Lecture #3

a) Nuclear structure - nuclear shell model
b) Nuclear structure -quasiparticle random phase approximation
c) Exactly solvable model
d) Dependence on the distance between neutrons (or protons)
e) Numerical results and sources of uncertainty

In double beta decay two neutrons bound in the ground state of an initial even-even nucleus are simultaneously transformed into two protons that again are bound in the ground state of the final nucleus.

The nuclear structure problem is therefore to evaluate, with a sufficient accuracy, **the ground state wave functions of both nuclei**, and evaluate the matrix element of the $0\nu\beta\beta$-decay operator connecting them.

This cannot be done exactly; some approximation and/or truncation is always necessary. Moreover, there is no other analogous observable that can be used to judge the quality of the result.

Can one use the $2\nu\beta\beta$-decay matris elements for that?
What are the similarities and differences?

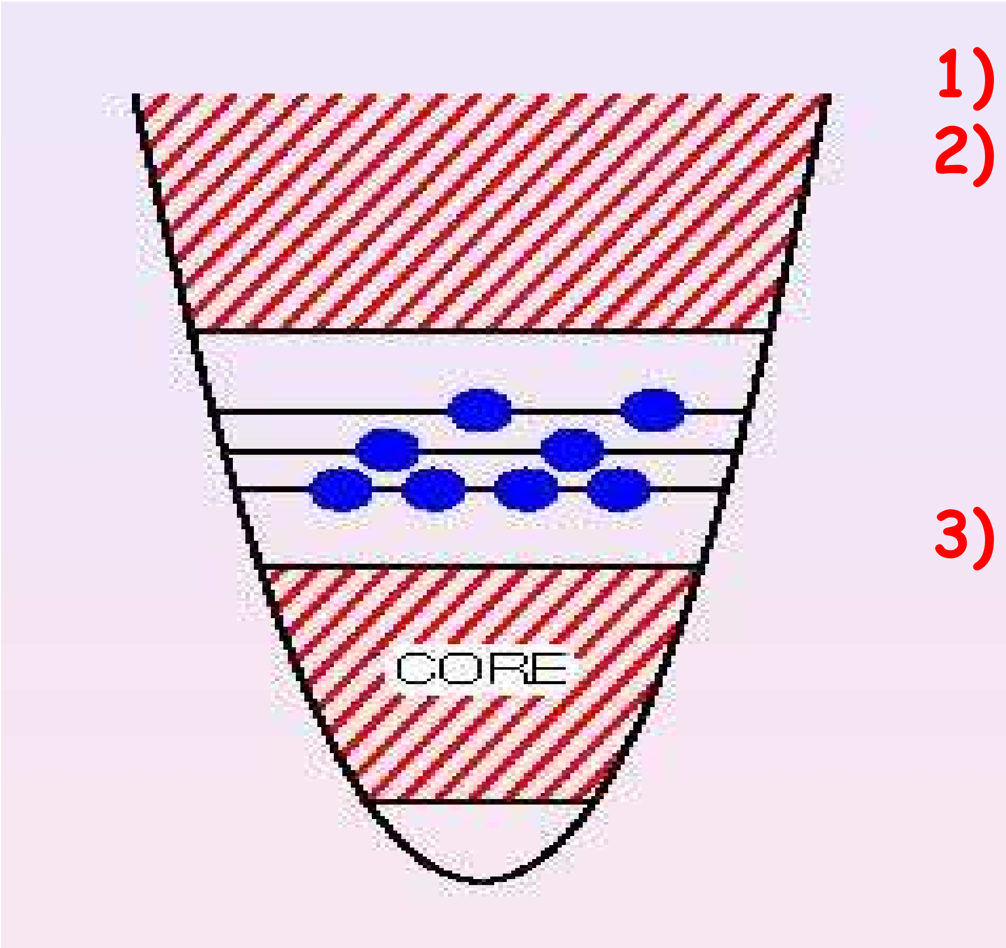
Both $2\nu\beta\beta$ and $0\nu\beta\beta$ operators connect the same states.
Both change two neutrons into two protons.

However, in $2\nu\beta\beta$ the momentum transfer $q <$ few MeV
And thus $e^{iqr} \sim 1$, **long wavelength approximation** is valid, only the GT operator $\sigma\tau$ need to be considered.

In $0\nu\beta\beta$ $q \sim 100$-200 MeV, $e^{iqr} = 1 +$ **many terms**, there is no natural cutoff in that expansion.

Explaining $2\nu\beta\beta$-decay rate is necessary but not sufficient

Basic procedures:



1) Define the valence space
2) Derive the effective hamiltonian H_{eff} using the nucleon-nucleon interaction plus some empirical nuclear data.
3) Solve the equations of motion to obtain the ground state wave functions

Two complementary procedures are commonly used:
a) Nuclear shell model (NSM)
b) Quasiparticle random phase approximation (QRPA)

In NSM a **limited** valence space is used but **all** configurations of valence nucleons are included. Describes well properties of low-lying nuclear states. Technically difficult, thus only few $0\nu\beta\beta$ calculations.

In QRPA a **large** valence space is used, but **only a class** of configurations is included. Describes collective states, but not details of dominantly few-particle states. Rather simple, thus many $0\nu\beta\beta$ calculations.

Illustration of capabilities of NSM (Nowacki 2004)
(see also the seminar by Alfredo Poves)

Nucleus	^{128}Sn	^{130}Sn	^{132}Sb	^{132}Te	^{133}Te
Transition	$0^+ \to 1^+$	$0^+ \to 1^+$	$4^+ \to 3,4,5^+$	$0^+ \to 1^+$	$\frac{3}{2}^+ \to \frac{1}{2},\frac{3}{2},\frac{5}{2}^+$
$T_{1/2}$exp.	59.07m	3.72m	2.79m	3.2d	12.5m
$T_{1/2}$calc. (0.74)	32.21m	2.47m	1.56m	1.73d	6.42m
Renorm.	0.54	0.6	0.55	0.54	0.53

^{134}Te	^{135}Xe	^{136}Cs
$0^+ \to 1^+$	$\frac{3}{2}^+ \to \frac{1}{2},\frac{3}{2},\frac{5}{2}^+$	$5^+ \to 4,5,6^+$
41.8m	9.14h	13.16d
29.19m	7.07h	8.1d
0.62	0.63	0.57

Ordinary β decay,
GT transitions,
well described
with renormalization
(so-called quenching)
by a factor ~0.57
(often included by
taking g_A = 1 since
$1/1.26^2 = 0.63$)

2νββ decay in NSM (an illustration using a talk by F. Nowacki 2004, might be somewhat obsolete)

Parent nuclei	^{48}Ca	^{76}Ge	^{82}Se	^{130}Te	^{136}Xe
$T^{2\nu}_{1/2}(g.s.)$ th.	$3.7E19$	$1.15E21$	$3.4E19$	$4E20$	$6E20$
$T^{2\nu}_{1/2}(g.s.)$ exp	$4.2E19$	$1.4E21$	$8.3E19$	$2.7E21$	$> 8.1E20$
New $T_{1/2}$ (exp)	3.9E19	1.7E21	9.6E19	7.6E20	> 1.0E22
Ratio th/exp	0.95	0.68	0.35	0.53	<0.06

QRPA proceeds in two steps.

1) First pairing between like nucleons is included in a simple fashion:

$$\begin{pmatrix} a^\dagger_{jm} \\ \tilde{a}_{jm} \end{pmatrix} = \begin{pmatrix} u_j c^\dagger_{jm} & + & v_j \tilde{c}_{jm} \\ -v_j c^\dagger_{jm} & + & u_j \tilde{c}_{jm} \end{pmatrix}$$

particles quasiparticles

Bogoliubov transformation, proton and neutron Fermi levels are smeared. However, particle numbers are conserved only in average.

2) Then the proton-neutron interaction is included

$$|J^\pi M; m\rangle = \Sigma_{pn} \left[X^m_{pn,J\pi} A^\dagger(pn; J^\pi M) + Y^m_{pn,J\pi} \tilde{A}(pn; J^\pi M) \right] |0^+_{QRPA}\rangle$$

two quasiparticle creation operator

two quasiparticle annihilation operator

correlated ground state, includes zero-point motion

The vectors X and Y are obtained by solving the equations of motion:

$$\begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \omega \begin{pmatrix} X \\ Y \end{pmatrix}$$

Eigenvalue equation for ω^2, unphysical solutions with $\omega^2 < 0$ possible

with

$$\begin{aligned} A^J_{pn,p'n'} &= \langle O|(c^\dagger_p c^\dagger_n)^{(JM)^\dagger} \hat{H} (c^\dagger_{p'} c^\dagger_{n'})^{(JM)}|O\rangle \\ &= \delta_{pn,p'n'}(E_p + E_n) \\ &\quad +(u_p v_n u_{p'} v_{n'} + v_p u_n v_{p'} u_{n'}) g_{ph} \langle pn^{-1}, J|V|p'n'^{-1}, J\rangle \\ &\quad +(u_p u_n u_{p'} u_{n'} + v_p v_n v_{p'} v_{n'}) g_{pp} \langle pn, J|V|p'n', J\rangle \ , \end{aligned}$$

$$\begin{aligned} B^J_{pn,p'n'} &= \langle O|\hat{H} (c^\dagger_p c^\dagger_n)^{(J-M)} (-1)^M (c^\dagger_{p'} c^\dagger_{n'})^{(JM)}|O\rangle \\ &\quad +(-1)^J (u_p v_n v_{p'} u_{n'} + v_p u_n u_{p'} v_{n'}) g_{ph} \langle pn^{-1}, J|V|p'n'^{-1}, J\rangle \\ &\quad -(-1)^J (u_p u_n v_{p'} v_{n'} + v_p v_n u_{p'} u_{n'}) g_{pp} \langle pn, J|V|p'n', J\rangle \ . \end{aligned}$$

particle-hole

particle-particle

Evaluate the $M^{2\nu}$ is relatively simple

overlap

$$M^{2\nu} = \Sigma_{k,m} \frac{\langle f||\vec{\sigma}\tau^+||1_k^+\rangle\langle 1_k^+|1_m^+\rangle\langle 1_m^+||\vec{\sigma}\tau^+||i\rangle}{\omega_m - (M_i + M_f)/2} ,$$

$$\langle 1_m^+||\vec{\sigma}\tau^+||i\rangle = \Sigma_{pn}\langle p||\vec{\sigma}||n\rangle(u_p v_n X_{pn}^m + v_p u_n Y_{pn}^m)$$

$$\langle f||\vec{\sigma}\tau^+||1_k^+\rangle = \Sigma_{pn}\langle p||\vec{\sigma}||n\rangle(\tilde{v}_p \tilde{u}_n \tilde{X}_{pn}^k + \tilde{u}_p \tilde{v}_n \tilde{Y}_{pn}^k) .$$

But the two `vacua' $|0^+_{QRPA}\rangle$ are not identical, hence the Overlap is included (this is an approximation).
but more importantly, how does one choose g_{pp}?

The usual practice is to give up on the predictability of the $2\nu\beta\beta$ decay, instead to choose g_{pp} such that the $M^{2\nu}$ has the correct value
(~ ±20% deviation from the nominal g_{pp} = 1)

Evaluation of $M^{0\nu}$ involves transformation to the relative coordinates of the nucleons (the operators $\boldsymbol{O_K}$ depend on r_{ij})

$$M_K = \sum_{J^\pi, k_i, k_f, \mathcal{J}} \sum_{pnp'n'} (-1)^{j_n + j_{p'} + J + \mathcal{J}} \times$$

$$\sqrt{2\mathcal{J}+1} \begin{Bmatrix} j_p & j_n & J \\ j_{n'} & j_{p'} & \mathcal{J} \end{Bmatrix} \times$$

$$\langle p(1), p'(2); \mathcal{J} \parallel \bar{f}(r_{12}) \tau_1^+ \tau_2^+ O_K \bar{f}(r_{12}) \parallel n(1), n'(2); \mathcal{J} \rangle \times$$

$$\langle 0_f^+ ||| \widetilde{[c_{p'}^+ \tilde{c}_{n'}]_J} || J^\pi k_f \rangle \langle J^\pi k_f | J^\pi k_i \rangle \langle J^\pi k_f i ||| [c_p^+ \tilde{c}_n]_J || 0_i^+ \rangle \, .$$

unsymmetrized two-body radial integral involves \`neutrino potentials'

From QRPA for final nucleus

overlap

From QRPA for initial nucleus

Note the two separate multipole decompositions. J^π refers to the virtual state in odd-odd nucleus, while $\mathcal{J}$ refers to the angular momentum of the neutron pair transformed into proton pair.

