Neutrino oscillations

Idea first introduced by B. Pontecorvo

Sov. Phys. JETP 6 (1958) 429

Ibidem 7 (1958) 172.

Basic idea:

in SM interaction $\Leftrightarrow$ flavour state
propagation $\Leftrightarrow$ massive states

$\Updownarrow$ {MIXING}

SM interaction Lagrangian:

$$\mathcal{L}_W \propto \frac{g}{\sqrt{2}} \bar{e}^0_{\alpha L} \gamma^\mu \nu^0_{\alpha L} W^-_\mu + \text{neutral current}$$

Diagonalize the mass terms for e and ν:

$$e^0_L = V_L e_L$$
$$\nu^0_L = U_L \nu_L$$

$$\mathcal{L}_W \propto \frac{g}{\sqrt{2}} \bar{e}_{iL} \gamma^\mu \underbrace{(V_L^\dagger U_L)_{ij}}_{} \nu_{Lj} W^-_\mu + \text{neutral} \qquad + \text{h.c.}$$

lepton mixing matrix
unitary
It is the Pontecorvo - Maki - Nakagawa - Sakata matrix: U_{PMNS}

U_{PMNS} relates a neutrino produced or absorbed alongside a charged lepton with the mass eigenstate.

$$|\nu^0_{aL}\rangle = U^*_{ai}|\nu_i\rangle$$

Neutrino oscillations

$\boxed{\text{Vacuum case}}$

Let's assume that at time $t=0$, a ν with flavour a is produced

$$|\nu(0)\rangle = |\nu_a\rangle = U^*_{aj}|\nu_j\rangle$$

The evolution equation (Schroedinger equation) is given by:

$$i\frac{d}{dt}|\nu(t)\rangle = H|\nu(t)\rangle$$

$$\Downarrow$$

$$|\nu(t)\rangle = e^{-iHt}|\nu(0)\rangle$$

$$= \sum_j U^*_{aj}\, e^{-iHt}|\nu_j\rangle$$

$|\nu_a\rangle = U^*|\nu_j\rangle$

$$= \sum_j U^*_{aj}\, e^{-iE_jt}$$

The probability of having a ν with flavour b at time t:

$$P(\nu_a \to \nu_b; t) = \left|\langle\nu_b|\nu(t)\rangle\right|^2 = \left|\langle\nu_i|\, U_{bi}U^*_{aj}\, e^{-iE_jt}|\nu_j\rangle\right|^2$$

question

$\langle\nu_i|\nu_j\rangle = \delta_{ij}$

$$= \left|U_{bi}U^*_{ai}\, e^{-iE_it}\right|^2$$

Relativistic approximation: $E_i = \sqrt{p^2+m_i^2} \simeq p + \frac{m_i^2}{2p}$

Comment on principle of indetermination

2 neutrino case

$$C \equiv \cos\theta$$

$$S \equiv \sin\theta$$

$$U = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

$$|\nu_e\rangle = c|\nu_1\rangle + s|\nu_2\rangle$$

$$|\nu_\mu\rangle = -s|\nu_1\rangle + c|\nu_2\rangle$$

$$P(\nu_e \to \nu_\mu ; t) = \left| -cs\, e^{-i\left(p+\frac{m_1^2}{2p}\right)t} + cs\, e^{-i\left(p+\frac{m_2^2}{2p}\right)t} \right|^2$$

$$= c^2 s^2 \cancel{\left|e^{-ipt}\right|^2} \cancel{\left|e^{-i\frac{m_1^2}{2p}t}\right|^2} \left| -1 + e^{-i\frac{\Delta m^2}{2p}t} \right|^2$$

$$= c^2 s^2 \left(+1 + \cos^2 \frac{\Delta m^2}{2E} t - 2\cos\frac{\Delta m^2}{2E} t + \sin^2 \frac{\Delta m^2}{2E} t \right)$$

