lambda^{-4})$ resulting from the redefinition. Seemingly, the Lagrangians in Eqs. (38) and (40) are different, as they contain different interaction terms. However, the equivalence theorem makes sure that field redefinitions cannot change the physical content of the theory. This way, the two Lagrangians give exactly the same predictions for physical observables, at any order in the perturbation theory, which one can always verify by an explicit calculation. Of course, the effect of the 2-derivative interaction proportional to δ_2 never vanishes, but just turns out to be equivalent to a shift of the SM-like triple Higgs coupling. When the SM interactions of h with the W, Z and fermions are taken into account, one should also include the shift of the corresponding coupling due to Eq. (39).

The redefinitions like the one in Eq. (39) allow one to bring the SM EFT mass eigenstate Lagrangian to a more convenient form where the calculations are somewhat simplified. One particular construction can be found in Refs. [30, 42].

Let me now turn to another topic which is very relevant for connecting the SM EFT operators to precision observables. There are important but less-than-obvious effects of the higher-dimensional operators which stem from shifting the SM input parameters. In the SM the values of the electroweak parameters g_L, g_Y, and v are customarily extracted from 3 precisely measured observables:

1. The *Fermi constant G_F* which is extracted from the muon lifetime,

2. The *electromagnetic structure constant $\alpha(0)$*, which parametrizes the interaction strength of charged particles in the deep Thomson limit where the exchanged 4-momentum approach zero,

3. The Z boson mass $m_Z^2(m_Z)$, which parametrizes the real part of the pole of the Z-boson propagator.

In the SM at tree level these input observables are related to the SM parameters as

$$\sqrt{2}G_F = \frac{1}{v^2}, \quad \alpha = \frac{g_L^2 g_Y^2}{4\pi(g_L^2 + g_Y^2)}, \quad m_Z^2 = \frac{(g_L^2 + g_Y^2)v^2}{4}. \tag{41}$$

Beyond the SM, already at tree level these observables may depend on other parameters of the theory.[14] For example, the Warsaw basis operator O_{HD} contributes to the to the Z boson mass after electroweak symmetry breaking:

$$\frac{c_{HD}}{\Lambda^2}|H^\dagger D_\mu H|^2 \rightarrow \frac{c_{HD}v^2}{2\Lambda^2}\frac{(g_L^2 + g_Y^2)v^2}{8}Z_\mu Z_\mu. \tag{42}$$

Thus, the Z boson mass term in the Lagrangian is affected by the dimension-6 operators. The observable $m_Z^2(m_Z)$ was measured at the LEP collider with the order 0.01% precision: $m_Z^{\exp} = 91.1876 \pm 0.0021$ GeV. It would be however wrong to conclude that the Wilson coefficient c_{HD} is constrained with that precision. The point is that the m_Z measurement is used to determine the electroweak parameters g_L, g_Y, v, and at this point c_{HD} just complicates that determination. Before we go on with constraining c_{HD}, we must first disentangle it from the SM input parameters.

It pays off to make this discussion a little bit more general than we need for the SM EFT sake. Let us parametrize the Fourier-transformed kinetic terms of the gauge boson as

$$\eta_{\mu\nu}\left(\Pi_{WW}(p^2)W_\mu^+ W_\mu^- + \frac{1}{2}\Pi_{ZZ}(p^2)Z_\mu Z_\mu + \frac{1}{2}\Pi_{\gamma\gamma}(p^2)A_\mu A_\mu + \Pi_{Z\gamma}(p^2)Z_\mu A_\mu\right) + p_\mu p_\nu(\dots) \tag{43}$$

For a canonically normalized gauge field we have $\Pi_{VV} = m_V^2 - p^2$. The form factors may represent SM loop corrections or BSM tree- or loop-level corrections. Next, we allow the couplings of the W and Z bosons to fermions to be shifted from the SM value. We

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[14]As a side remark, basically the same discussion applies to the SM itself beyond the tree-level, as the loop corrections affect the input observables such that they depend on other SM parameters and even on the regularization scheme.

parametrize the interactions between the electroweak gauge bosons and fermions as

$$\mathcal{L} \supset \frac{g_{L,0} g_{Y,0}}{\sqrt{g_{L,0}^2 + g_{Y,0}^2}} A_\mu \sum_f Q_f (\bar{e}_I \bar{\sigma}_\mu e_I + e_I^c \sigma_\mu \bar{e}_I^c)$$

$$+ \left[\frac{[g_L^{We}]_{IJ}}{\sqrt{2}} W_\mu^+ \bar{\nu}_I \bar{\sigma}_\mu e_J + W_\mu^+ \frac{[g_L^{Wq}]_{IJ}}{\sqrt{2}} \bar{u}_I \bar{\sigma}_\mu d_J + \frac{[g_R^{Wq}]_{IJ}}{\sqrt{2}} W_\mu^+ u_I^c \bar{\sigma}_\mu \bar{d}_J^c + \text{h.c.} \right]$$

$$+ Z_\mu \sum_{f=u,d,e,\nu} [g_L^{Zf}]_{IJ} \bar{f}_I \bar{\sigma}_\mu f_J + Z_\mu \sum_{f=u,d,e} [g_R^{Zf}]_{IJ} f_I^c \bar{\sigma}_\mu \bar{f}_J^c. \tag{44}$$

