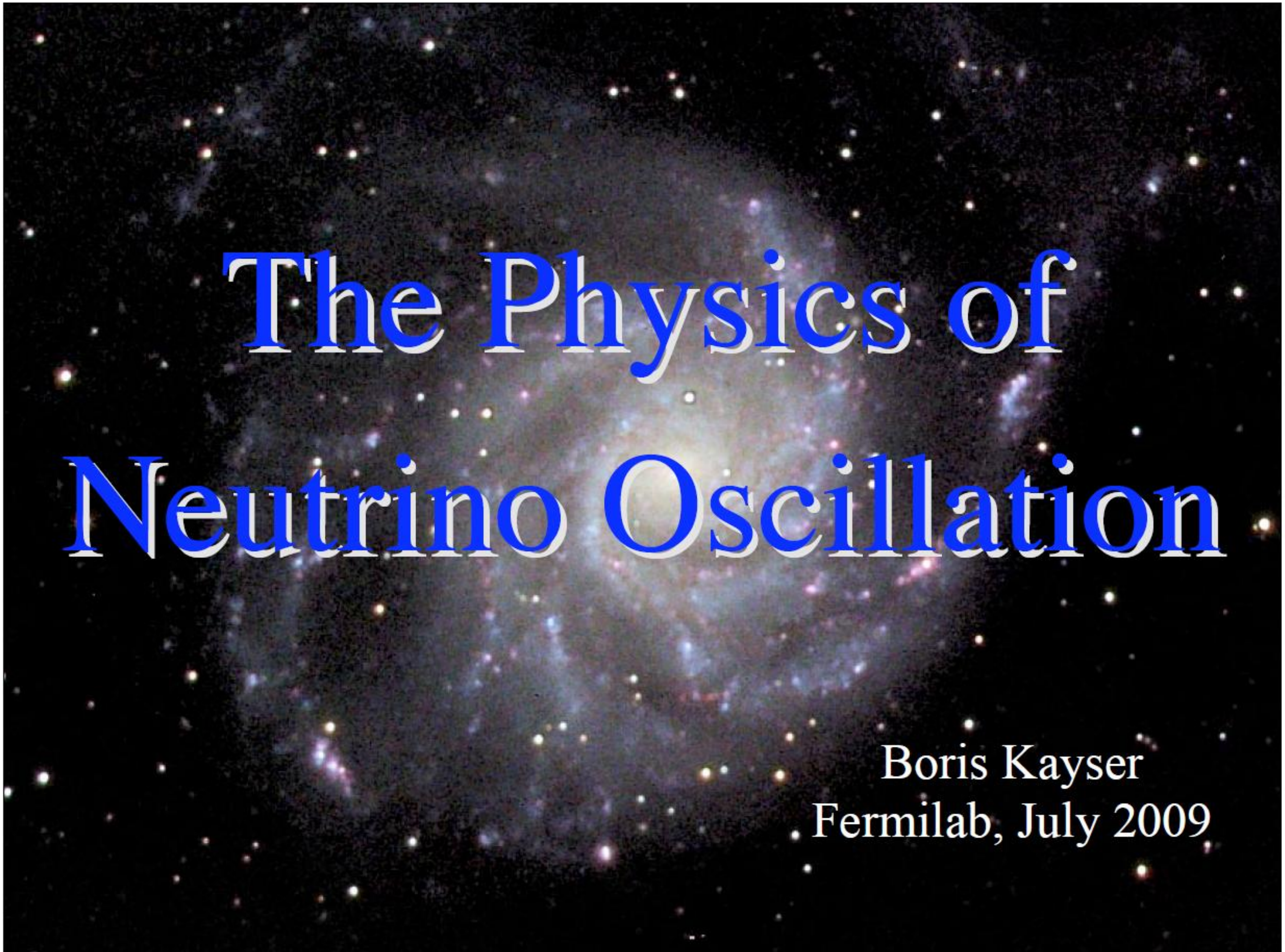
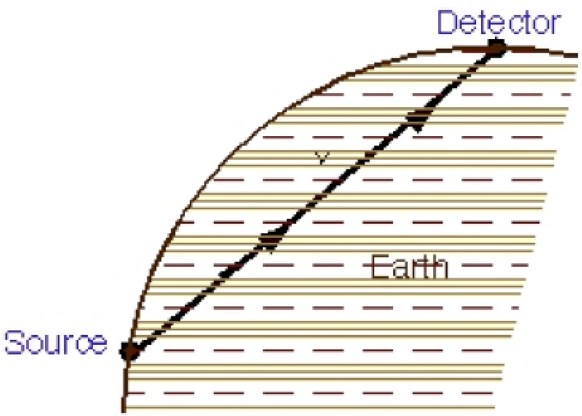


The Physics of Neutrino Oscillation

Boris Kayser
Fermilab, July 2009

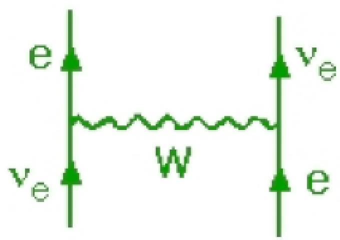
Neutrino Flavor Change in Matter



Coherent forward scattering from ambient matter can have a big effect.