$$= 2c^2 s^2 \left(1 - \cos\frac{\Delta m^2}{2E} t \right)$$

$$= \sin^2 2\theta \;\; \sin^2 \frac{\Delta m^2}{4E} t$$

$\cos 2\alpha =$
$= 1 - 2\sin^2\alpha$

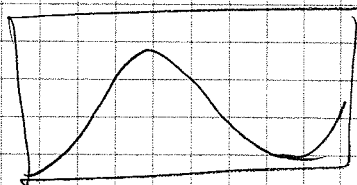
The oscillation length:

$$l_{osc} = \frac{4\pi E}{\Delta m^2} = 2.48\ \text{km}\ \frac{E\,(\text{GeV})}{\Delta m^2\,(\text{eV}^2)}$$

Check

Properties:

1. $P(\nu_e \to \nu_\mu ; t) = P(\nu_\mu \to \nu_e ; t)$ in $2-\nu$
2. $P(\nu_e \to \nu_\mu ; t) = P(\bar{\nu}_e \to \bar{\nu}_\mu ; t)$ in $2-\nu$
3. If L very small or E too large $P \to 0$
4. If L very large $P \to \frac{1}{2}\sin^2\theta$



3-ν OSCILLATIONS

For 3-ν mixing, we have:

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}$$

with
$$U_{PMNS} = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{-i\frac{\alpha_{21}}{2}} & \\ & & e^{-i\frac{\alpha_{31}}{2}} \end{pmatrix}$$

One has:
- 3 angles.
- 1 phase Dirac as for CKM
  2 Majorana phases.
  They do not enter in neutrino oscillations as $U_{ai} U_{bi}^*$.

CASE A

$$\frac{\Delta m^2_{21} L}{4E} \ll 1$$

This holds for atmospheric, reactor, accelerator ν experiments.

$$P(\nu_a \to \nu_b; t) = 4 |U_{a3}|^2 |U_{b3}|^2 \sin^2 \frac{\Delta m^2_{31} L}{4E}$$

In fact:

$$P(\nu_a \to \nu_b; t) = \left| U_{bi} U_{ai}^* e^{-i\frac{\Delta m^2_{i1} t}{2E}} \right|^2$$

$$= \left| U_{a1}^* U_{b1} + U_{a2}^* U_{b2} \cancel{e^{-i\frac{\Delta m^2_{21} t}{2E}}} + U_{a3}^* U_{b3} e^{-i\frac{\Delta m^2_{31} t}{2E}} \right|^2$$

$$= |U_{a3}|^2 |U_{b3}|^2 \left| 1 - e^{-i\frac{\Delta m^2_{31} t}{2E}} \right|^2$$

$$\uparrow$$

$$(U^\dagger U)_{ab} = 0 \implies U_{a1}^* U_{b1} + U_{a2}^* U_{b2} = - U_{a3}^* U_{b3}$$

$$= |U_{a3}|^2 |U_{b3}|^2 \, 4 \sin^2 \frac{\Delta m^2_{31} t}{4E}$$

Ex. $$P(\nu_e \to \nu_\mu; L) = 4 s_{13}^2 s_{23}^2 c_{13}^2 \sin^2 \frac{\Delta m^2_{31} L}{4E}$$

$$P(\nu_e \to \nu_\tau; L) = \sin^2 2\theta_{13} c_{23}^2 \sin^2 \frac{\Delta m^2_{31} L}{4E}$$

$$P(\nu_\mu \to \nu_\tau; L) = c_{13}^4 \sin^2 2\theta_{23} \sin^2 \frac{\Delta m^2_{31} L}{4E}$$

and

$$P(\nu_e \to \nu_e; L)$$

~~Consider the following experiments:~~

~~1) KamLAND, $\langle E \rangle \sim 3$ MeV, $\langle L \rangle \sim 80$ km~~

CASE B

$$\frac{\Delta m^2_{31} L}{4E} \gg 1 \qquad \text{True for KAMLAND, solar.}$$

In this case one averages over Δm^2_{31} and Δm^2_{32}.