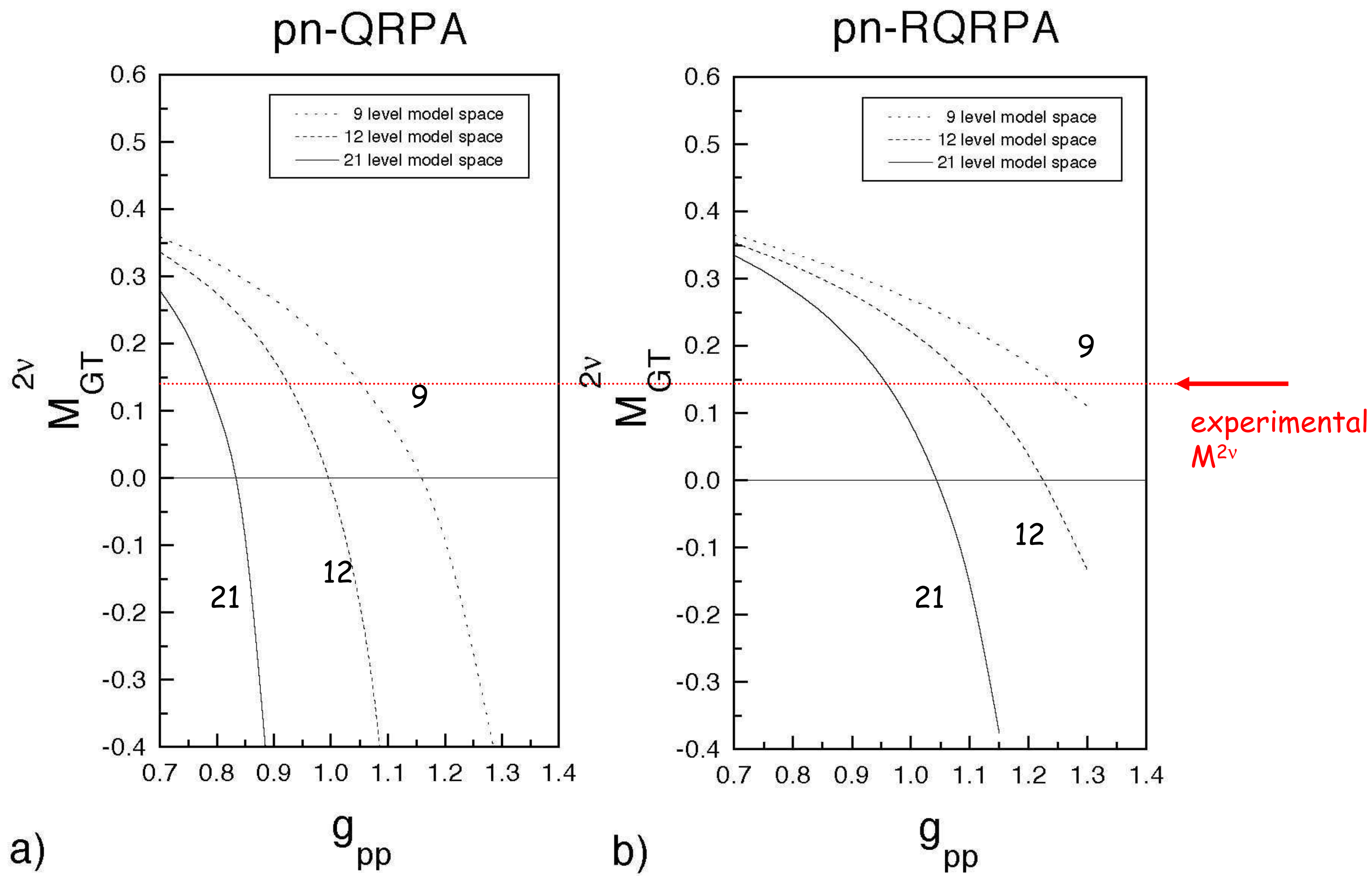
Note to (semi) experts: From QRPA to RQRPA

QRPA is a harmonic approximation, it assumes small amplitude excitations, i.e. that the number of quasiparticles in the correlated ground state $|0^+_{QRPA}\rangle$ in each nucleon orbit is small.

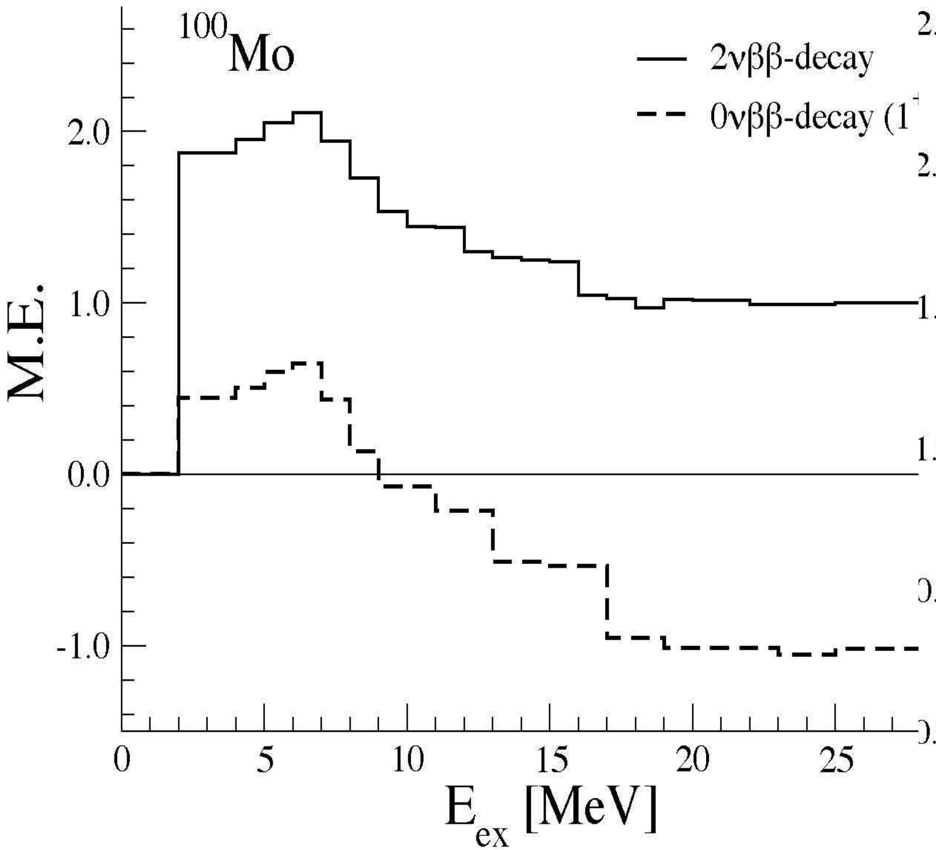
When that number cannot be neglected, deviations of the Pauli principle occur. The renormalized QRPA removes that issue approximately (as mean values)

$$\langle 0^+_{QRPA} | [A(pn,JM), A^+(p'n',JM)] | 0^+_{QRPA} \rangle = \delta_{pp'}\delta_{nn'} \times$$
$$\underbrace{\left\{1 - \frac{1}{\hat{j}_l} < 0^+_{QRPA} | [a^+_p \tilde{a}_p]_{00} | 0^+_{QRPA} > - \frac{1}{\hat{j}_k} < 0^+_{QRPA} | [a^+_n \tilde{a}_n]_{00} | 0^+_{QRPA} > \right\}}_{\mathcal{D}_{pn,J^\pi}}$$

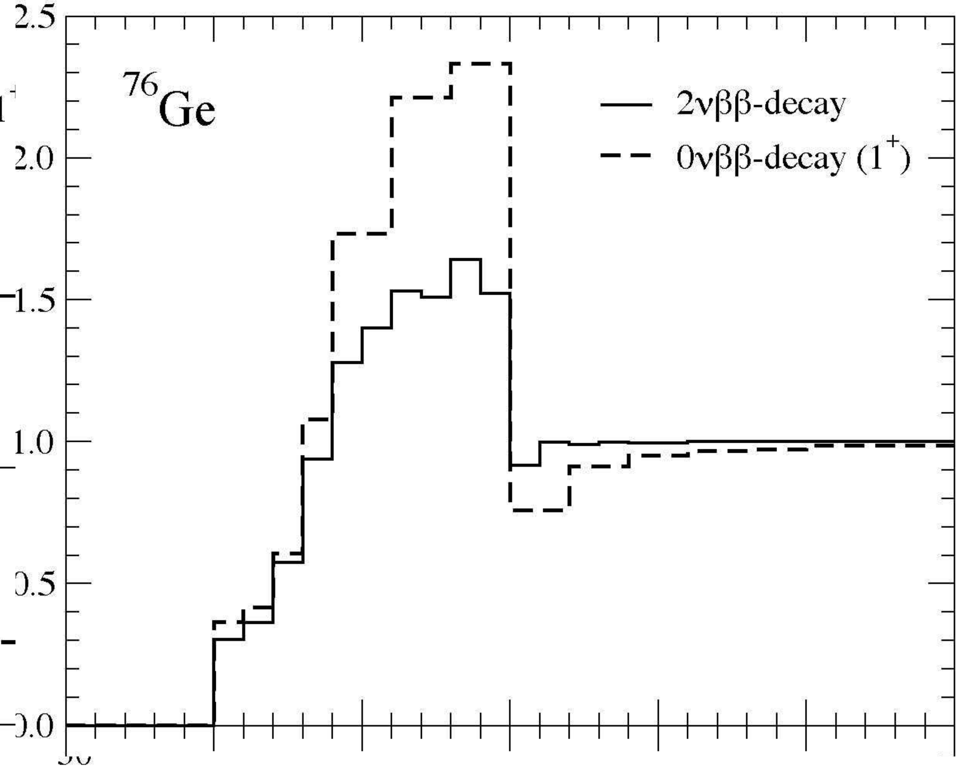
$2\nu\beta\beta$ matrix elements for ^{76}Ge as a function of g_{pp} in QRPA and RQRPA, calculation performed with 9,12, and 21 orbits. Note the crossing of zero and approach to collapse (infinite slope)



Bone of contention: Should one fix g_{pp} from $M^{2\nu}$ or using the data on β decay involving the first 1^+ state?
In other words, is the `single state dominance' always a good approximation?