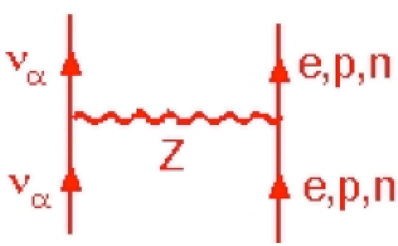
Interaction

Interaction Potential Energy



$$V_W = +\sqrt{2} G_F N_e \qquad (-\text{ for } \overline{\nu_e})$$

N_e: #e/vol



$$V_Z = -\frac{\sqrt{2}}{2} G_F N_n \qquad (+\text{ for } \overline{\nu_\alpha})$$

N_n: #n/vol

$$i\frac{\partial}{\partial t}\begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix} = \begin{bmatrix} & & \\ & H & \\ & & \end{bmatrix}\begin{bmatrix} f_e(t) \\ f_\mu(t) \end{bmatrix}$$

In general

In vacuum

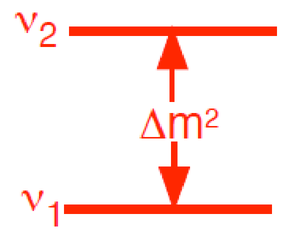
$$< \nu_\alpha|H|\nu_\beta > = < \sum_i U^*_{\alpha i}\nu_i|H|\sum_j U^*_{\beta j}\nu_j > = \sum_j U_{\alpha j}U^*_{\beta j}\sqrt{p^2 + m_j^2}$$

Momentum of the beam

In flavor change, only relative phases, hence relative energies, matter.

$\therefore$ In H, any multiple of the Identity Matrix I may be omitted.

In Vacuum



$$U = \begin{array}{c} \\ \nu_e \\ \nu_\mu \end{array} \begin{array}{c} \begin{array}{cc} \nu_1 & \nu_2 \end{array} \\ \left[\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right] \end{array} \quad ; \quad \begin{array}{l} \nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta \\ \nu_\mu = \nu_1(-\sin\theta) + \nu_2 \cos\theta \end{array}$$

It follows that, omitting a piece $\propto$ I,

$$H_{\mathrm{Vac}} = \frac{\Delta m^2}{4E} \left[\begin{array}{cc} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{array} \right] .$$

With Schrödinger's Equation, this gives the usual $P(\nu_e \to \nu_\mu)$.

The eigenvalues of H_{Vac} are —

$$\pm\frac{\Delta m^2}{4E} \equiv \pm\lambda \quad .$$

With $c \equiv \cos\theta$, $s \equiv \sin\theta$,

$$\nu_e = \nu_1 c + \nu_2 s \xrightarrow{t} \nu(t) = \nu_1 c e^{i\lambda t} + \nu_2 s e^{-i\lambda t}$$

$$P(\nu_e \to \nu_\mu) = \ | < \nu_\mu|\nu(t) > |^2 = |sc(-e^{i\lambda t} + e^{-i\lambda t})|^2$$

$$= \sin^2 2\theta \sin^2(\Delta m^2 \frac{L}{4E})$$

In Matter

$$H_M = H_{\text{Vac}} + V_W \begin{array}{c} \nu_e \quad \nu_\mu \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right] \begin{array}{c} \nu_e \\ \nu_\mu \end{array} \end{array} + \underbrace{V_Z \begin{array}{c} \nu_e \quad \nu_\mu \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \begin{array}{c} \nu_e \\ \nu_\mu \end{array} \end{array}}_{\propto \text{ I, so drop}}$$

$$H_M = H_{\text{Vac}} + \frac{V_W}{2} \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] + \overbrace{\frac{V_W}{2} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]}$$

$$H_M = \frac{\Delta m^2}{4E} \left[\begin{array}{cc} -(\cos 2\theta - x) & \sin 2\theta \\ \sin 2\theta & \cos 2\theta - x \end{array}\right] ,$$

$$\text{with} \quad x \equiv \frac{V_W/2}{\Delta m^2/4E} = \frac{2\sqrt{2} G_F N_e E}{\Delta m^2} .$$

The Effective Splitting and Mixing in Matter

If we define —

$$\Delta m_M^2 \equiv \Delta m^2 \sqrt{\sin^2 2\theta + (\cos 2\theta - x)^2}$$

and

$$\sin^2 2\theta_M \equiv \frac{\sin^2 2\theta}{\sin^2 2\theta + (\cos 2\theta - x)^2} \quad ,$$

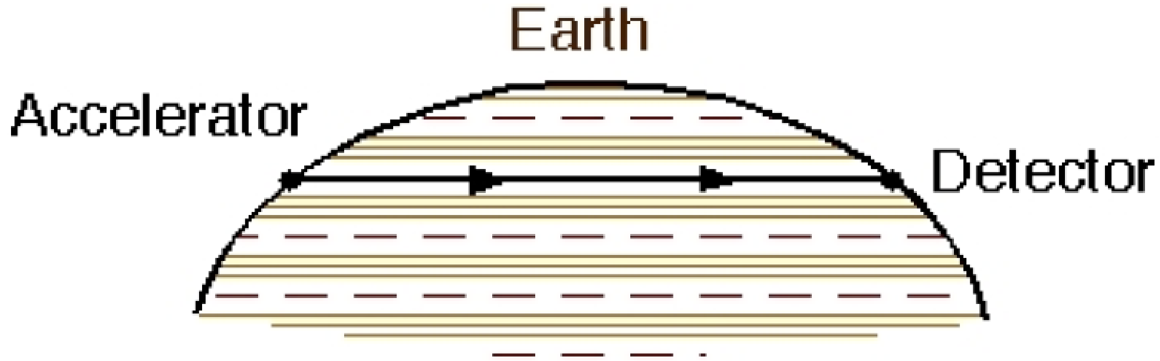
then

$$H_M = \frac{\Delta m_M^2}{4E} \begin{bmatrix} -\cos 2\theta_M & \sin 2\theta_M \\ \sin 2\theta_M & \cos 2\theta_M \end{bmatrix} \quad .$$

This is H_{Vac} with $(\Delta m^2, \theta) \rightarrow (\Delta m_M{}^2, \theta_M)$.