$$P(\nu_e \to \nu_e ; L) \cong c_{13}^2 P_{2\nu} + s_{13}^2$$

with $P_{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta m^2_{21} L}{4E}$

In fact

$$P(\nu_e \to \nu_e ; L) = \left| U_{ai}^* U_{ai}\, e^{-i \frac{\Delta m^2_{i1}}{2E} t} \right|^2$$

$$= \left| |U_{e1}|^2 + |U_{e2}|^2 e^{-i \frac{\Delta m^2_{21}}{2E} t} + |U_{e3}|^2 e^{-i \frac{\Delta m^2_{31}}{2E} t} \right|^2$$

$$= \left(|U_{e1}|^2 + |U_{e2}|^2 \cos \frac{\Delta m^2_{21} t}{2E} + |U_{e3}|^2 \cos \frac{\Delta m^2_{31} t}{2E} \right)^2$$

$$+ \left(|U_{e2}|^2 \sin \frac{\Delta m^2_{21} t}{2E} + |U_{e3}|^2 \sin \frac{\Delta m^2_{31} t}{2E} \right)^2$$

$$= \left(|U_{e1}|^2 + |U_{e2}|^2 \cos \frac{\Delta m^2_{21} t}{2E} \right)^2 + |U_{e3}|^2 \frac{1}{2}$$

$\sin, \cos \frac{\Delta m^2_{31}}{2E} t \to 0$

$\sin^2, \cos^2 \frac{\Delta m^2_{31}}{2E} t \to \frac{1}{2}$

$$+ |U_{e2}|^4 \sin^2 \frac{\Delta m^2_{21} t}{2E} + |U_{e3}|^2 \frac{1}{2}$$

$$= |U_{e3}|^2 + P_{2\nu}(\nu_e \to \nu_e) \cdot c_{13}^2$$

$$= s_{13}^2 + c_{13}^2 \left(1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta m^2_{21} L}{4E} \right)$$

Ex. Consider the following experiments:

① KamLAND

$\langle E \rangle \sim 3\,\text{MeV}$

$\langle L \rangle \sim 80\,\text{km}$

$P(\bar{\nu}_e \to \bar{\nu}_e; L)$

② CHOOZ

$\langle E \rangle \sim 3\,\text{MeV}$

$L \sim 1\,\text{km}$

$P(\bar{\nu}_e \to \bar{\nu}_e; L)$

③ MINOS

$E \sim 3\,\text{GeV}$

$L = 735\,\text{km}$

$P(\nu_\mu \to \nu_e; L)$

$P(\nu_\mu \to \nu_\tau; L)$

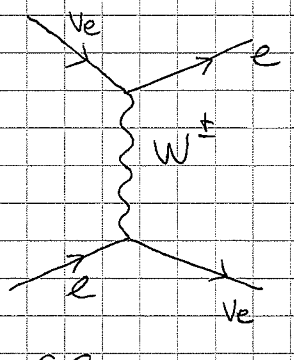
$P(\nu_\mu \to \nu_\mu; L)$

Figure out yourself Δm^2_{21} and Δm^2_{31}.

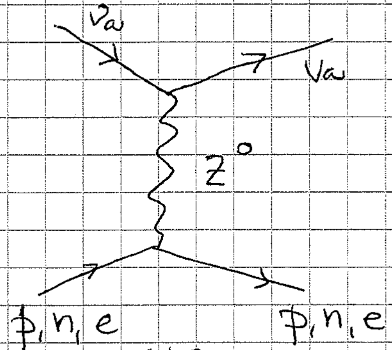
(*) Compute the CP-asymmetry $P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e)$.

MATTER EFFECTS

When ν_L travel through the Earth (e, p, n), they interact with the background particles via forward elastic scattering.



