Exercises for Advanced Quantum Field Theory (2018/19)

Problem 1. Let $T^a \equiv (T^a)^{ij}$, $a = 1, \cdots, N^2-1$, $i, j = 1, \cdots, N$, be an antihermitian basis for the fundamental representation R of the Lie algebra of $SU(N)$, with

$$[T^a, T^b] = f^{abc}T^c, \quad tr(T^aT^b) = -\frac{\delta^{ab}}{2}, \quad tr(f^af^b) = -N\delta^{ab}, \quad tr(f^af^bf^c) = \frac{N}{2}\, f^{abc}.$$

a) Show that the completely symmetric tensor

$$d^{abc} \equiv i\, tr(T^{(a}T^bT^{c)}) \tag{1}$$

- which can be defined for an arbitrary Lie group G and for any representation - is an *invariant* real tensor.
b) Noting that the $N \times N$ matrices

$$i\left(T^{(a}T^{b)} + \frac{\delta^{ab}}{2N}\,\mathbf{1}\right)$$

are antihermitian and traceless, and that the T^a form a basis for such matrices, prove the identity

$$T^aT^b = \frac{1}{2}\, f^{abc}T^c - \frac{\delta^{ab}}{2N}\,\mathbf{1} + 2id^{abc}T^c. \tag{2}$$

Verify this relation explicitly for $SU(2)$.
c) Using (2) verify that the *Casimir* invariant of the fundamental representation, defined by $T^aT^a = -C_R\mathbf{1}$, is $C_R = (N^2-1)/2N$.
d) Suppose that C^{ijmn} is an *invariant* tensor, where the indices i and j transform in the fundamental representation R, while m and n transform in the conjugate representation $\overline{R}$. Prove that one has

$$C^{ijmn} = a\,\delta^{im}\delta^{jn} + b\,\delta^{in}\delta^{jm}, \tag{3}$$

where a and b are constants. *Hint*: analyze the multiple product $R \times R \times \overline{R} \times \overline{R}$ using that $R \times \overline{R} = 1 + adj$ - where adj denotes the adjoint representation - and remember that the product $R_1 \times R_2$ contains a (unique) singlet only if $R_1 = \overline{R}_2$.
e) Show that one has the relation between invariant tensors

$$(T^a)^{im}(T^a)^{jn} = \frac{1}{2}\left(\frac{1}{N}\,\delta^{im}\delta^{jn} - \delta^{in}\delta^{jm}\right). \tag{4}$$

Problem 2. Le $\varphi \equiv \varphi^i$ be complex scalar fields transforming in the fundamental representation R of $SU(N)$.
a) Show that there is only *one* independent quartic (particle number preserving) polynomial invariant under $SU(N)$, *i.e.* a polynomial of the kind $\varphi^\dagger\varphi^\dagger\varphi\varphi$. Explain why the result suggested by the decomposition $(R\times\overline{R})\times(R\times\overline{R}) = (1+adj)\times(1+adj) = 1+1+\cdots$,

i.e. *two* singlets, is not correct.
b) Show that the polynomials

$$(\varphi^\dagger\varphi)(\varphi^\dagger\varphi), \quad (\varphi^\dagger T^a\varphi)(\varphi^\dagger T^a\varphi), \quad (\varphi^\dagger T^aT^b\varphi)(\varphi^\dagger T^aT^b\varphi), \tag{5}$$

corresponding respectively to the invariant tensors $\delta^{im}\delta^{jn}$, $(T^a)^{im}(T^a)^{jn}$ and $(T^aT^b)^{im}(T^aT^b)^{jn}$ contracting $\varphi^{\dagger i}\varphi^m\varphi^{\dagger j}\varphi^n$, are $SU(N)$-invariant.
c) Are the three polynomials in (5) linearly independent? If not find the relations between them.

Problem 3. Let $\varphi \equiv \varphi^a$, $a = 1, \cdots, 8$, be complex scalar fields transforming in the adjoint representation 8 of $SU(3)$.
a) Using that $8 \times 8 = (1 + 8 + 27)_S + (8 + 10 + \overline{10})_A$, see *e.g.* [Slansky, Phys. Rep.], determine the number of independent invariant quartic interactions of the kind $\varphi^\dagger\varphi^\dagger\varphi\varphi$, as in Problem 2.
b) Write three independent invariant quartic interaction terms.