Thus, $\Delta m_M{}^2$ and θ_M are the effective splitting and mixing angle in matter.

Travel Through the Earth



The matter density encountered en route is ~ constant.

Thus, H_M is position-independent, just like H_{Vac}.

Therefore, in the earth (but not too deep),

$$P_M(\nu_e \to \nu_\mu) = \sin^2 2\theta_M \sin^2(\Delta m_M^2 \frac{L}{4E})$$

↑ In matter

The Size and Consequence of the Matter Effect

The matter effect depends on —

$$x = \frac{2\sqrt{2}G_F N_e E}{\Delta m^2} \propto E \quad .$$

The denominator contains a Sign

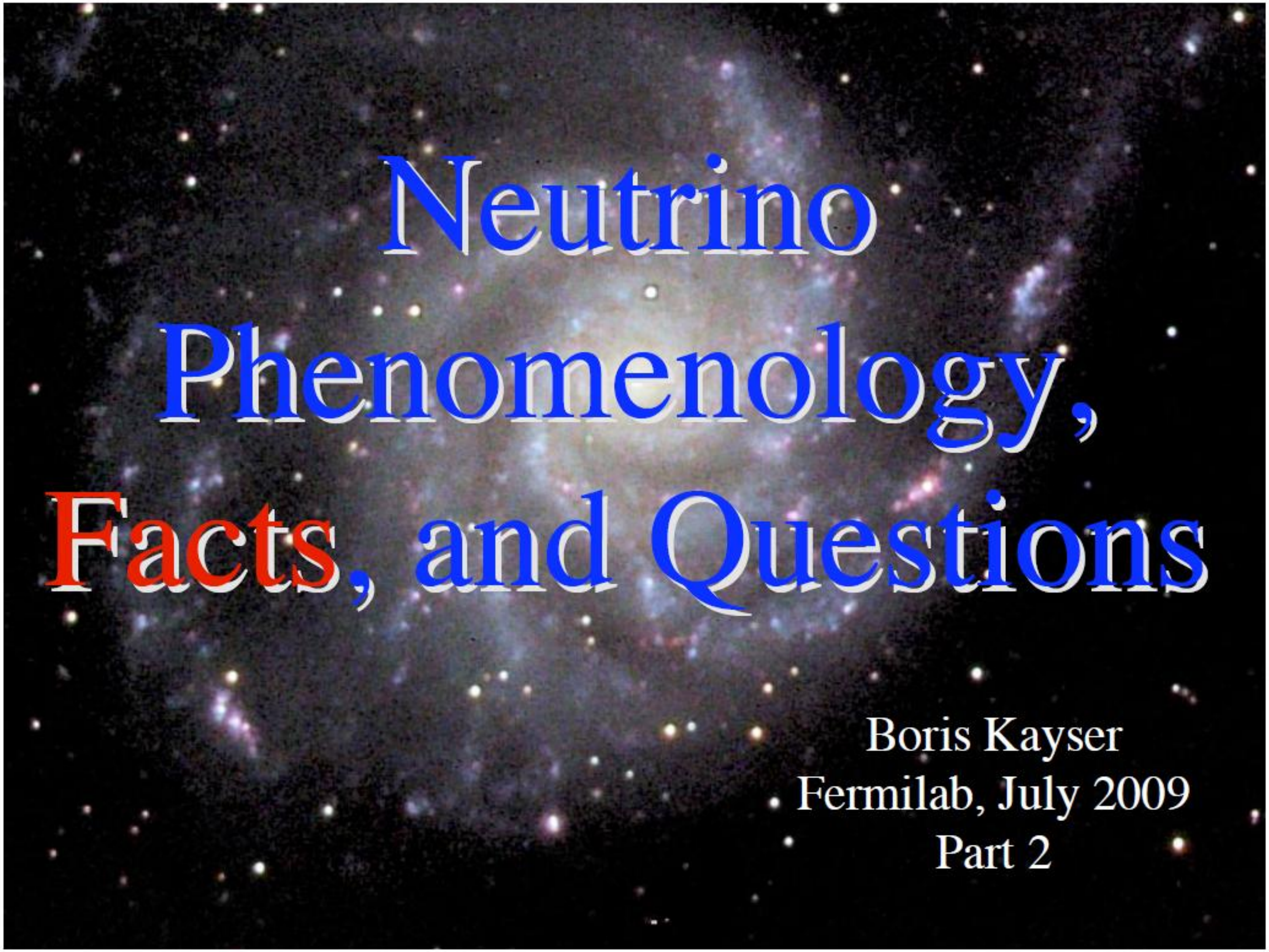
In the earth's mantle, for $|\Delta m^2| = |\Delta m^2(\text{atmospheric})|$ $\cong 2.4 \times 10^{-3}\ \text{eV}^2$,

$$|x| \simeq \frac{E}{11\text{GeV}}$$

Since $V_W(\bar{\nu}) = -V_W(\nu)$, $x(\bar{\nu}) = -x(\nu)$.