Here, $g_{L,0}$, $g_{Y,0}$ are the gauge couplings of the $SU(2)_L \times U(1)_Y$ local symmetry. The subscript zero is to mark that they are *not* related to the input observables by the relations in Eq. (41), even at tree level. The SM fermions f_J, f_J^c are in the basis where the mass terms are diagonal, and then the unitary CKM matrix V_0 enters through the quark doublets as $q_I = (u_I, V_{IJ,0} d_J)$. The gauge coupling strength are parametrized as

$$
\begin{aligned}
[g_L^{We}]_{IJ} &= g_{L,0} \left(\delta_{IJ} + [\delta g_L^{We}]_{IJ} \right), \\
[g_L^{Wq}]_{IJ} &= g_{L,0} \left([V]_{IJ,0} + [\delta g_L^{Wq}]_{IJ} \right), \\
[g_R^{Wq}]_{IJ} &= [\delta g_R^{Wq}]_{IJ} \\
[g^{Zf}]_{IJ} &= \sqrt{g_{L,0}^2 + g_{Y,0}^2} \left(T_3^f - Q_f \frac{g_{Y,0}^2}{g_{L,0}^2 + g_{Y,0}^2} + [\delta g^{Zf}]_{IJ} \right).
\end{aligned} \tag{45}
$$

In the SM limit all δg vanish, and then Eq. (41) is recovered at tree level. Finally, we also allow for new contributions to observables from 4-fermion operators.

Now I discuss how the observables G_F, $\alpha(0)$ and $m_Z^2(m_Z)$ are related at tree level to the parameters of such an extension of the SM. We have

$$
\begin{aligned}
2\sqrt{2} G_F &= \frac{g_L^{We} g_L^{W\mu}}{2\Pi_{WW}(0)} - [c_{\ell\ell}]_{1221} - 2[c_{\ell\ell}^{(3)}]_{1122}, \\
\alpha(0) &= \frac{g_{L,0}^2 g_{Y,0}^2}{4\pi(g_{L,0}^2 + g_{Y,0}^2)} \frac{-1}{\Pi'_{\gamma\gamma}(0)}, \\
m_Z^2(m_Z) &= \Pi_{ZZ}(m_Z^2).
\end{aligned} \tag{46}
$$

Here $[c_{\ell\ell}^{(3)}]_{IIJJ}/v^2$ multiplies the 4-fermion operator $(\bar{\ell}_I \bar{\sigma}_\mu \sigma^i \ell_I)(\bar{\ell}_J \bar{\sigma}_\mu \sigma^i \ell_J)$ in the Lagrangian.

Let us now assume that the propagators and couplings are close to the SM ones,

$$
\begin{aligned}
\Pi_{WW} &= \frac{g_{L,0}^2 v_0^2}{4} - p^2 + \delta\Pi_{WW}(p^2), \\
\Pi_{ZZ} &= \frac{(g_{L,0}^2 + g_{Y,0}^2) v_0^2}{4} - p^2 + \delta\Pi_{ZZ}(p^2), \\
\Pi_{\gamma\gamma} &= -p^2 + \delta\Pi_{\gamma\gamma}(p^2), \\
g_L^{Wf} &= g_{L,0}(1 + \delta g_L^{Wf}),
\end{aligned} \tag{47}
$$

and expand Eq. (46) to the linear order in the deformations:

$$
2\sqrt{2}G_F \approx \frac{2}{v_0^2}\left(1 - \frac{\delta\Pi_{WW}(0)}{m_W^2} + \delta g_L^{We} + \delta g_L^{W\mu} - \frac{1}{2}[c_{\ell\ell}]_{1221} - [c_{\ell\ell}^{(3)}]_{1122}\right),
$$

$$
\alpha(0) = \frac{g_{L,0}^2 g_{Y,0}^2}{4\pi(g_{L,0}^2 + g_{Y,0}^2)}\left(1 + \delta\Pi_{\gamma\gamma}'(0)\right),
$$

$$
m_Z^2(m_Z) = \frac{(g_{L,0}^2 + g_{Y,0}^2)v_0^2}{4} + \delta\Pi_{ZZ}(m_Z^2). \tag{48}
$$