Problem 4. Consider the Lagrangian of scalar QCD with gauge group $SU(N)$

$$\mathcal{L} = -\frac{1}{4}\,F^{a\mu\nu}F^a{}_{\mu\nu} + (D_\mu\varphi)^\dagger D^\mu\varphi - m^2\varphi^\dagger\varphi - P(\varphi,\varphi^\dagger) \equiv -\frac{1}{4}\,F^{a\mu\nu}F^a{}_{\mu\nu} + \mathcal{L}_\varphi, \tag{6}$$

where P is an invariant quartic polynomial and the scalars transform in a generic irreducible representation Θ^a of $SU(N)$:

$$D_\mu\varphi = (\partial_\mu - gA_\mu)\varphi, \quad A_\mu = A^a_\mu\Theta^a.$$

a) Determine the covariantly conserved color currents $J^{a\mu}$ and the conserved Nöther color currents $j^{a\mu}$.
b) Writing the renormalized scalar Lagrangian as

$$\begin{aligned}\mathcal{L}^{ren}_\varphi \;&=\; Z_\varphi(\partial_\mu\varphi^\dagger\partial^\mu\varphi - (m^2+\delta m^2)\varphi^\dagger\varphi) + gZ_{1\varphi}(\varphi^\dagger A^\mu\partial_\mu\varphi - \partial_\mu\varphi^\dagger A^\mu\varphi)\\ &\quad -g^2Z_{2\varphi}\,\varphi^\dagger\,A^\mu A_\mu\,\varphi - \widetilde{P}(\varphi,\varphi^\dagger),\end{aligned} \tag{7}$$

derive the Slavnov-Taylor identities for Z_φ, $Z_{1\varphi}$ and $Z_{2\varphi}$, analogous to $Z_3/Z_1 = Z_\Psi/Z_{1\Psi}$ etc.
c) Draw all one-loop Feynman diagrams that contribute to the renormalization of the four-gluon correlation function $\langle AAAA\rangle$.
d) Assuming that the scalars transform in the fundamental representation R of $SU(N)$, discuss the relation between $\widetilde{P}(\varphi,\varphi^\dagger)$ and $P(\varphi,\varphi^\dagger)$. In particular, how many independent coupling constants appear in these polynomials, if you want the theory to be strictly renormalizable?
e) Draw all one-loop Feynman diagrams contributing to the renormalization of the quartic scalar correlation function $\langle\varphi^\dagger\varphi^\dagger\varphi\,\varphi\rangle$. Determine the group-theoretical structure of their divergent parts and check explicitly if they are consistent with the answer to question d).

Problem 5. Consider a gauge theory invariant under a generic gauge group G, with N_f fermions and N_s complex scalars transforming respectively in the representations T^a and

Θ^a of the Lie algebra of G, whose dynamics is governed by the Lagrangian

$$\mathcal{L}_{fs} = -\frac{1}{4}\, F^{a\mu\nu} F^a{}_{\mu\nu} + \overline{\Psi}(i\gamma^\mu D_\mu - M)\Psi + (D_\mu\varphi)^\dagger D^\mu\varphi - m^2\varphi^\dagger\varphi - P(\varphi,\varphi^\dagger), \qquad (8)$$

where

$$D_\mu\varphi = (\partial_\mu - gA^a_\mu\Theta^a)\varphi, \quad D_\mu\Psi = (\partial_\mu - gA^a_\mu T^a)\Psi, \quad P(\varphi,\varphi^\dagger) = \frac{1}{4}\, C_{IJMN}\varphi^{\dagger I}\varphi^J\varphi^{\dagger M}\varphi^N.$$

In $P(\varphi,\varphi^\dagger)$ the indices I, J etc. label the representation associated with Θ^a, the quantities C_{IJMN} are constants with suitable symmetry properties, and a sum over the N_s scalars is understood.

a) Write down the Feynman rules of the theory.