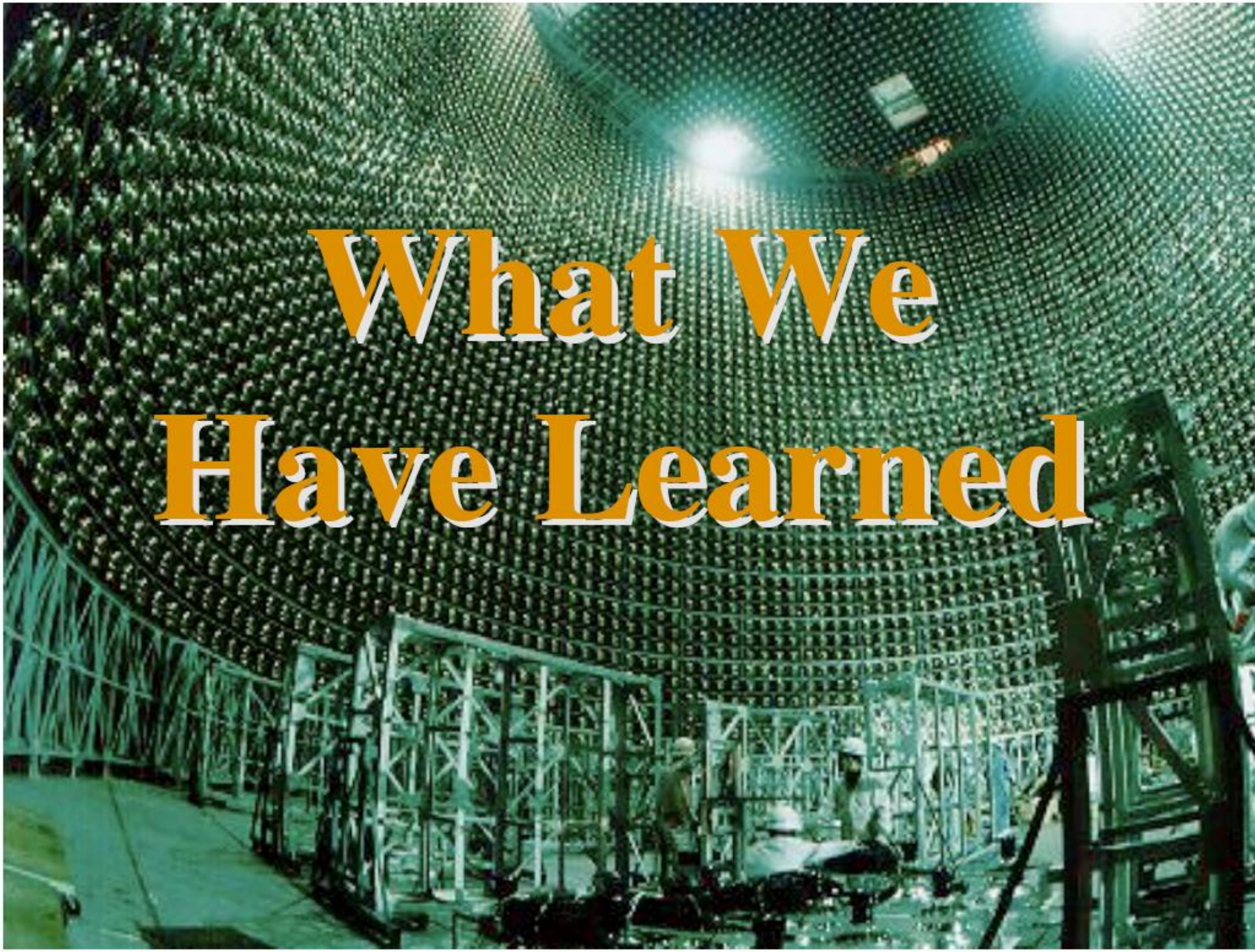
Thus $\overline{\Delta m_M^{\ 2}} \neq \Delta m_M^{\ 2}$ and $\sin^2 2\bar{\theta}_M \neq \sin^2 2\theta_M$.

The matter effect causes an asymmetry between $\bar{\nu}$ and ν oscillation. This must be separated from the genuine $\not{C}\!\!P$ asymmetry.