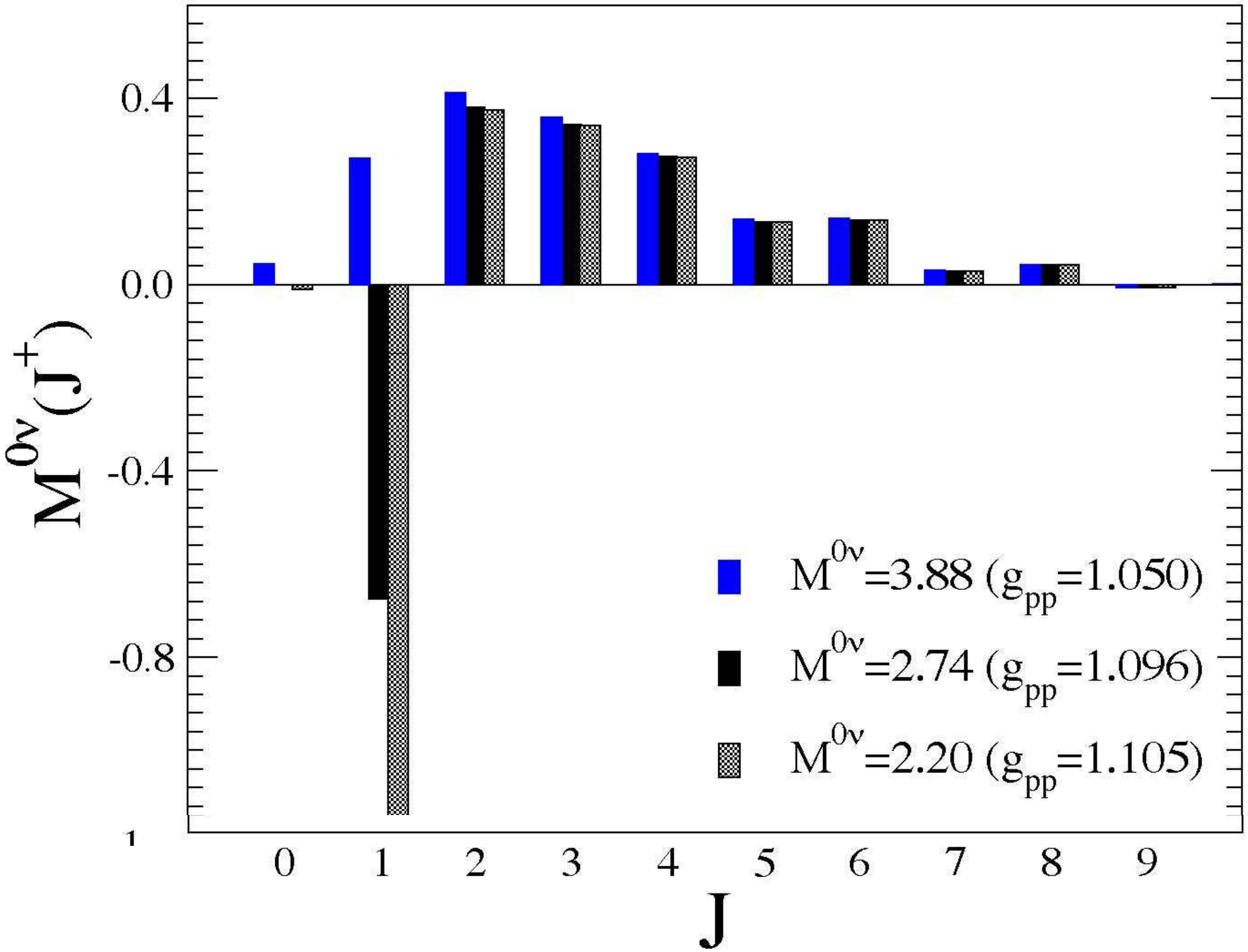


In ^{100}Mo the first 1^+ state gives a major contribution



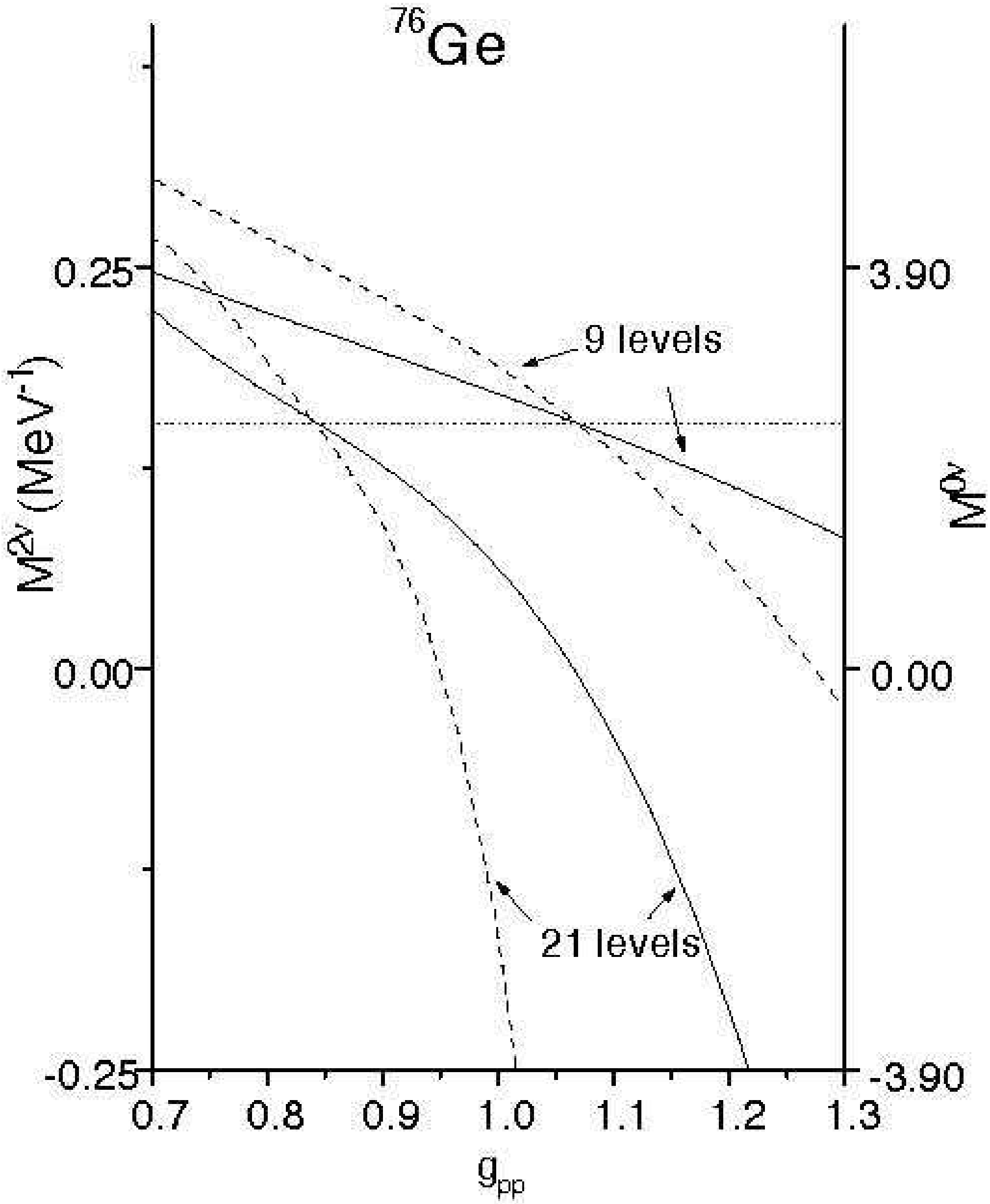
In ^{76}Ge the first 1^+ state does not dominate

Another issue: Should one use the same g_{pp} for all multipoles?



Contribution of different J^+ in the virtual odd-odd nucleus to $M^{0\nu}$ in ^{100}Mo. Three g_{pp} values that differ by 5%. Only 1^+ (GT) changes very rapidly. Fixing that multipole stabilizes $M^{0\nu}$.

Important bonus: Our prescription also essentially removes the dependence on the size of the s.p. basis.



$M^{0\nu}$ full lines,
$M^{2\nu}$ dashed lines.

By fixing g_{pp} to $M^{2\nu}$ we get the same $M^{0\nu}$ with 9 and 21 levels, but with different g_{pp} for the two cases, 1.05 vs. 0.85

How good is QRPA? Can we check its validity?

To do that (approximately) we use a two-level model that san be solved exactly using the algebra based on SO(5)xSO(5). It has many features analogous to real nuclei. The hamiltonian is

$$H = \epsilon \hat{N}_2 - G \sum_{a,b=1}^{2} \left(S_{pp}^{\dagger a} S_{pp}^{b} + S_{nn}^{\dagger a} S_{nn}^{b} + g_{pp} S_{pn}^{\dagger a} S_{pn}^{b} - g_{ph} \boldsymbol{T}_a \cdot \boldsymbol{T}_b \right)$$