b) Show that, in Lorenz-Feynman gauge with $\lambda = 1$, using dimensional regularization and relying on the minimal subtraction scheme, at one loop the gluon wave-function renormalizes according to

$$Z_3 = 1 + \frac{g^2}{(4\pi)^2\varepsilon}\left(\frac{10}{3}\, C_{adj} - \frac{8}{3}\, N_f T_f - \frac{2}{3}\, N_s T_s\right),$$

where T_f and T_s are the *Dynkin* indices of the representations T^a and Θ^a respectively and C_{adj} is the *Casimir* invariant of the adjoint representation. *Hint*: use the known results of a theory without scalars.

c) Using the known results of a theory without scalars, derive the one-loop β-function

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}\, C_{adj} - \frac{4}{3}\, N_f T_f - \frac{1}{3}\, N_s T_s\right).$$

d) Derive the one-loop β-function $\beta(\alpha)$ for the strong coupling constant $\alpha = g^2/4\pi$.

e) How many color-scalars N_s in the adjoint (or fundamental) representation of $SU(3)$ could we add at most to the Standard Model, to keep QCD asymptotically free?

Problem 6. Consider the Lagrangian (8) of Problem 5, where $P(\varphi,\varphi^\dagger)$ is the most general G-invariant quartic polynomial in the scalars, and take $N_f = N_s = 1$.

a) Is $\mathcal{L}_{fs}$ *strictly* renormalizable?

b) Suppose henceforth that the scalars are real and transform in the adjoint representation of G, i.e. $\varphi \equiv \varphi^a$ and $(\Theta^a)^{bc} = -f^{abc}$. Choose the G-invariant quartic polynomial

$$P(\varphi) = \frac{\alpha}{4}\,(\varphi^a\varphi^a)^2 + \frac{\beta}{4!}\, d^{abcd}\varphi^a\varphi^b\varphi^c\varphi^d, \qquad (9)$$

where α and β are coupling constants, and d^{abcd} is the *unique* independent completely symmetric invariant tensor with all indices in the adjoint representation[1]. In this case, a

[1] In every simple Lie algebra $\mathcal{G}$ there is a one-to-one correspondence between *completely symmetric* algebraically independent *invariant* tensors with all indices in the adjoint representation, and algebraically independent Casimir operators: for a Lie algebra of rank r there exist precisely r of them. In particular every $\mathcal{G}$ admits a *unique* quadratic Casimir operator, that for a compact $\mathcal{G}$ is given by $\tau^a\tau^b\delta^{ab} = \tau^a\tau^a$. There exists a *unique* cubic Casimir operator $d^{abc}\tau^a\tau^b\tau^c$ for the algebras $A_r = su(r)$ $(r > 2)$, see the definition (1), and none for the others; remember that $D_3 = so(6) = su(4)$. There exists a *unique* quartic Casimir operator $d^{abcd}\tau^a\tau^b\tau^c\tau^d$ for A_r $(r > 2)$, B_r $(r > 1)$, C_r $(r > 1)$ and D_r $(r > 2$ and $r \neq 4)$, there exist *two* of them for $D_4 = so(8)$, while none of them exist for the remaining ones, *i.e.* $A_1 = su(2)$, $A_2 = su(3)$, G_2, F_4, E_6, E_7 and E_8; see *e.g.* T.v. Ritbergen *et. al.*, Int. J. Mod. Phys. **A4** (1999) 41, hep-th/9802376.

more natural normalization for the terms quadratic in the scalars of the Lagrangian (8) is

$$\frac{1}{2}\left(D_\mu\varphi^a D^\mu\varphi^a - m^2\varphi^a\varphi^a\right).$$

Consider the interaction terms

$$X = i\overline{\Psi}T^a\Psi\varphi^a, \qquad Y = \frac{1}{3!}d^{abc}\varphi^a\varphi^b\varphi^c,$$

where d^{abc} is the invariant tensor given in equation (1) of Problem 1. Suppose for simplicity that the product $R \times \overline{R} = 1 + adj + \cdots$ contains the adjoint representation just once, where R denotes the representation corresponding to T^a. Can (the Yukawa coupling) X and (the triple scalar interaction) Y appear as divergent one-loop counterterms? Can they appear at higher loops? Answer using symmetry arguments.
c) Consider the Lagrangians

$$\mathcal{L}_1 = \mathcal{L}_{fs} + \gamma X, \qquad \mathcal{L}_2 = \mathcal{L}_{fs} + \mu Y,$$

where γ and μ are coupling constants. Are the Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$ strictly renormalizable? *Hint*: remember the concept of renormalizable and super-renormalizable interactions. If the answer is negative, draw some divergent one-loop or two-loop Feynman diagrams, whose renormalization requires the introduction of interaction terms not present respectively in $\mathcal{L}_1$ and $\mathcal{L}_2$. *Hint:* it may be useful to deal first with Problem 7.
d) Specify the discussion of the questions above to the particular case $G = SU(2)$.