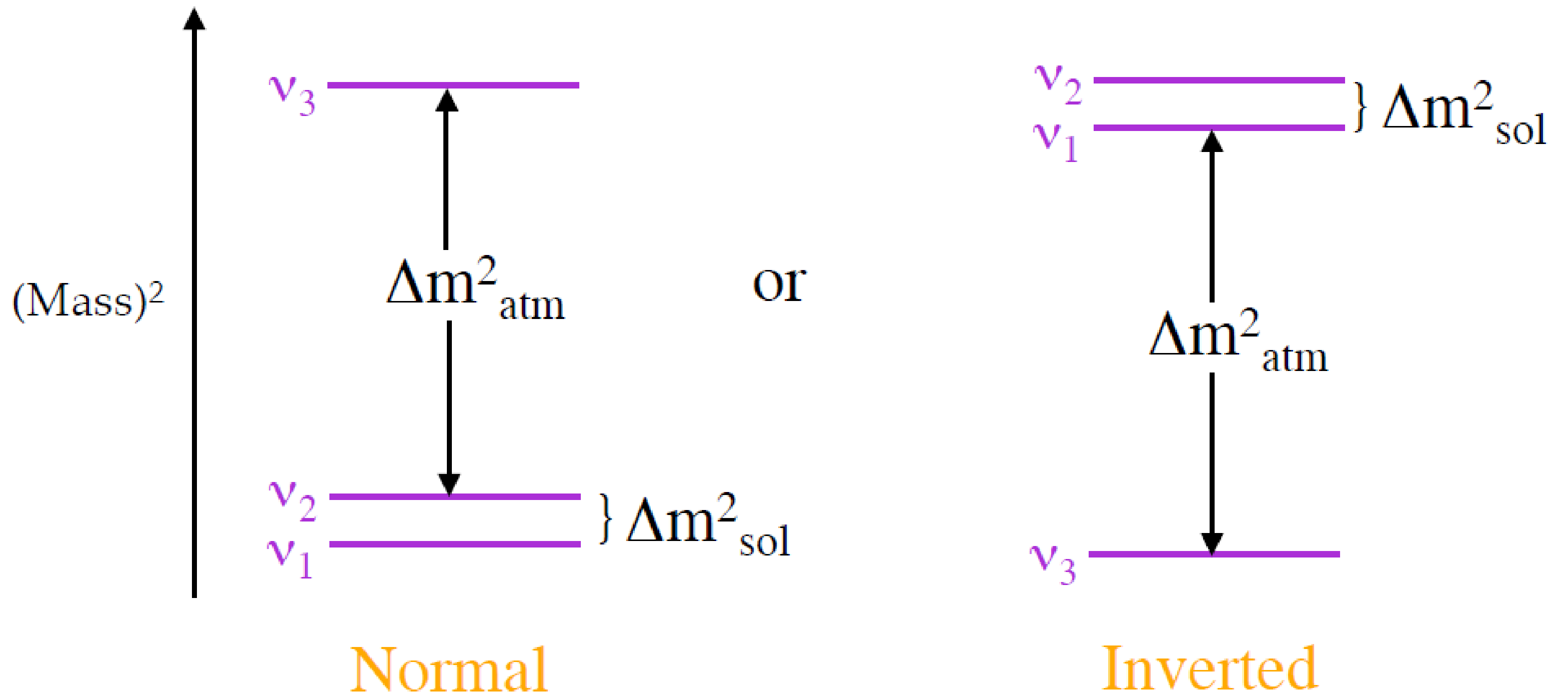

Neutrino
Phenomenology,
Facts, and Questions
Boris Kayser
Fermilab, July 2009
Part 2

Evidence For Flavor Change

Neutrinos	Evidence of Flavor Change
Solar	Compelling
Reactor (L ~ 180 km)	Compelling
Atmospheric	Compelling
Accelerator (L = 250 and 735 km)	Compelling
Stopped μ^+ Decay $\begin{pmatrix} \text{LSND} \\ L \approx 30\ \text{m} \end{pmatrix}$	Unconfirmed by MiniBooNE

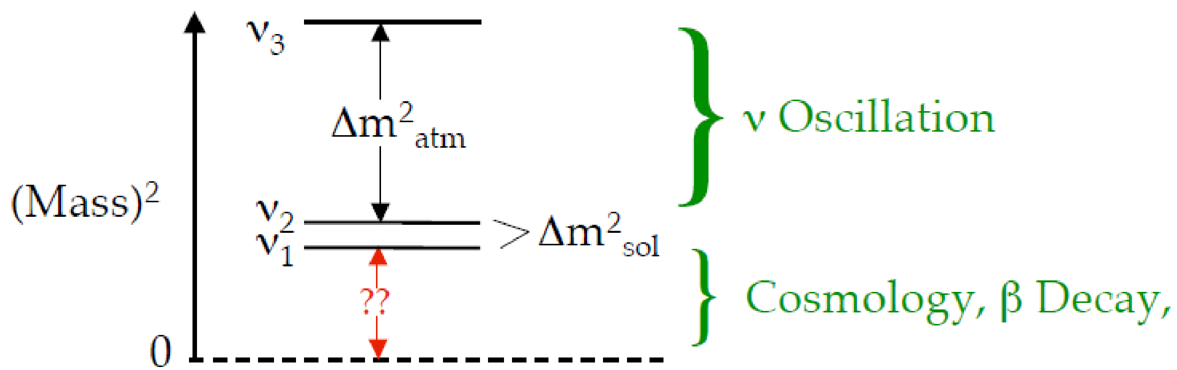

What We
Have Learned

The $(\text{Mass})^2$ Spectrum



$\Delta m^2_{sol} \cong 7.6 \times 10^{-5}\ \mathrm{eV}^2$, $\Delta m^2_{atm} \cong 2.4 \times 10^{-3}\ \mathrm{eV}^2$

The Absolute Scale of Neutrino Mass



How far above zero is the whole pattern?

Oscillation Data $\Rightarrow \sqrt{\Delta m^2_{atm}} <$ Mass[Heaviest ν_i]

The Upper Bound From Cosmology

Neutrino mass affects large scale structure.

Cosmological Data + Cosmological Assumptions $\Rightarrow$

$$\Sigma\, m_i < (0.17 - 1.0)\ \text{eV}\ .$$

$\uparrow$ Mass(ν_i)

(Seljak, Slosar, McDonald
Hannestad; Pastor)

If there are only 3 neutrinos,

$$0.04\ \text{eV} \lesssim \text{Mass}[\text{Heaviest}\ \nu_i] < (0.07 - 0.4)\ \text{eV}$$

$\uparrow$ $\sqrt{\Delta m^2_{atm}}$ $\qquad$ Cosmology $\uparrow$

The Upper Bound From Tritium

Cosmology is wonderful, but there are known loopholes in its argument concerning neutrino mass.

The absolute neutrino mass can in principle also be measured by the kinematics of β decay.

Tritium decay: $^3H \to {}^3He + e^- + \overline{\nu_i}$; $i = 1, 2,$ or 3

$$BR\left({}^3H \to {}^3He + e^- + \overline{\nu_i} \right) \propto |U_{ei}|^2$$

In $^3H \to {}^3He + e^- + \overline{\nu_i}$, the bigger m_i is, the smaller the maximum electron energy is.