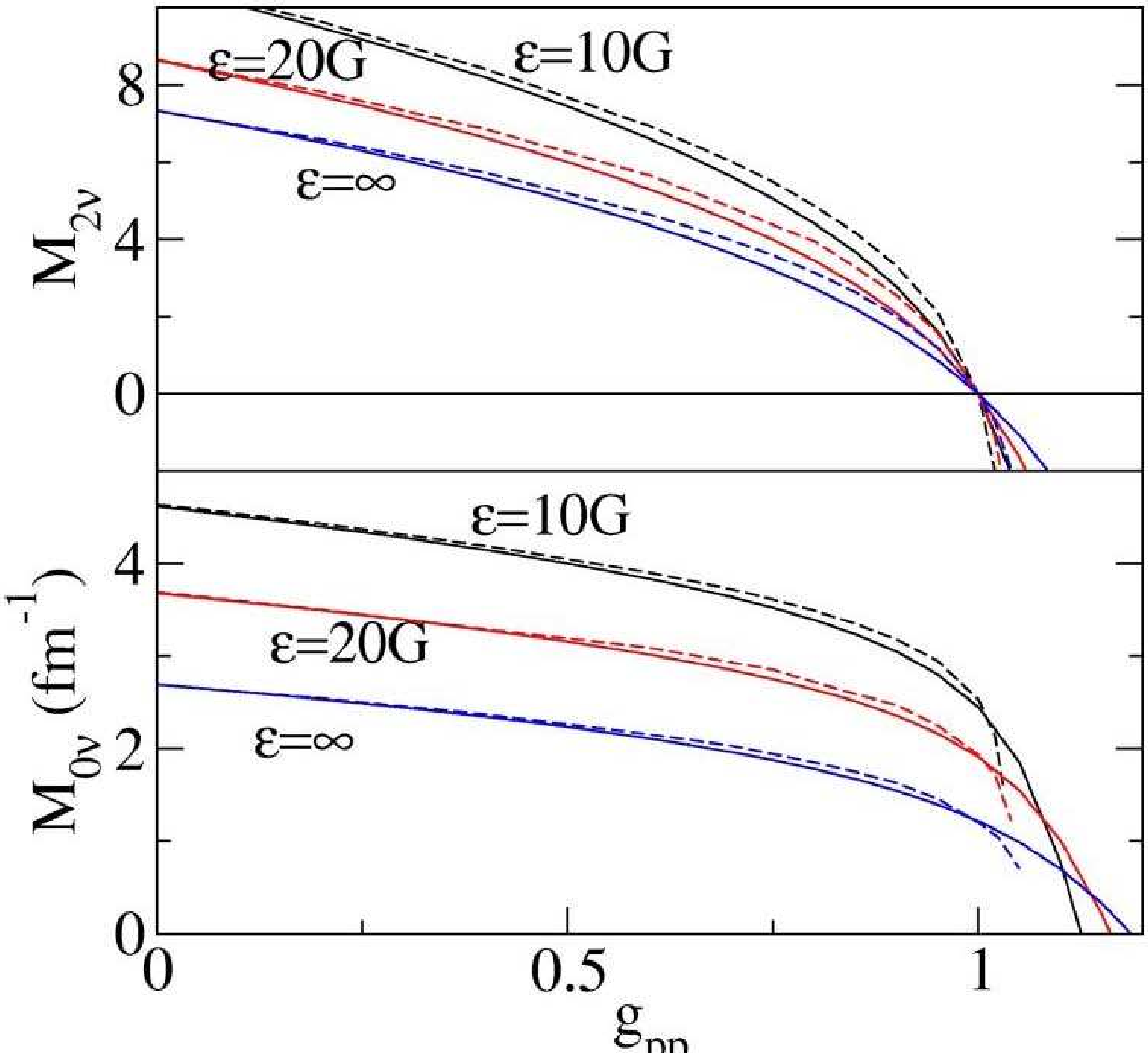
number operator

pair operators

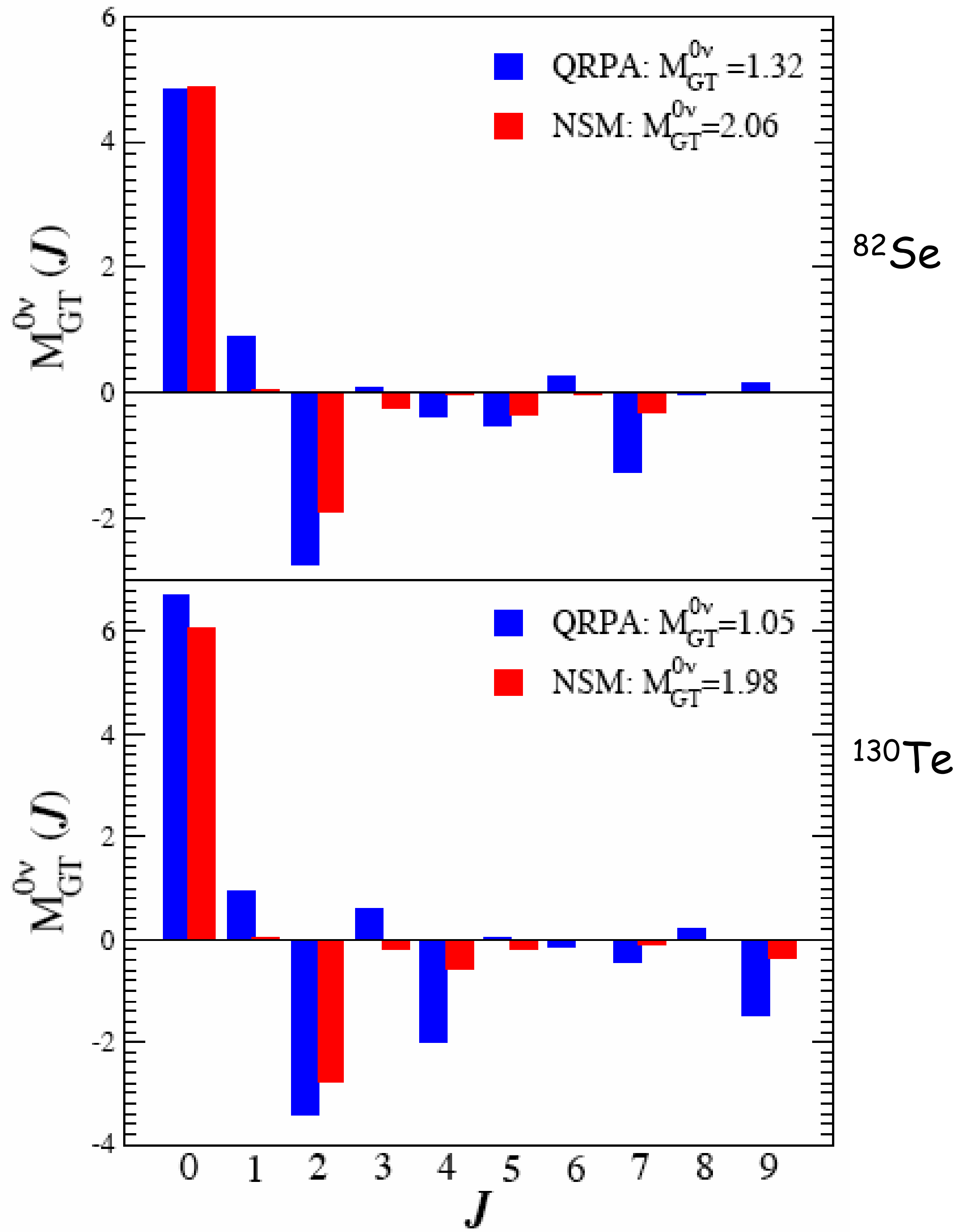
strength of n-p interaction

distance between the two shells

From Engel & Vogel, PRC<u>69</u>,034304(2004)



Comparison of exact (solid lines) and QRPA (dashed) $\mathbf{M^{2\nu}}$ and $\mathbf{M^{0\nu}}$, for different level spacings ε. **In this model QRPA works perfectly.**



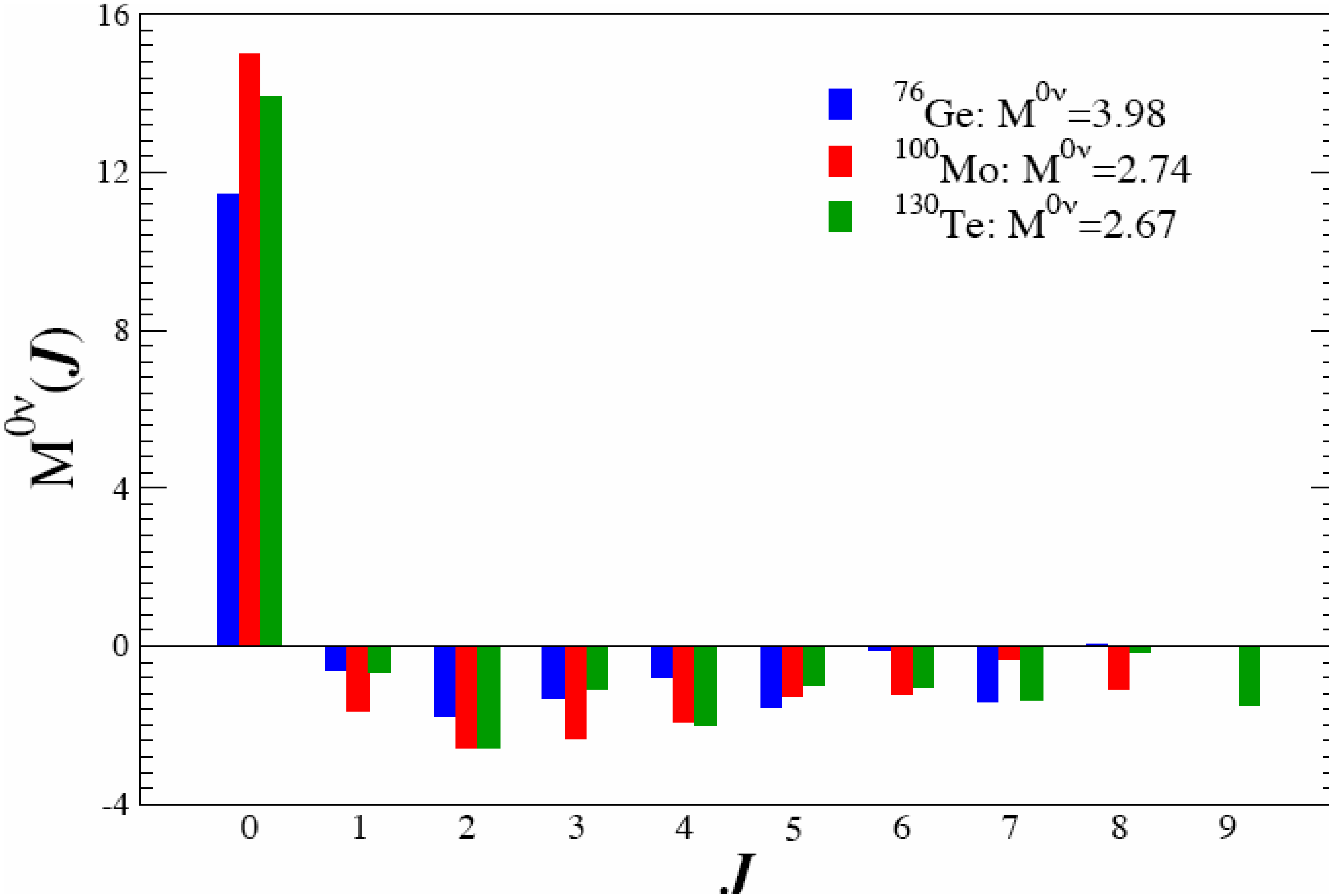
Why it is difficult to calculate the matrix elements accurately?

Contributions of different angular momenta J of the neutron pair that is transformed in the decay into the proton pair with the same J.

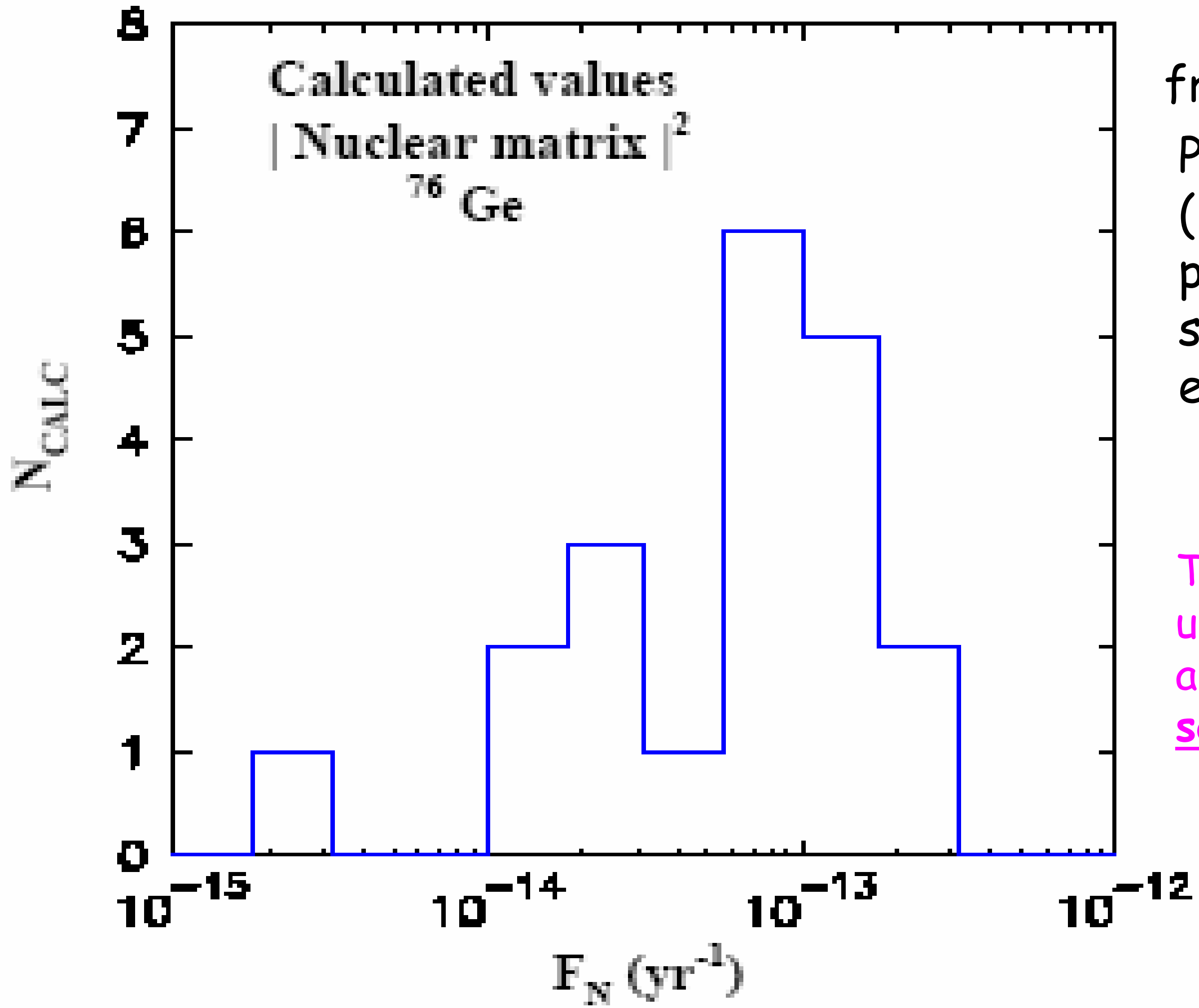
Note the opposite signs, and thus tendency to cancel, between the $J = 0$ (pairing) and the $J \neq 0$ (ground state correlations) parts.

The same restricted s.p. space is used for QRPA and NSM. There is a reasonable agreement between the two methods

The opposite signs, and similar magnitudes of the $J = 0$ and $J \neq 0$ parts is universal. Here for three nuclei with coupling constant g_{pp} adjusted so that the $2\nu\beta\beta$ rate is correctly reproduced. Now two oscillator shells are included.