Problem 7. Consider the Lagrangian $\mathcal{L}_2$ of Problem 6.
a) Are there *divergent* one-loop Feynman diagrams contributing to the correlation functions $\langle AA\varphi\rangle$ and $\langle AAA\varphi\rangle$?
b) Discuss the renormalizability, in the strict sense, of the theory associated with $\mathcal{L}_2$ in the light of the answer to the previous question.

Problem 8. Consider the Lagrangian (see Problems 5 and 6)

$$\mathcal{L}_{XY} = \mathcal{L}_{fs} + \gamma X + \mu Y.$$

a) Are there *divergent* one-loop Feynman diagrams contributing to the correlation functions $\langle AA\varphi\rangle$ and $\langle AAA\varphi\rangle$?
b) Discuss the renormalizability, in the strict sense, of the theory associated with $\mathcal{L}_{XY}$ in the light of the answer to the previous question.

Problem 9. Consider the Lagrangian $\mathcal{L}_{XY}$ of Problem 8.
a) Draw all one-loop Feynman diagrams that contribute to the renormalization of the correlation function $\langle\overline{\Psi}\Psi A\rangle$.
b) Introducing the renormalized minimal-interaction Lagrangian $\mathcal{L}^{ren} = -igZ_{1\Psi}\overline{\Psi}\gamma^\mu A_\mu\Psi$, compute the renormalization constant $Z_{1\Psi}$ at one-loop order. *Hint*: using the known result in the absence of scalars, one concludes that

$$Z_{1\Psi} = 1 - \frac{g^2}{(4\pi)^2\varepsilon}\left(2C_{adj} + 2C_f\right) + O\left(\gamma^2\right),$$

where C_f is the *Casimir* invariant of the fermion representation.
c) Draw all one-loop Feynman diagrams contributing to the renormalization of the correlation function $\langle \varphi\varphi A \rangle$.
d) Draw all one-loop Feynman diagrams contributing to the renormalization of the correlation function $\langle \varphi\varphi AA \rangle$.
e) Introduce the renormalization constants $Z_{1\varphi}$ and $Z_{2\varphi}$ for the scalar interactions according to equation (7) of Problem 4. Show that they have the general one-loop structure

$$Z_{1\varphi} = 1 + \frac{1}{\varepsilon}\left(a_1 g^2 + b_1 \gamma^2 + c_1 \alpha + d_1 \beta\right), \qquad Z_{2\varphi} = 1 + \frac{1}{\varepsilon}\left(a_2 g^2 + b_2 \gamma^2 + c_2 \alpha + d_2 \beta\right),$$

where a_i, b_i, c_i and d_i are numerical constants.
f) Prove the equalities $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$. *Hint*: use the Salvnov-Taylor identities derived in Problem 4, together with the fact that at one loop scalar fields do not contribute to the renormalization of the ghost Lagrangian $\partial_\mu \overline{C}^a D^\mu C^a$.

Problem 10 (optional). Consider the gauge group $G = SO(N)$ and denote the YM potentials by A^{IJ}_μ, where $A^{IJ}_\mu = -A^{JI}_\mu$ and $I, J = 1, \cdots, N$. Choose the fermions in the fundamental (vector) representation of $SO(N)$, $\Psi \equiv \Psi^I$, and denote the local $SO(N)$ transformation parameters by $\Lambda^{IJ}(x) = -\Lambda^{JI}(x)$. For an infinitesimal transformation of the fermions one has thus

$$\delta \Psi^I = \Lambda^{IJ} \Psi^J. \tag{10}$$

a) Derive the expression δA^{IJ}_μ of the infinitesimal transformation of the YM potentials, using that the covariant derivative is given by