There are 3 separate thresholds in the β energy spectrum.

The β energy spectrum is modified according to —

$$(E_0 - E)^2 \Theta[E_0 - E] \Rightarrow \sum_i |U_{ei}|^2 (E_0 - E)\sqrt{(E_0 - E)^2 - m_i^2}\, \Theta[(E_0 - m_i) - E]$$

Maximum β energy when there is no neutrino mass

β energy

Present experimental energy resolution is insufficient to separate the thresholds.

Measurements of the spectrum bound the average neutrino mass —

$$\langle m_\beta \rangle = \sqrt{\sum_i |U_{ei}|^2 m_i^2}$$

Presently: $\langle m_\beta \rangle < 2.3$ eV

Leptonic Mixing

This has the consequence that —

Mass eigenstate → $|\nu_i> = \sum_{\alpha} U_{\alpha i} |\nu_\alpha>$. ← Flavor eigenstate

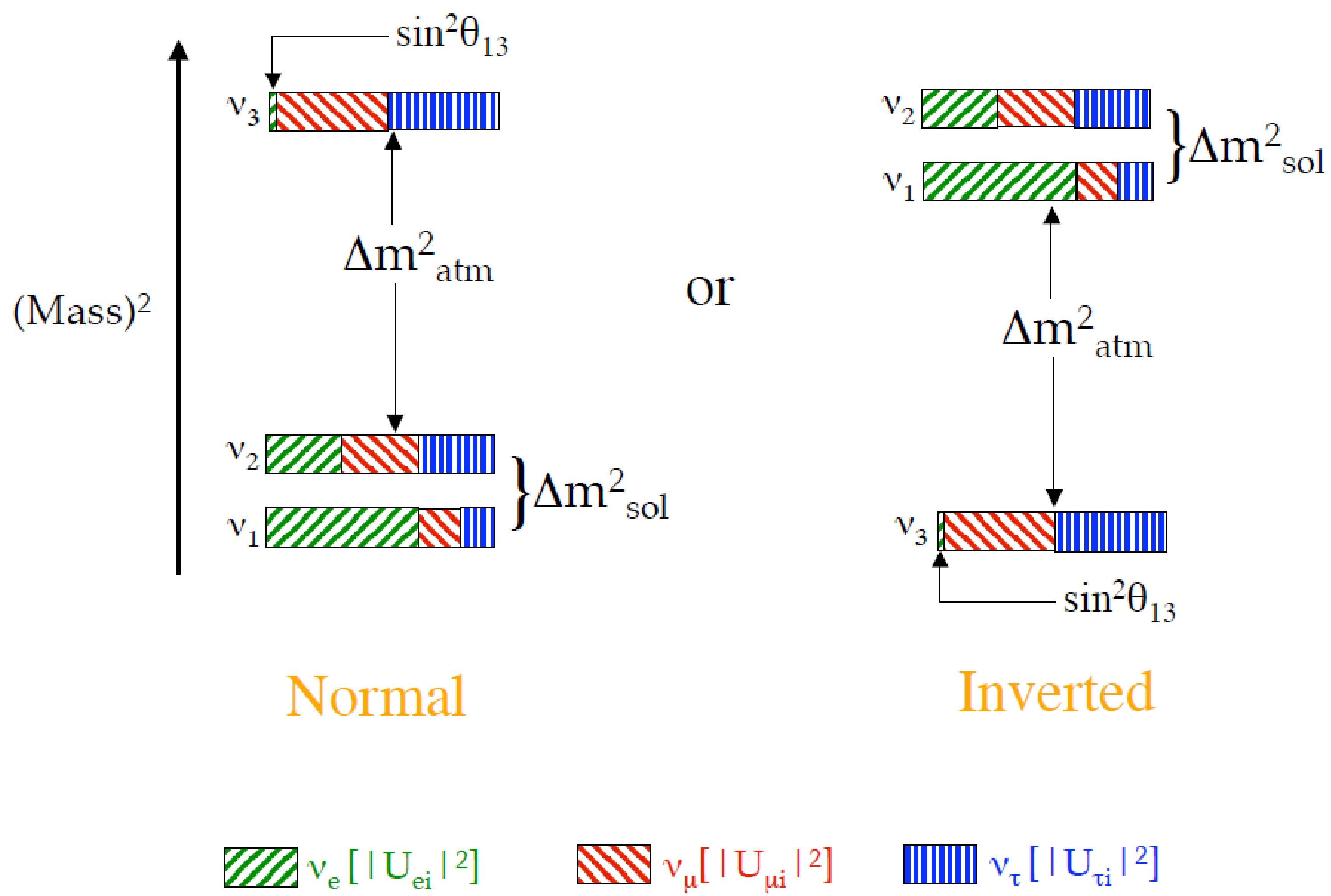
α: e, μ, or τ

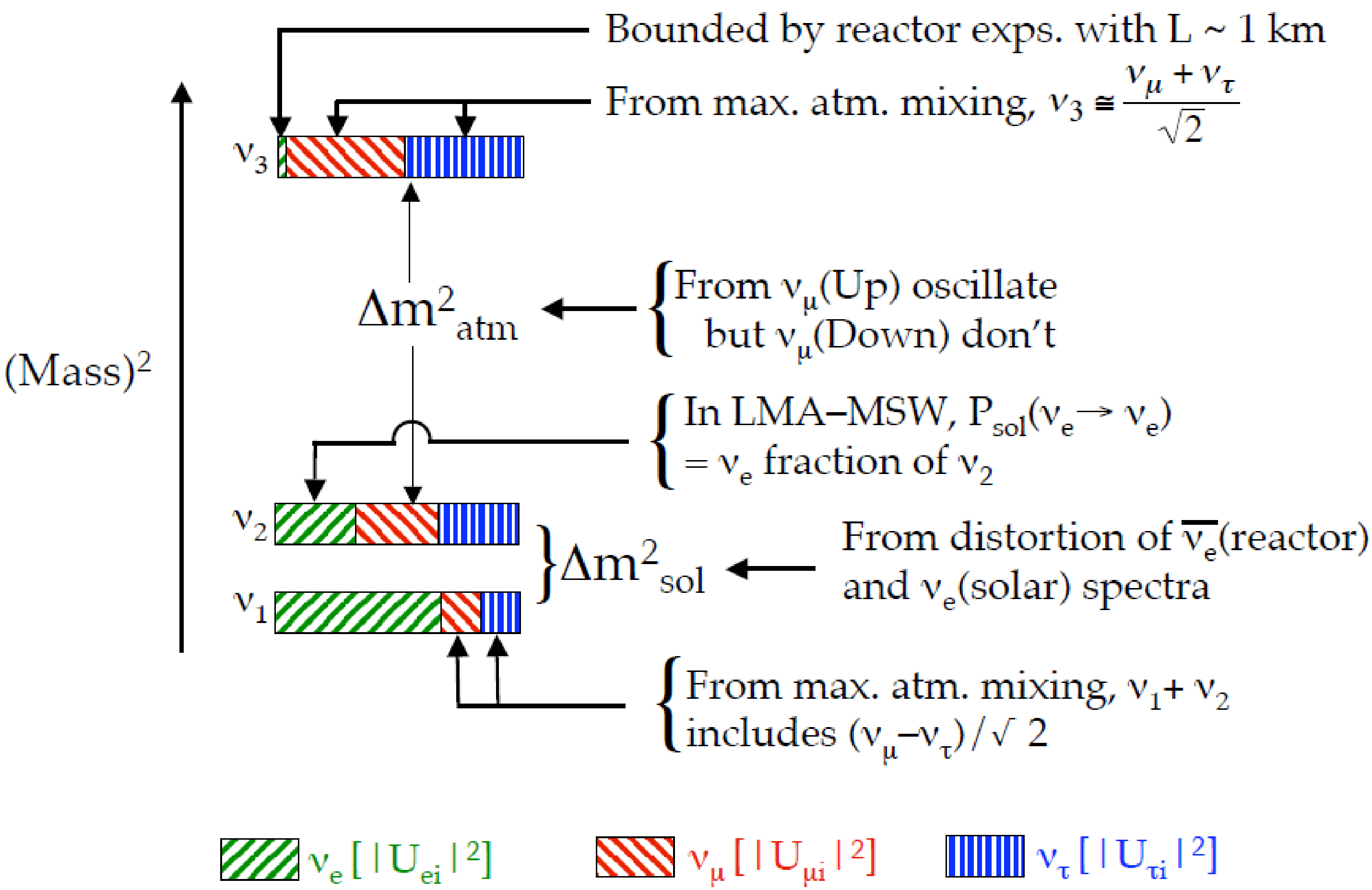
$U_{\alpha i}$: PMNS Leptonic Mixing Matrix

Flavor-α fraction of $\nu_i = |U_{\alpha i}|^2$.

When a ν_i interacts and produces a charged lepton, the probability that this charged lepton will be of flavor α is $|U_{\alpha i}|^2$.

The spectrum, showing its approximate flavor content, is




Bounded by reactor exps. with L ~ 1 km
From max. atm. mixing, $\nu_3 \cong \frac{\nu_\mu + \nu_\tau}{\sqrt{2}}$