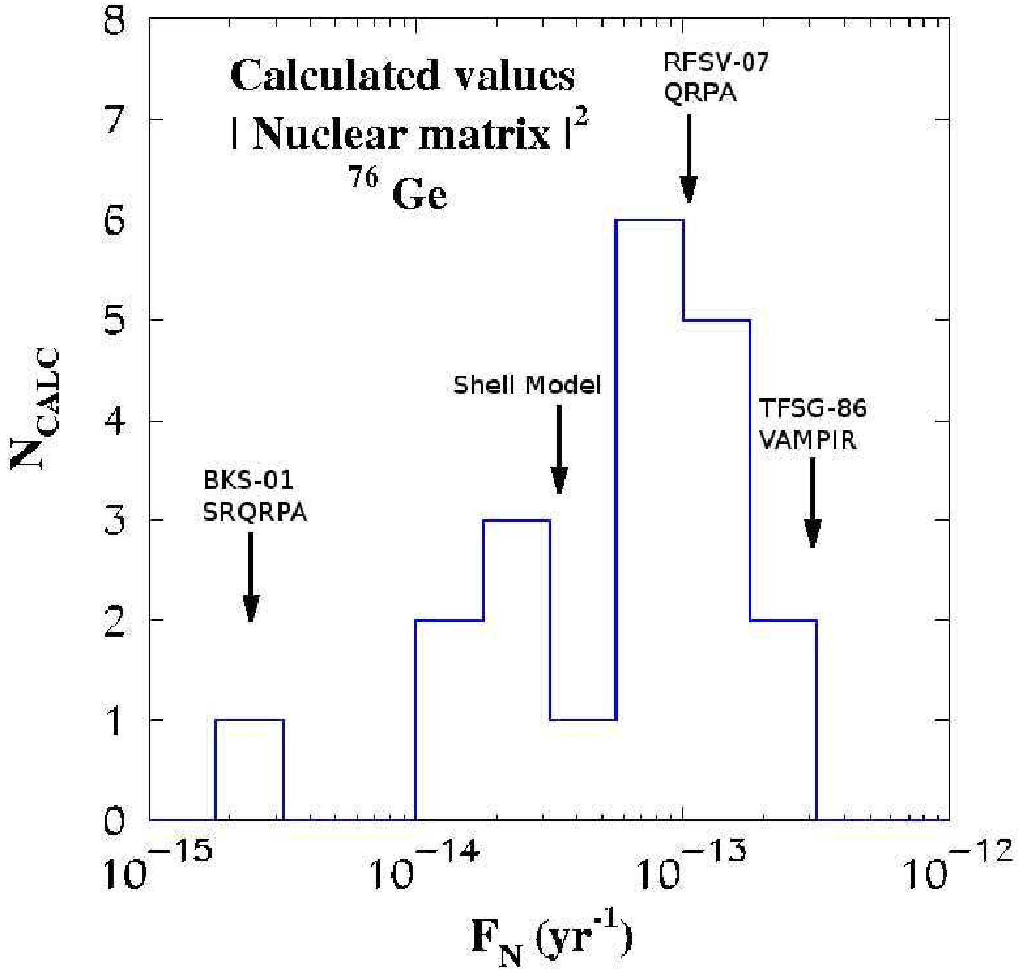


Different authors obtain different $M^{0\nu}$,
most calculations use QRPA. Why the differences?



from Bahcall et al Phys.Rev.D**70**,033012, (2004) , spread of published values of the squared nuclear matrix element for ^{76}Ge

This suggests an uncertainty of as much as a factor of 5. **Is it really so bad?**



The outliers do not describe relevant physics.

BKS-01 = A. Bobyk, W. Kaminski, F. Simkovic, PRC**63** (2001) ⇐ $2\nu\beta\beta$ 20 times too slow

Shell Model = E. Caurier, G. Martinez-Pinedo, F. Nowacki, A. Poves A. Zuker, RMP**77** (2005)

RFSV-07 = V.R., A. Faessler, F. Simkovic, P. Vogel, NPA**793** (2007) (2003) ⇐ $2\nu\beta\beta$ fitted

TFSG-86 = T. Tomoda, A. Faessler, K. W. Schmid, F. Grummer, NPA**452** (1986)

⇐ $2\nu\beta\beta$ 8 times too fast

Summary so far:

a) We understand why different authors got different $M^{0\nu}$ even though they use the same method.

b) Our choice of fixing g_{pp} does several good things.
It fixes the contribution of 1^+ multipole, the only one that varies quickly with g_{pp}.
It removes the dependence on the number of s.p. states included.
It gives at least semi-quantitative agreement with NSM concerning the `pairing' vs. `broken pairs' competition.

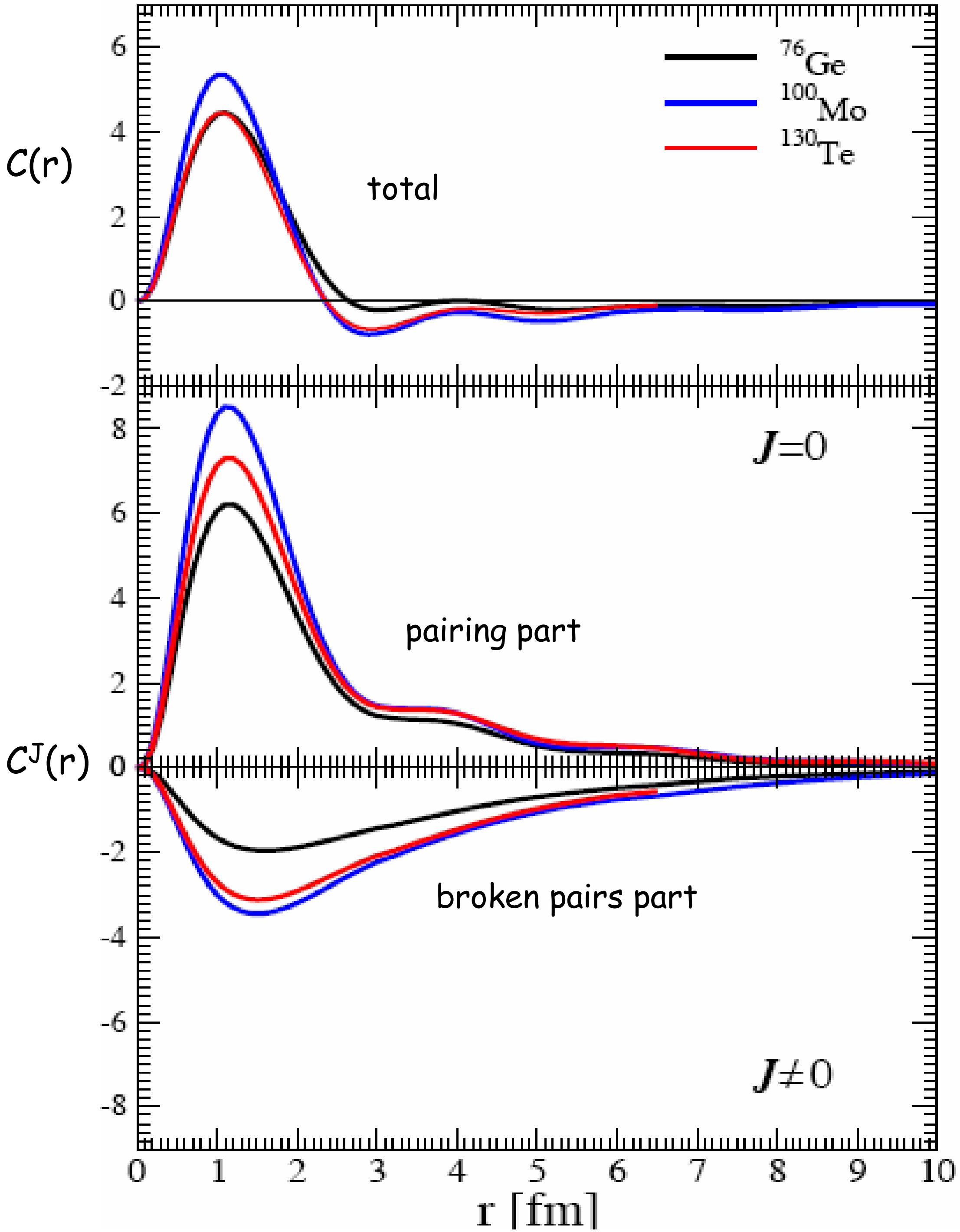
c) But there is a price to pay - we describe but do not predict $M^{2\nu}$.

d) And we have not exhausted all reasons for the variability of the calculated $M^{0\nu}$. We need to consider effects that exist only because of the high momentum transfer involved in $M^{0\nu}$.
In order to reveal these effects, consider the dependence of $M^{0\nu}$ on the distance *r* between the transformed neutrons.

Dependence of the $M^{0\nu}$ on the distance r between the two neutrons that are transformed into the two protons.

The "neutrino potential" is $H(r)= R/r\ \Phi(\omega r)$ where $\Phi(\omega r)$ is rather slowly varying function. This is a long range potential, more or less like a Coulomb potential. Thus, naively, one expect that the matrix element will get its main contribution from $r \sim R$, i.e. the mean distance between the nucleons in a nucleus.

This is not so. Due to the "pairing" and "broken pairs" competition, only distances $r < 2$-3 fm contribute, i.e., only nearest neighbors.



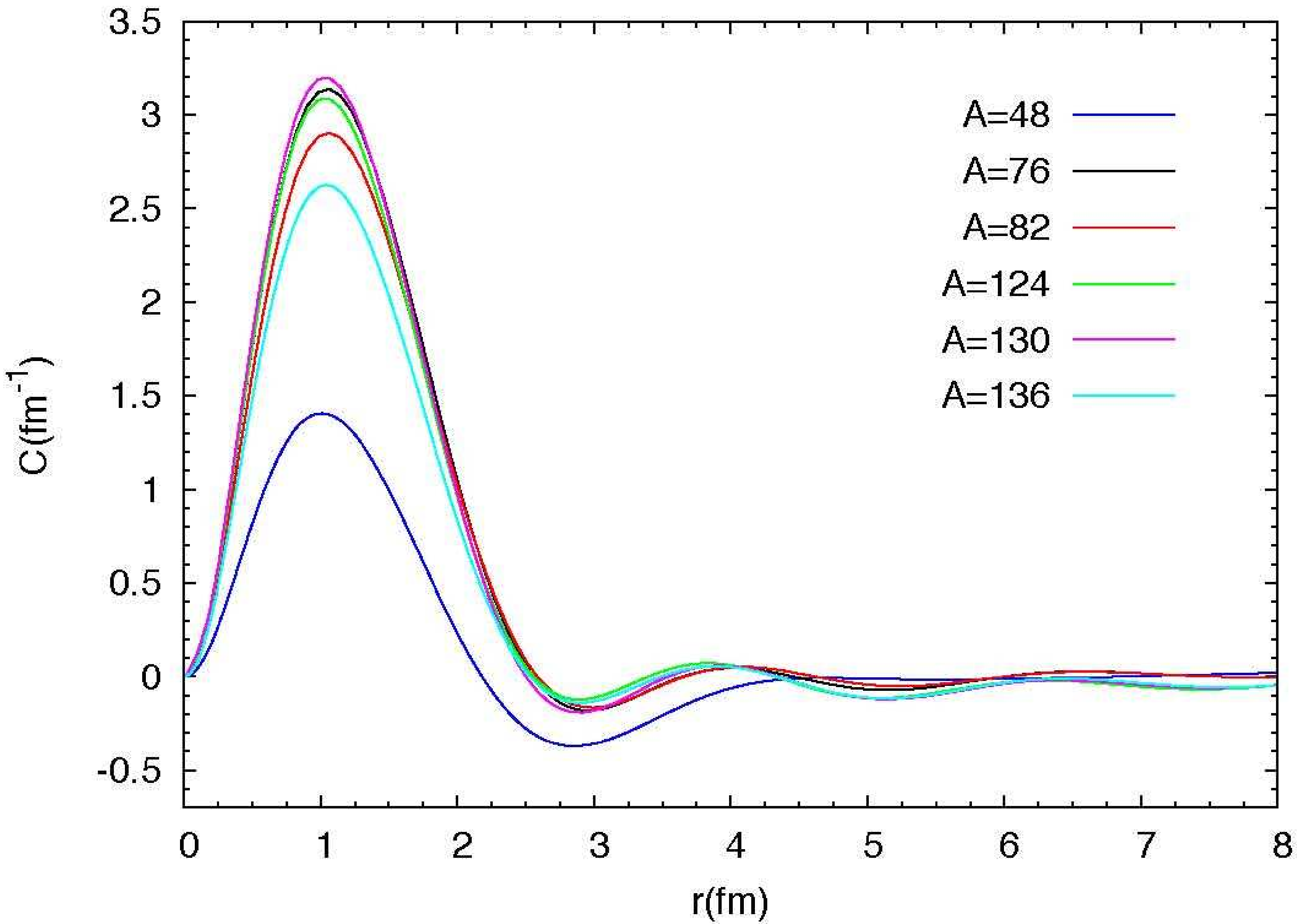
The radial dependence of $M^{0\nu}$ for the three indicated nuclei. The contributions summed over all components ss shown in the upper panel. The `pairing' $J = 0$ and `broken pairs' $J \neq 0$ parts are shown separately below.