CC



NC

The interaction hamiltonian is:

$$H_{CC} = 2\sqrt{2}\, G_F \left[\bar{e}\, \gamma_\mu (1-\gamma_5)\, \nu_e\right]\left[\bar{\nu}_e\, \gamma^\mu (1-\gamma_5)\, e\right]$$

$$= 2\sqrt{2}\, G_F \left[\bar{e}\, \gamma_\mu (1-\gamma_5)\, e\right]\left[\bar{\nu}_e\, \gamma^\mu (1-\gamma_5)\, \nu_e\right]$$

Fierz transformation

We study the equation of motion for one electron neutrino state in presence of a non-relativistic electron background. We need to compute

$$\langle \bar{e}_L \gamma_\mu e_L \rangle = \delta_{\mu 0} \frac{N_e}{2}$$

where $N_e \equiv e^+ e$ is the average electron density ($\frac{1}{2}$ from the fact that half of the " " " is RH and does not interact with ν_L).

Note: $\langle \bar{e}_L \gamma_0 e_L \rangle = \langle e^+ e \rangle = N_e$, $\langle \bar{e}\, \vec{\gamma}\, e \rangle = \langle \vec{v}_e \rangle$,

$\langle \bar{e}\, \gamma_0 \gamma_5\, e \rangle = \dfrac{\langle \vec{\sigma}_e \cdot \vec{p}_e \rangle}{E_e}$, $\langle \bar{e}\, \vec{\gamma}\, \gamma_5\, e \rangle = \langle \vec{\sigma}_e \rangle$

See De Gouvea.

Ignoring mass terms for the time being, the Dirac equation for a ν_e in a cold electron gas, is:

$$\left(i\partial^\mu \gamma_\mu - \sqrt{2}\, G_F N_e \gamma_0\right) |\nu_e\rangle = 0$$

It is not Lorentz-invariant.

The solution is still a plane-wave one. For $\vec{p} \gg m_\nu$ and $E \gg \sqrt{2}\, G_F N_e$, the neutrino dispersion relation becomes

$$E \simeq |\vec{p}| \pm \sqrt{2}\, G_F N_e$$

where + is for neutrinos (positive energy solutions)
− for antineutrinos.
This is similar to the modified dispersion relations of photons in matter (index of refraction).

$\sqrt{2}\, G_F N_e$ is called the matter potential in analogy to

$$E = \underset{\substack{\uparrow \\ \text{kinetic} \\ \text{energy}}}{T} + \underset{\substack{\uparrow \\ \text{potential}}}{V}$$

The resulting effective hamiltonian is:

$$H^{I}_{eff} = \bar{\nu}_e V \nu_e$$

with $(V_e)_{CC} = \sqrt{2}\, G_F N_e$ $\qquad (V_e)_{NC} = \sqrt{2}\, G_F \left(-\frac{N_n}{2}\right)$

The new hamiltonian will be:

$$H^m = H^0 + H^I$$

$$i \frac{d}{dt} |\nu_m\rangle = H^m |\nu_m\rangle$$

Let's study the ν-propagation in the flavour basis

In vacuum:
$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \left(p + \frac{m_1^2 + m_2^2}{4E}\right) - \frac{\Delta m^2}{4E} \cos 2\theta_0 & \frac{\Delta m^2}{4E} \sin 2\theta_0 \\ \dots & \underbrace{\left(\dots\dots \right)}_{\text{common phase}} + \frac{\Delta m^2}{4E} \cos 2\theta_0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\Delta m^2}{4E} \cos 2\theta_0 & \frac{\Delta m^2}{4E} \sin 2\theta_0 \\ \frac{\Delta m^2}{4E} \sin 2\theta_0 & \frac{\Delta m^2}{4E} \cos 2\theta_0 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

In matter:
$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \dots + \sqrt{2}\, G_F N_e & \dots \\ \dots & \dots \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

CONSTANT DENSITY CASE

One needs to diagonalise H^m.