This shows that in the presence of new physics the values of the parameters v_0, $g_{L,0}$, $g_{Y,0}$ have to be modified with respect to the SM. However, it is more convenient to absorb these corrections by redefining the parameters as

$$
v_0 = v(1 + \delta v), \qquad g_{L,0} = g_L(1 + \delta g_L), \qquad g_{Y,0} = g_Y(1 + \delta g_Y), \tag{49}
$$

such that v, g_L, g_Y satisfy Eq. (41). This is achieved by the following redefinition

$$
\delta v = \frac{1}{2}\left(-\frac{\delta\Pi_{WW}(0)}{m_W^2} + \delta g_L^{We} + \delta g_L^{W\mu} - \frac{1}{2}[c_{\ell\ell}]_{1221} - [c_{\ell\ell}^{(3)}]_{1122}\right),
$$

$$
\delta g_L = \frac{g_L^2}{4(g_L^2 - g_Y^2)v^2}\left[-\frac{2\delta\Pi_{ZZ}(m_Z^2)}{m_Z^2} + \frac{2\delta\Pi_{WW}(0)}{m_W^2} + \frac{2g_Y^2\delta\Pi_{\gamma\gamma}'(0)}{g_L^2}\right.
$$
$$
\left. + [c_{\ell\ell}]_{1221} + 2[c_{\ell\ell}^{(3)}]_{1122} - 2\delta g_L^{We} - 2\delta g_L^{W\mu}\right],
$$

$$
\delta g_Y = \frac{g_Y^2}{4(g_L^2 - g_Y^2)v^2}\left[\frac{2\delta\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{2\delta\Pi_{WW}(0)}{m_W^2} - \frac{2g_L^2\delta\Pi_{\gamma\gamma}'(0)}{g_Y^2}\right.
$$
$$
\left. - [c_{\ell\ell}]_{1221} - 2[c_{\ell\ell}^{(3)}]_{1122} + 2\delta g_L^{We} + 2\delta g_L^{W\mu}\right]. \tag{50}
$$

The rationale for this redefinition is that the observables on the left-hand side in Eq. (41) are measured with a fantastic precision (especially G_F and α but also m_Z is known an order of magnitude more precisely than typical electroweak observables). If we worked with the original parameters, we would have to add them to the fit together with the new physics parameters. Then G_F and α and m_Z would impose highly constraints on v_0, $g_{L,0}$, and $g_{Y,0}$ which would be highly correlated with the constraints on the new physics parameters. It is more transparent and more practical to do the redefinition in Eq. (49) and Eq. (50), assign the SM values to v, g_L, and g_Y, remove G_F, α, m_Z from the set of observables we fit to, and fit just the new physics parameters to other observables.

As an example, let us write down how this affect the W mass observable. In the original variables the observable is given by

$$
m_W^2(m_W) = \frac{g_{L,0}^2 v_0^2}{4} + \delta\Pi_{WW}(m_W^2). \tag{51}
$$

After the redefinition in Eq. (49) and Eq. (50) the formula for the W mass becomes

$$
m_W^2 = \frac{g_L^2 v^2}{4} + \frac{1}{g_L^2 - g_Y^2}\left(g_L^2\delta\Pi_{WW}(0) - \frac{g_L^4}{g_L^2 + g_Y^2}\delta\Pi_{ZZ}(m_Z^2) + g_Y^2 m_W^2\delta\Pi_{\gamma\gamma}'(0)\right) + \delta\Pi_{WW}(m_W^2). \tag{52}
$$

In the above, g_L, g_Y, v take the numerical values calculated from Eq. (41). Moreover, on the r.h.s. one can replace m_W and m_Z with their tree-level expressions, since the difference is higher order in the loop expansion. Note that 2-point function can be complex if light particles run in the loops (which is the case for light SM fermions); in all formulas here and below $\mathrm{Re}\,\delta\Pi$ is implicit.

The formulas in Eq. (50) and Eq. (52) are valid for any BSM scenario. In particular, we can apply them to the SM EFT with dimension-6 operators. Let's come back to our example withe the the O_{HD} operator from the Warsaw basis. From Eq. (42), it shifts the Z propagator as $\delta\Pi_{ZZ} = \frac{c_{HD}v^2}{2\Lambda^2}m_Z^2$. Once we redefine away the contribution to the Z mass, c_{HD} will re-emerge as a shift of the W mass (and many other electroweak observables), even though the operator itself does not contribute directly to the W mass term in the Lagrangian. From Eq. (52) one obtains

$$\frac{\delta m_W}{m_W} = \frac{\delta m_W^2}{2m_W^2} = -\frac{c_{HD}g_L^2 v^2}{4(g_L^2 - g_Y^2)\Lambda^2}. \tag{53}$$