Note that these two parts essentially cancel each other for r > 2-3 fm. This is a generic behavior. Hence the treatment of small values of *r* and large values of *q* are quite important.

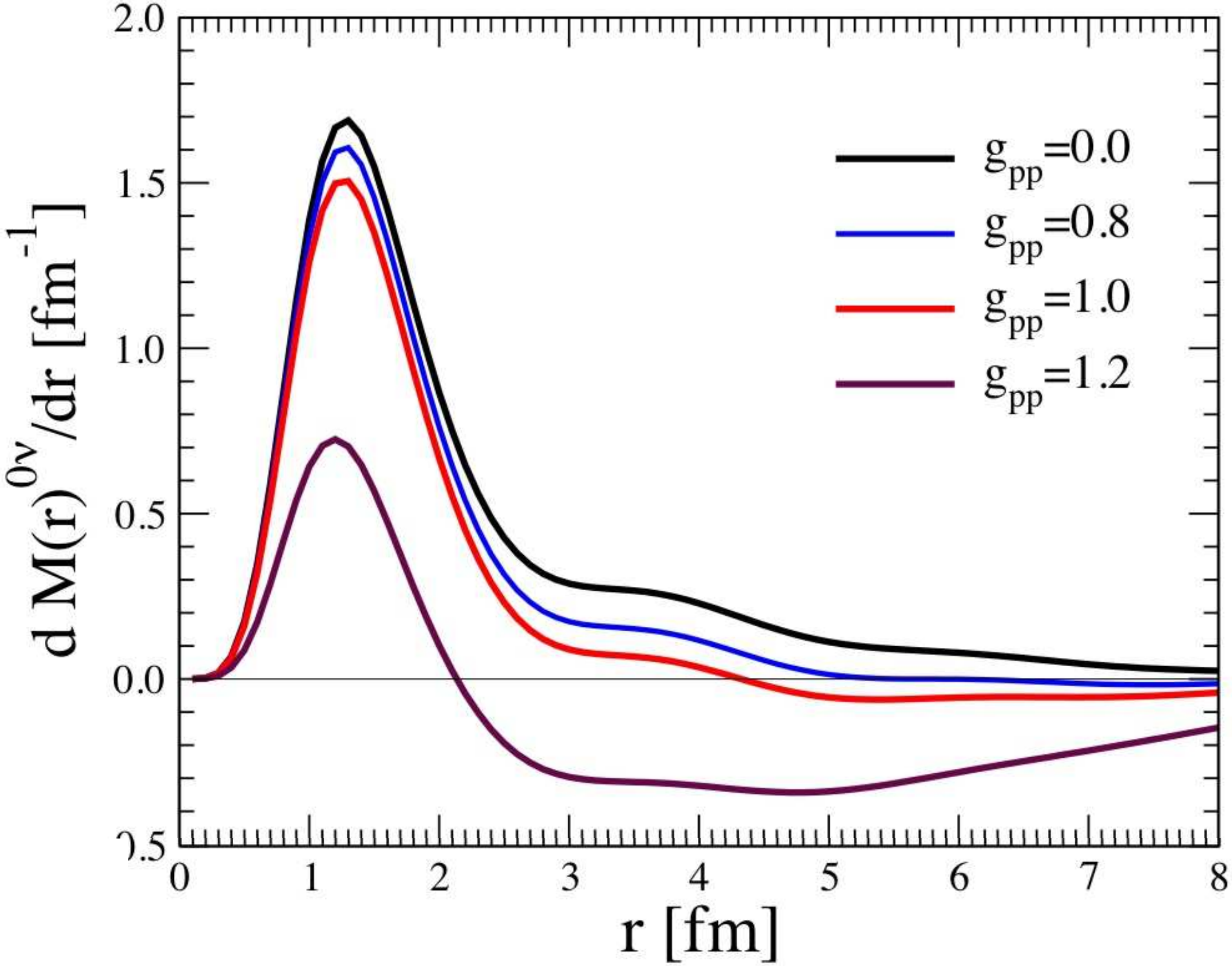
$$M^{0\nu} = \int C(r)dr$$

The radial dependence of $M^{0\nu}$ for the indicated nuclei, evaluated in the **nuclear shell model**. (Menendes et al, arXiv:0801.3760).

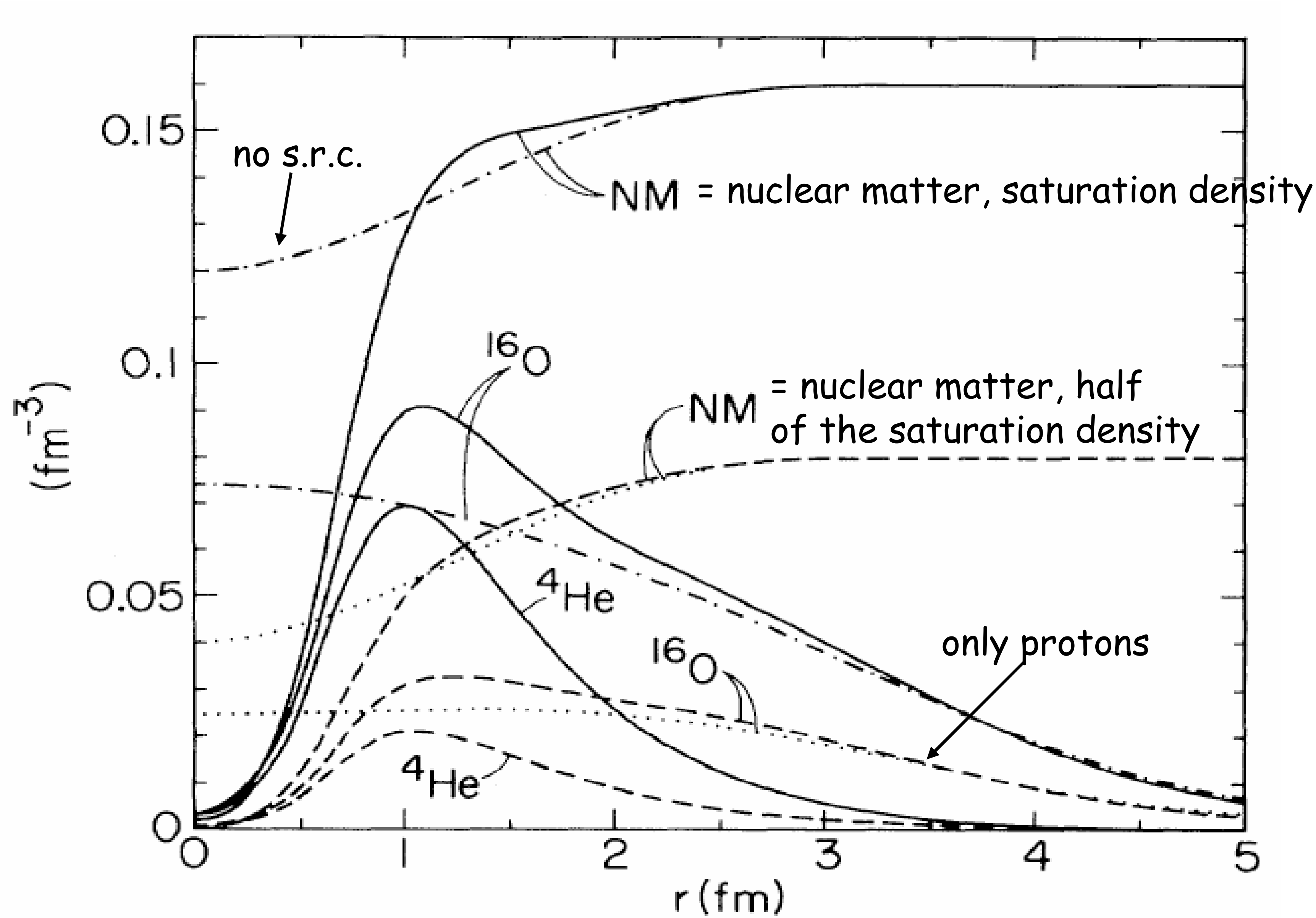
Note the similarity to the QRPA evaluation of the same function.



The radial dependence of $M^{0\nu}$ evaluated in the exactly solvable model described earlier. Note that the cancellation at r > 2-3 fm appears only near g_{pp} = 1.

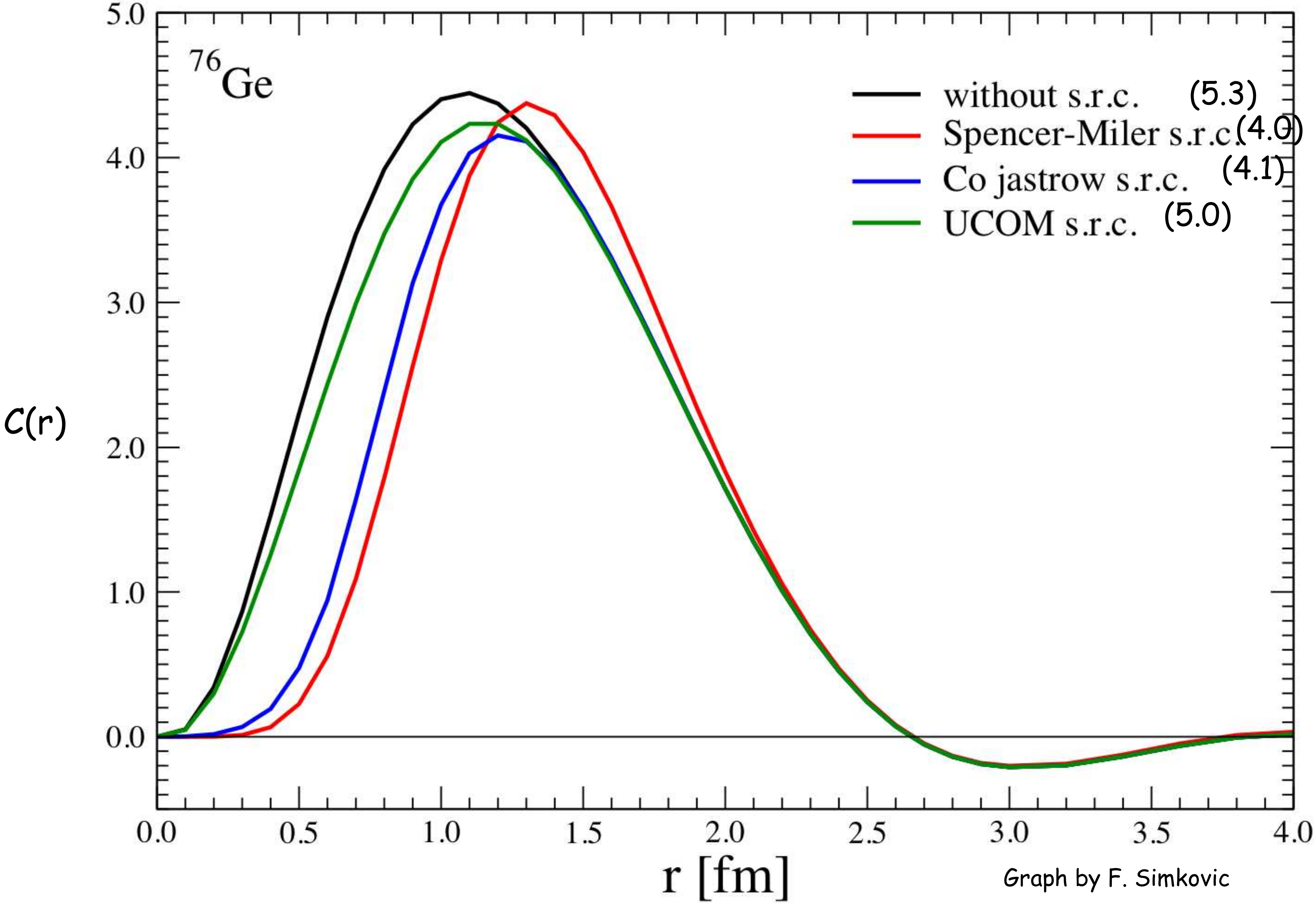


Two-nucleon probability distribution, with and without correlations, MC with realistic interaction. O. Benhar et al. RMP**65**,817(1993)