$$D_\mu \Psi^I = \partial_\mu \Psi^I - A^{IJ}_\mu \Psi^J.$$

b) Derive the form of the YM field-strength $F^{IJ}_{\mu\nu}$.
c) In which way are the expressions $D_\mu \Psi^I$, δA^{IJ}_μ, $F^{IJ}_{\mu\nu}$ related to the corresponding expressions of the *conventional* construction of non-abelian gauge theories, where one introduces a Lie algebra-valued YM potential $A^\mu = T^a A^a_\mu$?
d) Add real scalar fields φ^I transforming in the fundamental representation, too, and construct the most general renormalizable Lagrangian using the fields A^{IJ}_μ, Ψ^I and φ^I. Is a Yukawa coupling allowed? *Hint:* $SO(N)$ is the euclidean version of the N-dimensional Lorentz-group, and as the latter it has only two independent invariant tensors for vector indices, *i.e.* δ^{IJ} and $\varepsilon^{I_1 \cdots I_N}$.
e) Consider the gauge group $G = SO(10)$ with fermions in the "spinor" representation 16 of G, $\Psi \equiv \Psi^i$, $i = 1, \cdots, 16$. If you want to couple these fermions to scalar fields via a Yukawa coupling, which representations can you choose for the scalars? In particular, would a scalar multiplet of the form φ^{IJKM} - completely *antisymmetric* in all four vector indices - do the job? How many independent Yukawa couplings can you construct for each chosen representation of the scalars? *Hint*: use the products of $SO(10)$ irreducible representation listed in *Slansky*.
f) Find the BRST-transformation δC^{IJ} of the ghost field C^{IJ} - replacing the transformation parameter Λ^{IJ} - and verify that it is nihilpotent, *i.e.* $\delta^2 C^{IJ} = 0$.

Problem 11 (optional). Prove that an antisymmetric massless two-index potential $B_{\mu\nu}$ in four dimensions is physically equivalent to a scalar massless field φ, proceeding along

the following lines based on the functional integral approach.
a) Start from the partition function (g is a coupling constant)

$$Z = \int \mathcal{D}B \, e^{iI[B]}, \qquad I[B] = \frac{1}{2g^2} \int \frac{1}{6} \, H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \, d^4x, \qquad H_{\alpha\beta\gamma} = 3\partial_{[\alpha} B_{\beta\gamma]},$$

and show that Z can be rewritten as ($F^{\alpha\beta\gamma}$ is a completely antisymmetric tensor)

$$Z = \int \mathcal{D}F \, \mathcal{D}\varphi \, e^{iI[F,\varphi]}, \qquad I[F,\varphi] = \int \left(\frac{1}{12g^2} \, F_{\alpha\beta\gamma} F^{\alpha\beta\gamma} - \frac{1}{6} \, \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha \varphi \, F_{\beta\gamma\delta} \right) d^4x. \quad (11)$$

Hint: use the functional integral identities

$$\int \mathcal{D}\varphi \, e^{-i \int \frac{1}{6} \, \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha \varphi F_{\beta\gamma\delta} \, d^4x} = \delta \left(\partial_{[\alpha} F_{\beta\gamma\delta]} \right) = \frac{1}{det(\partial)} \int \mathcal{D}B \, \delta(F_{\alpha\beta\gamma} - 3\partial_{[\alpha} B_{\beta\gamma]}),$$

generalizations of the finite-dimensional identities

$$\int e^{ikx} \, dk = 2\pi\delta(x), \qquad \delta(f(x)) = \sum_j \frac{\delta(x - x_j)}{|f'(x_j)|}.$$

b) Perform in (11) the gaussian functional integral over $F^{\alpha\beta\gamma}$ to get

$$Z = \int \mathcal{D}\varphi \, e^{iI[\varphi]}, \qquad I[\varphi] = \frac{g^2}{2} \int \partial_\mu \varphi \, \partial^\mu \varphi \, d^4x.$$

Give an interpretation of the occurrence of the inversion $g^2 \leftrightarrow 1/g^2$.
c) Generalize the above procedure to the duality between massless p-form potentials $B_{\mu_1 \cdots \mu_p}$ and their dual $(D - p - 2)$-form potentials in D dimensions. What is the physical meaning of the particular case $p = 1$, $D = 4$?