ν_3
Δm^2_{atm}
From ν_μ(Up) oscillate but ν_μ(Down) don't
(Mass)^2
In LMA–MSW, $P_{sol}(\nu_e \to \nu_e)$ = ν_e fraction of ν_2
ν_2
ν_1
Δm^2_{sol}
From distortion of $\overline{\nu_e}$(reactor) and ν_e(solar) spectra
From max. atm. mixing, $\nu_1 + \nu_2$ includes $(\nu_\mu - \nu_\tau)/\sqrt{2}$
ν_e [$|U_{ei}|^2$]
ν_μ [$|U_{\mu i}|^2$]
ν_τ [$|U_{\tau i}|^2$]

The 3 X 3 Unitary Mixing Matrix

Caution: We are *assuming* the mixing matrix U to be 3 x 3 and unitary.

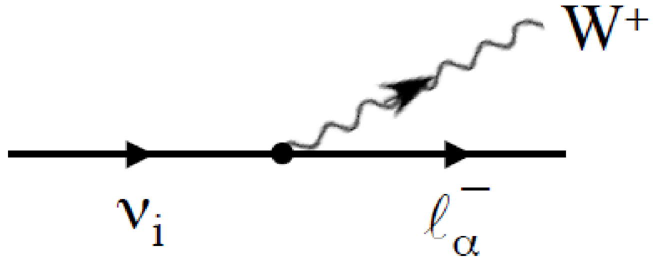
$$L_{SM} = -\frac{g}{\sqrt{2}} \sum_{\alpha=e,\mu,\tau} \left(\overline{\ell}_{L\alpha} \gamma^{\lambda} \nu_{L\alpha} W_{\lambda}^{-} + \overline{\nu}_{L\alpha} \gamma^{\lambda} \ell_{L\alpha} W_{\lambda}^{+} \right)$$

$$= -\frac{g}{\sqrt{2}} \sum_{\substack{\alpha=e,\mu,\tau \\ i=1,2,3}} \left(\overline{\ell}_{L\alpha} \gamma^{\lambda} U_{\alpha i} \nu_{Li} W_{\lambda}^{-} + \overline{\nu}_{Li} \gamma^{\lambda} U_{\alpha i}^{*} \ell_{L\alpha} W_{\lambda}^{+} \right)$$

$$(CP)\left(\overline{\ell}_{L\alpha} \gamma^{\lambda} U_{\alpha i} \nu_{Li} W_{\lambda}^{-} \right)(CP)^{-1} = \overline{\nu}_{Li} \gamma^{\lambda} U_{\alpha i} \ell_{L\alpha} W_{\lambda}^{+}$$