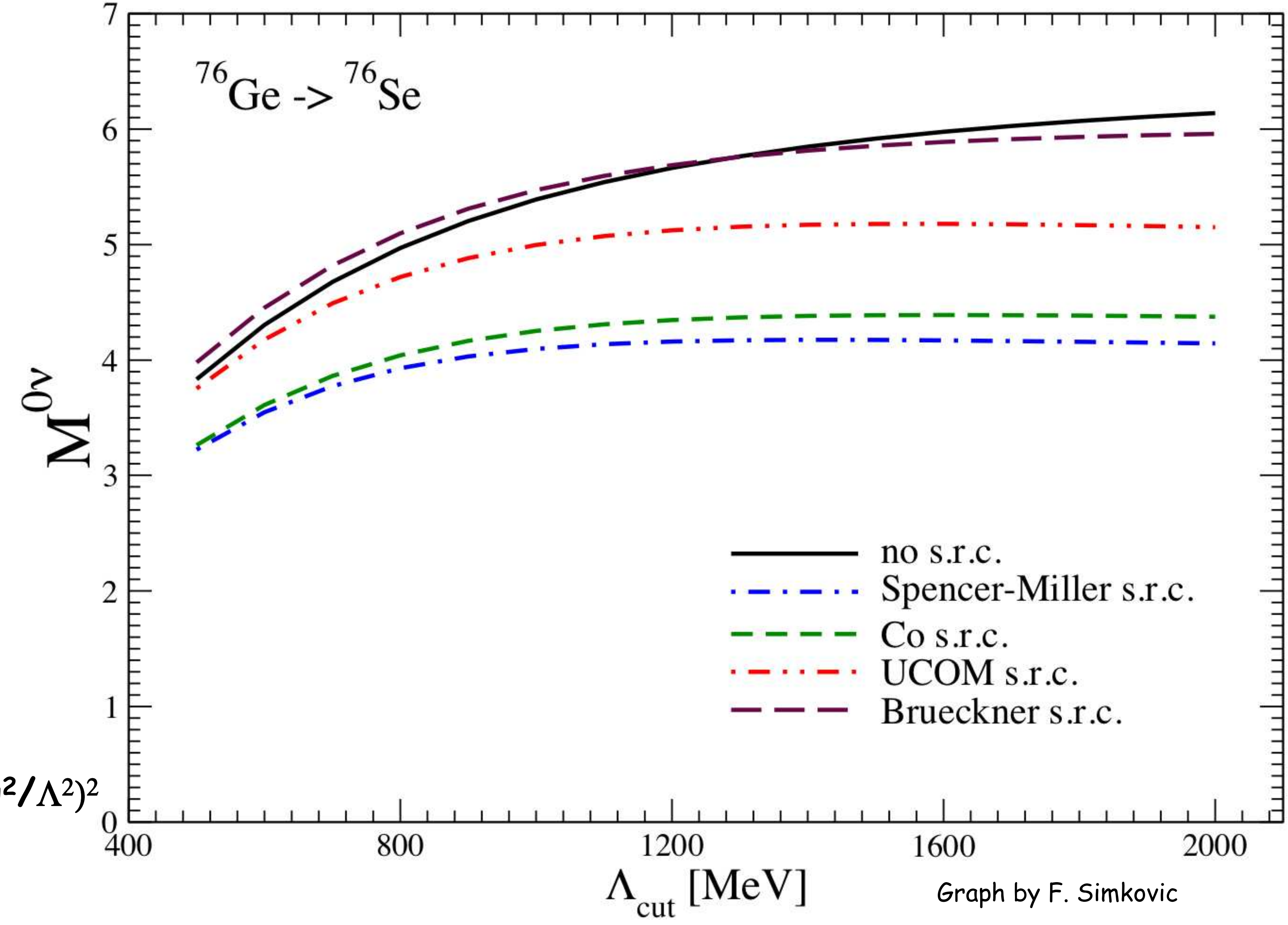
See also Bisconti et al., Phys. Rev. C73, 054304(2006) for the more modern version of this

Dependence on the distance between the two transformed nucleons and the effect of different treatments of short range correlations. This causes changes of $M^{0\nu}$ by ~ 20%.



Graph by F. Simkovic

Dependence of the $0\nu\beta\beta$ matrix element on the $M_A = M_V = \Lambda_{cut}$ parameter in the usual dipole nucleon form factor . When correction for short range correlations is included the $M^{0\nu}$ changes little for $\Lambda_{cut} \geq 1000$ MeV.



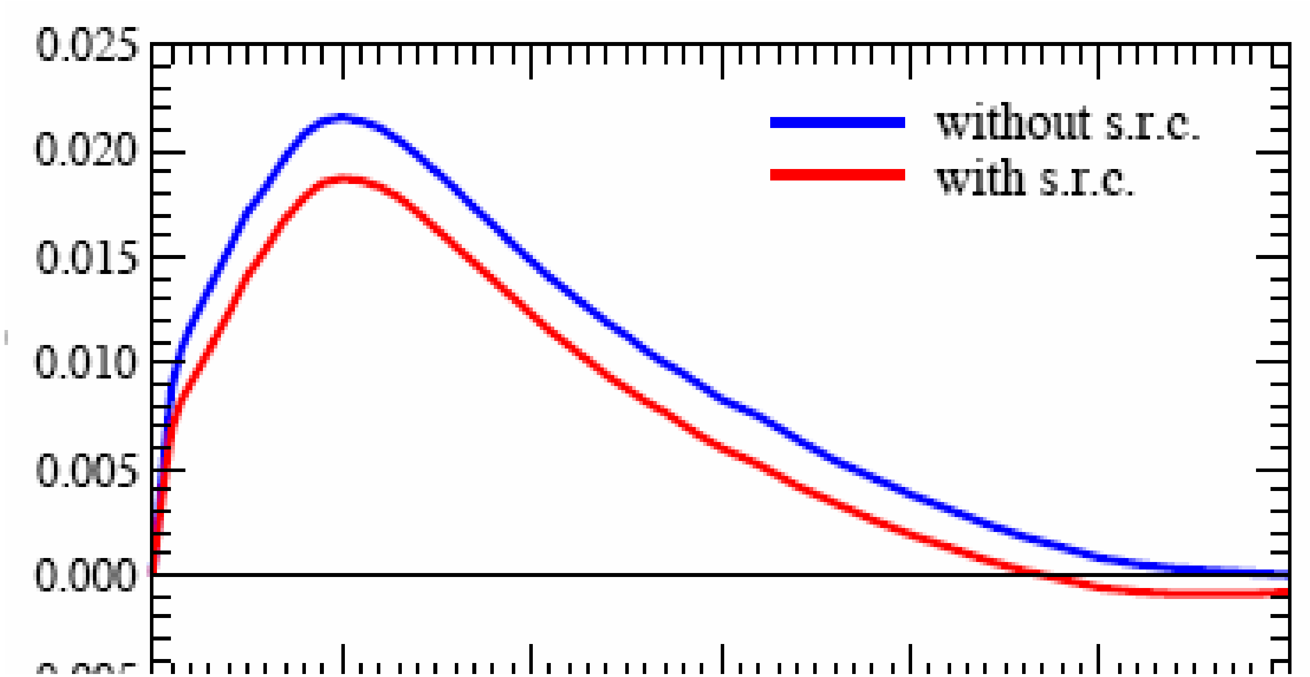
$f(q^2)$

$\sim 1/(1+q^2/\Lambda^2)^2$

Graph by F. Simkovic

Contribution of different momentum transfers ***q*** to the $0\nu\beta\beta$ matrix element in $^{76}Ge \rightarrow {}^{76}Se$ decay.

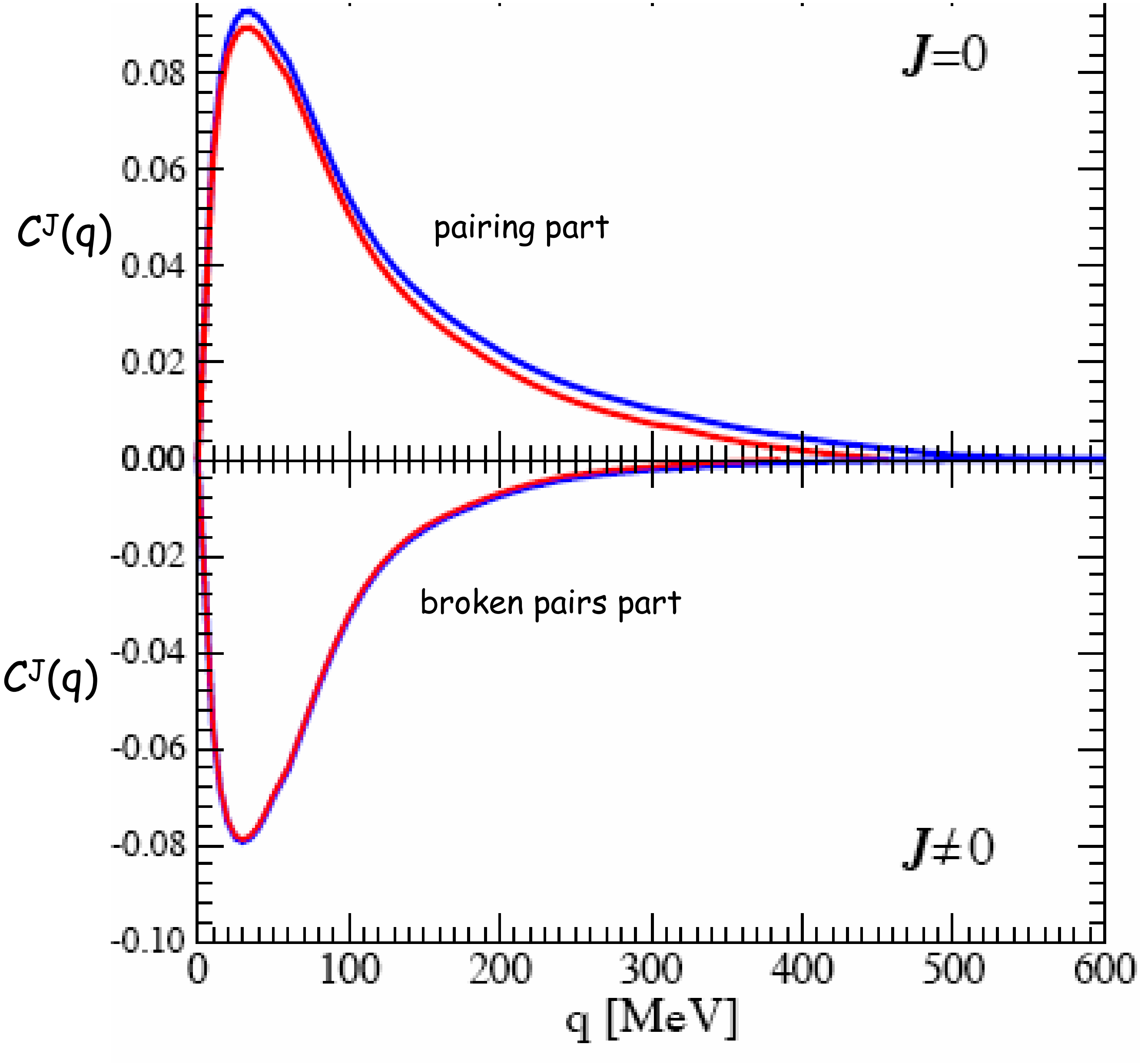
C(q)



Here the curves peak at q ~ 100 MeV, with a long tail extending to ~ 500 MeV.

$M^{0\nu} = \int C(q) dq$

Competition between the $J = 0$ and $J \neq 0$ parts as a function of momentum transfer q. Note the change of scale compared to the previous slide.



These curves peak at ~ 40 MeV. For $q > 200$ MeV the J = 0 part is dominating.