$$tg\,2\theta^m = \frac{\frac{\Delta m^2}{2E}\sin 2\theta_0}{\frac{\Delta m^2}{2E}\cos 2\theta_0 - \sqrt{2}\,G_F N_e}$$

and

$$E_A - E_B = \sqrt{\left(\frac{\Delta m^2}{2E}\cos 2\theta_0 - \sqrt{2}\,G_F N_e\right)^2 + \left(\frac{\Delta m^2}{2E}\right)^2 \sin^2 2\theta_0}$$

The probability will be simply given by:

$$P(\nu_e \to \nu_\mu; t) = \sin^2 2\theta^m \sin^2\left(\frac{L}{2}(E_A - E_B)\right)$$

Notice that for antineutrinos, $V \to -V$ and

$$tg\,2\theta^m = \frac{\Delta m^2}{2E}\sin 2\theta_0 \Big/ \left(\frac{\Delta m^2}{2E}\cos 2\theta_0 + \sqrt{2}\,G_F N_e\right)$$

CP and CPT are violated by the background.

We study 3 regimes:

① vacuum case: $V \ll \frac{\Delta m^2}{2E}\cos 2\theta_0$

one recovers the vacuum case

$$tg\,2\theta^m \to tg\,2\theta_0$$

$$E_A - E_B \to \frac{\Delta m^2}{2E}$$

(2) matter suppression case: $V \gg \frac{\Delta m^2}{2E} \cos 2\theta_0$

$$\operatorname{tg} 2\theta^{m} \sim -\frac{\frac{\Delta m^2}{2E}}{V} \sin 2\theta_0 \ll \operatorname{tg} 2\theta_0$$

Effectively suppressed oscillations.

(3) resonance : $V \cong \frac{\Delta m^2}{2E} \cos 2\theta_0$

$\operatorname{tg} 2\theta^{m} \to \infty$ which means $\theta^{m} = \frac{\pi}{4}$

or maximal mixing.

$$E_A - E_B \longrightarrow \frac{\Delta m^2}{2E} \sin 2\theta_0$$

so the interaction length is much larger than in vacuum:

$$L_m^{osc} = \frac{L_0^{osc}}{\sin 2\theta_0}$$

If the oscillations are enhanced for neutrinos $\left(\frac{\Delta m^2}{2E} \cos 2\theta_0 > 0\right)$, they will be suppressed for antineutrinos and viceversa. For $\cos 2\theta_0 > 0$, the enhancement and suppressions of the probabilities can tell what is the sign of Δm^2.

VARYING DENSITY and MSW effect.

Let's consider now a varying density $N_e(t)$.
For ex., a neutrino travelling from the center of the Sun to its surface.

$$N_e(t) \Longrightarrow H^m(t) \Longrightarrow \theta^m(t)$$

Let's define the "mass eigenstates in matter" as a time t as:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U^m(t) \begin{pmatrix} \nu_1^m(t) \\ \nu_2^m(t) \end{pmatrix}$$

usual mixing matrix which contains θ^m

We have that

$$\frac{d}{dt}\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \dot{U}^m(t) \begin{pmatrix} \nu_1^m(t) \\ \nu_2^m(t) \end{pmatrix} + U^m(t) \begin{pmatrix} \dot{\nu}_1^m(t) \\ \dot{\nu}_2^m(t) \end{pmatrix}$$

$$= -i H^m \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}$$

We obtain the equation of motion for ν_1^m and ν_2^m.

$$i \begin{pmatrix} \dot{\nu}_1^m(t) \\ \dot{\nu}_2^m(t) \end{pmatrix} = -i \left(U^m(t)\right)^\dagger \dot{U}^m(t) \begin{pmatrix} \nu_1^m(t) \\ \nu_2^m(t) \end{pmatrix} + \left(U^m(t)\right)^\dagger H^m(t) U^m(t) \begin{pmatrix} \nu_1^m(t) \\ \nu_2^m(t) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} E_A & -i\dot{\theta}^m(t) \\ i\dot{\theta}^m(t) & E_B \end{pmatrix}}_{\text{propagation matrix}} \begin{pmatrix} \nu_1^m(t) \\ \nu_2^m(t) \end{pmatrix}$$