Phases in U will lead to CP violation, unless they are removable by redefining the leptons.

$U_{\alpha i}$ describes —



$$U_{\alpha i} \sim \langle \ell_\alpha^- W^+ \,|\, H \,|\, \nu_i \rangle$$

$$\text{When } |\nu_i\rangle \rightarrow |e^{i\varphi}\nu_i\rangle,\; U_{\alpha i} \rightarrow e^{i\varphi}\, U_{\alpha i}$$

$$\text{When } |\ell_\alpha^-\rangle \rightarrow |e^{i\varphi}\ell_\alpha^-\rangle,\; U_{\alpha i} \rightarrow e^{-i\varphi}\, U_{\alpha i}$$

Thus, one may multiply any column, or any row, of U by a complex phase factor without changing the physics.

Some phases may be removed from U in this way.

Exception: If the neutrino mass eigenstates are their own antiparticles, then —

Charge conjugate

$$\nu_i = \nu_i^{\,c} = C\overline{\nu_i}^{\,T}$$

One is no longer free to phase-redefine ν_i without consequences.

U can contain additional CP-violating phases.

How Many Mixing Angles and $\not{CP}$ Phases Does U Contain?

Real parameters before constraints:	18
Unitarity constraints — $\sum_i U_{\alpha i}{}^{*} U_{\beta i} = \delta_{\alpha\beta}$	
Each row is a vector of length unity:	– 3
Each two rows are orthogonal vectors:	– 6
Rephase the three ℓ_α :	– 3
Rephase two ν_i, if $\overline{\nu_i} \neq \nu_i$:	– 2
Total physically-significant parameters:	4
Additional (Majorana) $\not{CP}$ phases if $\overline{\nu_i} = \nu_i$:	2

How Many Of The Parameters Are Mixing Angles?

The *mixing angles* are the parameters in U when it is *real*.

U is then a three-dimensional rotation matrix.

Everyone knows such a matrix is described in terms of 3 angles.

Thus, U contains 3 mixing angles.

Summary

Mixing angles	$\not{C}\not{P}$ phases if $\bar{\nu}_i \neq \nu_i$	$\not{C}\not{P}$ phases if $\bar{\nu}_i = \nu_i$
3	1	3

The Mixing Matrix

$$U = \overset{\text{Atmospheric}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix}} \times \overset{\text{Cross-Mixing}}{\begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix}} \times \overset{\text{Solar}}{\begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$c_{ij} \equiv \cos \theta_{ij}$$
$$s_{ij} \equiv \sin \theta_{ij}$$

$$\times \begin{bmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Majorana ~~CP~~ phases

$\theta_{12} \approx \theta_{sol} \approx 34°$, $\theta_{23} \approx \theta_{atm} \approx 38\text{-}52°$, $\theta_{13} \lesssim 10°$

δ would lead to $P(\overline{\nu_\alpha} \rightarrow \overline{\nu_\beta}) \neq P(\nu_\alpha \rightarrow \nu_\beta)$. ~~CP~~

But note the crucial role of $s_{13} \equiv \sin \theta_{13}$.

Good Luck

Because $(\Delta m^2_{sol} / \Delta m^2_{atm}) << 1$ and $\theta_{13} << 1$, all confirmed flavor change processes seen so far are effectively two-neutrino processes.

Because $\theta_{13} << 1$, $\theta_{atm} \approx \theta_{23}$ and $\theta_{sol} \approx \theta_{12}$.

This has greatly simplified the analysis of what is happening.

The Majorana $\not{CP}$ Phases

The phase α_i is associated with neutrino mass eigenstate ν_i:

$$U_{\alpha i} = U^0_{\alpha i} \exp(i\alpha_i/2) \text{ for all flavors } \alpha.$$

$$\text{Amp}(\nu_\alpha \rightarrow \nu_\beta) = \sum_i U_{\alpha i}^* \exp(-\, i m_i^2 L/2E)\, U_{\beta i}$$

is insensitive to the Majorana phases α_i.

Only the phase δ can cause CP violation in neutrino oscillation.

There Is Nothing Special About θ_{13}

All mixing angles must be nonzero for $\not{CP}$ in oscillation.

For example —

$$P(\bar{\nu}_\mu \to \bar{\nu}_e) - P(\nu_\mu \to \nu_e) = 2\cos\theta_{13}\sin 2\theta_{13}\sin 2\theta_{12}\sin 2\theta_{23}\sin\delta$$
$$\times \sin\left(\Delta m^2{}_{31}\frac{L}{4E}\right)\sin\left(\Delta m^2{}_{32}\frac{L}{4E}\right)\sin\left(\Delta m^2{}_{21}\frac{L}{4E}\right)$$

In the factored form of U, one can put δ next to θ_{12} instead of θ_{13}.