The recent experimental measurements yield $m_W^{\mathrm{exp}} = 80.385 \pm 0.015$ with the SM prediction being $m_W^{\mathrm{SM}} = 80.364 \pm 0.004$. This translates to $\frac{\delta m_W}{m_W} = (2.6 \pm 1.9)\cdot 10^{-4}$. Assuming O_{HD} is the only higher-dimensional operators present in the SM EFT Lagrangian we obtain the constraint on its Wilson coefficient:

$$\frac{c_{HD}}{\Lambda^2} = \frac{-1.2 \pm 0.9}{(10\ \mathrm{TeV})^2}. \tag{54}$$

Since $c_{HD} \sim g_*^2$, the W mass measurement probes weakly coupled ($g_* \sim 1$) new physics at the scales of order 10 TeV, and strongly coupled new physics ($g_* \sim 4\pi$) at the scales of of order 100 TeV. In general, electroweak precision measurements have a much better reach than direct searches at the LHC for some classes of new physics scenarios.

Exercise: Determine the phenomenological constraints on the Warsaw basis operator $O_{H\square}$ from the Higgs measurements at the LHC. Use the constraint for the overall Higgs signal strength $\frac{\sigma(pp\to h)}{\sigma(pp\to h)_{\mathrm{SM}}} = 1.09 \pm 0.11$. What is maximum mass scale of new physics that can be probed by LHC Higgs measurements?

A   Method of regions

To understand the method of regions it is instructive to first go through a simple example. Consider the dimensionally regularized integral with two disparate mass scales $m \ll M$,

$$I_0 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} \frac{1}{k^2 - M^2}. \tag{A.55}$$

It is trivial to calculate the integral exactly

$$I_0 = \frac{i}{16\pi^2} \left[\frac{1}{\bar{\epsilon}} + \frac{m^2}{M^2 - m^2} \log\left(\frac{m^2}{M^2}\right) + \log\left(\frac{\mu^2}{M^2}\right) + 1 \right]. \tag{A.56}$$

We can know expand the exact results in powers of $1/M$,

$$I_0 = \frac{i}{16\pi^2} \left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right) + 1 + \frac{m^2}{M^2} \log\left(\frac{m^2}{M^2}\right) \right] + \mathcal{O}(M^{-4}). \tag{A.57}$$

Let us go back to the original integral and expand the integrand in $1/M$ *before* doing the integral. Formally, we make that expansion assuming $k^2 \sim m^2 \ll M^2$ after rotating to Euclidean momenta, and then integrate term by term the resulting series. This leads to

$$
\begin{aligned}
I_{\text{soft}} &= -\frac{1}{M^2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2} + \mathcal{O}(M^{-4}) \\
&= -\frac{i}{16\pi^2} \frac{m^2}{M^2} \left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 1 \right] + \mathcal{O}(M^{-4}).
\end{aligned}
\tag{A.58}
$$

Clearly, $I_{\text{soft}} \neq I_0$; even the $\mathcal{O}(M^0)$ terms in I_0 are not reproduced in I_{soft}. This is because I_0 is actually dominated by $k^2 \sim M^2$. Let us then try to expand the integrand for $k^2 \sim M^2 \gg m^2$ and integrate terms by term:

$$
\begin{aligned}
I_{\text{hard}} &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \frac{1}{k^2 - M^2} + m^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4} \frac{1}{k^2 - M^2} + \mathcal{O}(M^{-4}) \\
&= \frac{i}{16\pi^2} \left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right) + 1 \right] + \frac{i}{16\pi^2} \frac{m^2}{M^2} \left[\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right) + 1 \right] + \mathcal{O}(M^{-4}).
\end{aligned}
\tag{A.59}
$$

Now the first term in I_{hard} correctly reproduces the $\mathcal{O}(M^0)$ part in I_0. Yet the $\mathcal{O}(M^{-2})$ parts do not quite match. Magically, however, the *sum* of I_{hard} and I_{soft} correctly reproduces the $\mathcal{O}(M^{-2})$ part of I_0 as well:

$$I_0 = I_{\text{soft}} + I_{\text{hard}}. \tag{A.60}$$

One can check that the above equality is in fact true to all orders in $1/M$. Notice that, for the second integral in I_{hard}, the pole is due to an IR singularity, which then cancels in the sum against a UV pole in I_{soft}.

More generally, when integrand contains two disparate mass scales $m \ll M$, the momentum integral can be calculated as a sum of two contributions: the soft one where the integrand is expanded for $k \sim m$, and the hard one with the integrand expanded for $k \sim M$. It is understood that, the original integral as well as soft and the hard parts are calculated using dimensional regularization, in case they are UV or IR divergent.

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