Hadronic current expressed in terms of nucleon fields Ψ:

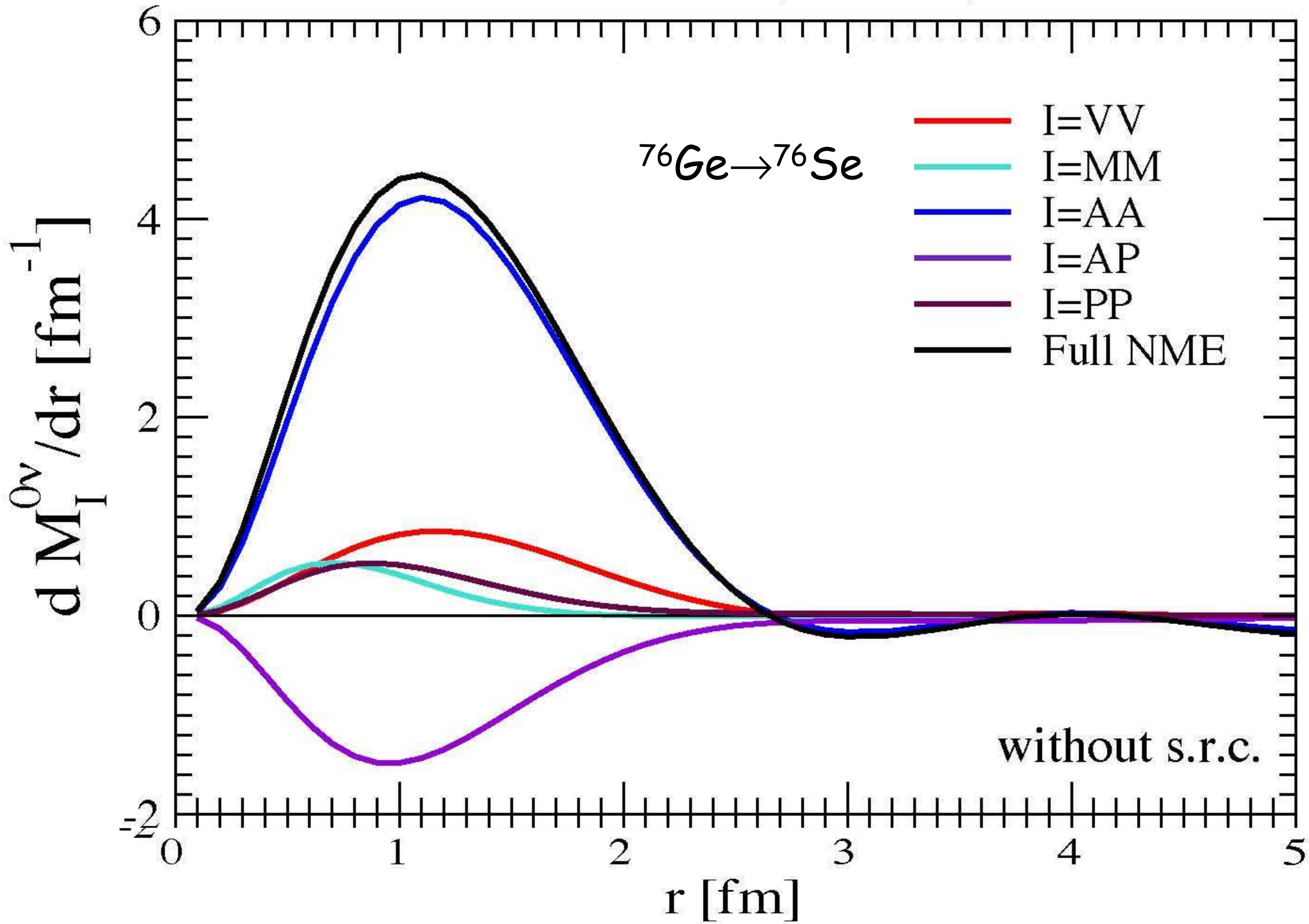
$$j^{\rho\dagger} = \overline{\Psi}\tau^{+}\left[g_V(q^2)\gamma^{\rho} + i g_M(q^2)\frac{\sigma^{\rho\nu}}{2m_p}q_{\nu} - g_A(q^2)\gamma^{\rho}\gamma_5 - g_P(q^2)q^{\rho}\gamma_5\right]\Psi,$$

Vector $g_V(q^2) = g_V/(1 + q^2/M_V^2)^2$, $g_V = 1$, $M_V = 0.85$ GeV
Axial vector $g_A(q^2) = g_A/(1 + q^2/M_A^2)^2$, $g_A = 1.25$, $M_A = 1.09$ GeV
Weak Magnetism $g_M(q^2) = (\mu_p - \mu_n)\, g_V(q^2)$
Induced pseudoscalar $g_P(q^2) = 2m_p g_A(q^2)/(q^2 + m_\pi^2)$

After the nonrelativistic reduction the space part of the current is

$$\vec{J}_n(\vec{q}^2) = g_M(\vec{q}^2) i \frac{\vec{\sigma}_n \times \vec{q}}{2m_p} + g_A(\vec{q}^2)\vec{\sigma} - g_P(\vec{q}^2)\frac{\vec{q}\,\vec{\sigma}_n \cdot \vec{q}}{2m_p}$$

Contributions of different parts of the nucleon current. Note that the AP (axial-pseudoscalar interference) contains $q^2/(q^2 + m_\pi^2)$, and MM contains $q^2/4M_p^2$.



Various sources of uncertainty in QRPA,
due to uncertainty in the input parameters.

- Mean field
- Size of the model space
- Residual nucleon-nucleon interaction

  schematic and realistic interactions (Brueckner G-matrix from Bonn, Argonne, Nijmegen OBEP)

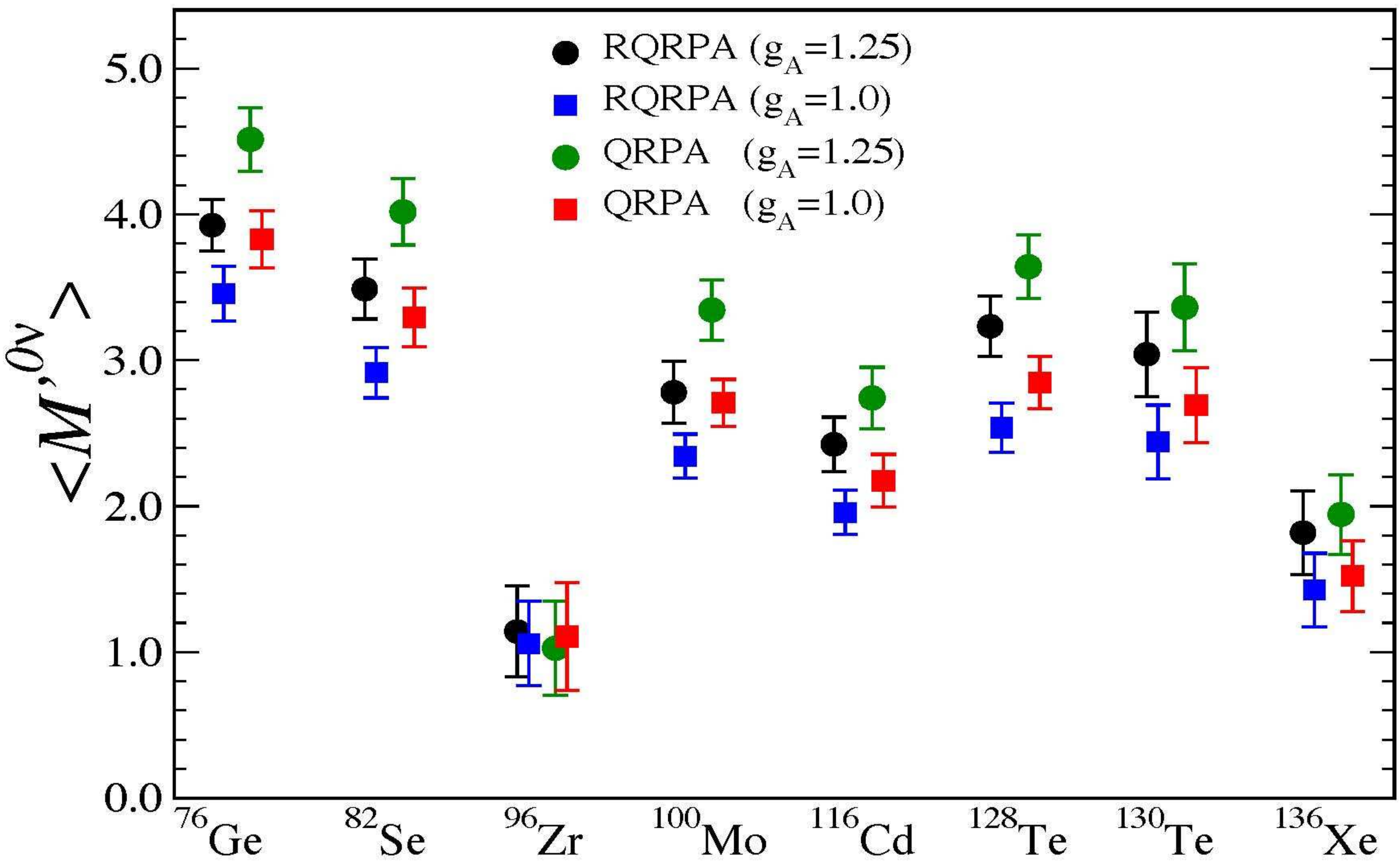
  Renormalizations within QRPA:

  – In pairing channel g_{pair} — BCS model
  – In particle-hole channel g_{ph} — the energy of GTR ($g_{ph} \approx 1$).
  – In particle-particle channel g_{pp}

  The renormalization depends on the basis size (cut-off dependence)

- Renormalization of axial-vector coupl. constant g_A=1.0–1.25

Ranges of calculated $M'^{0\nu}$ due to the QRPA uncertainties



Additional sources of variations and/or uncertainties due to the modifications of the neutrino potential

- Finite size of the nucleon
- Two-nucleon short-range correlations (s.r.c.) (Jastrow factor, UCOM, etc.)
- Higher order terms of the nucleon weak current (induced pseudoscalar and weak magnetism)

Effects of treatment of s.r.c. and comparison with NSM

QuickTime™ and a
decompressor
are needed to see this picture.

Note the smooth dependence on A and Z, both in QRPA and NSM

$0\nu\beta\beta$ half-lives for $\langle m_{\beta\beta}\rangle$ = 100 meV based on the QRPA matrix elements of Simkovic et al. (arXiv:0710.2055).

This is a conservative full range based on the estimated QRPA uncertainty in the nuclear matrix elements.
The estimates are highly correlated, if one of them is indeed near its upper edge, all of them are.

^{76}Ge	**(1-3) x 10^{26} y**	GERDA plans, Phase II, to reach 2×10^{26}y
^{82}Se	**(0.5 - 1.2) x 10^{26} y**	
^{100}Mo	**(0.25 - 1) x 10^{26} y**	
^{130}Te	**(0.25 - 1) x 10^{26} y**	CUORE plans to reach $(2\text{-}6)\times10^{26}$y
^{136}Xe	**(0.5 - 4) x 10^{26} y**	EXO-200 plans to reach 6×10^{25}y

Note: The sensitivity to $\langle m_{\beta\beta}\rangle$ scales as $1/(T_{1/2})^{1/2}$

Conclusions of this lecture:

- Various physics effects that influence the magnitude of the $0\nu\beta\beta$ nuclear matrix elements have been identified.
- The corresponding corrections, within QRPA, were estimated.
- In particular, the competition between the `pairing', $J = 0$, and the `broken pairs', $J \neq 0$, contributions causes almost complete cancellation for the internucleon distance $r \geq$ 2-3 fm, hence making the short range behavior important.
- Thus the treatment of the nucleon finite size, induced weak currents and the short range nucleon-nucleon repulsion causes visible changes in the nuclear matrix elements.
- There is little independent information about such effects (for analogous charge-changing operators). Thus, the prudent approach is to include them in the corresponding systematic error.
- The total range, assuming the basic validity of QRPA, is reasonable, and the qualitative agreement with the ISM is encouraging.

Results obtained in collaboration with Fedor Simkovic, Vadim Rodin, Amand Faessler and Jonathan Engel.