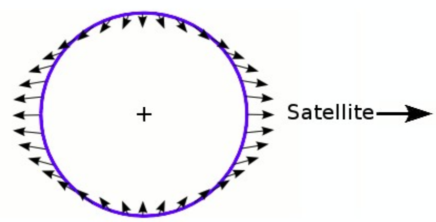
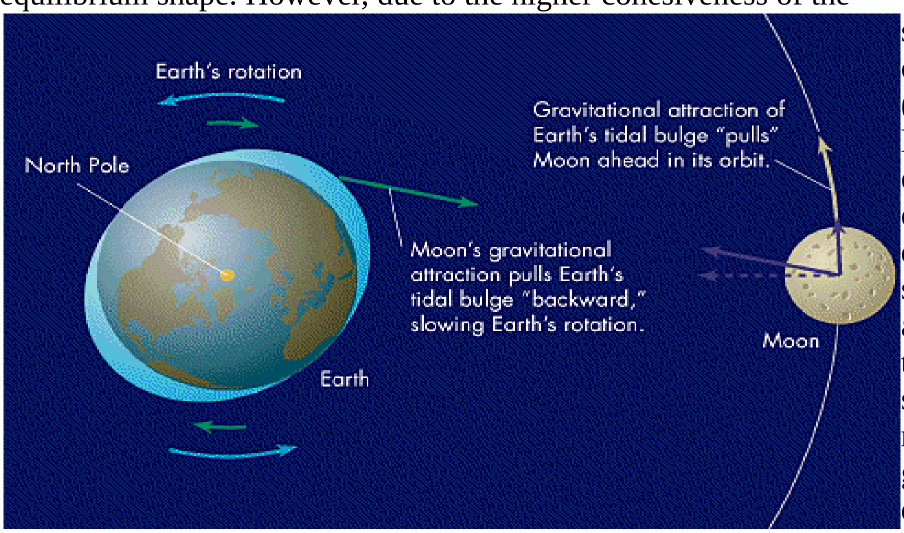
## Tidal effects

Tidal forces depend on the gradient of a gravitational field and they act in different astrophysical scenarios. On large scales tides are relevant in galaxy collisions, affect the motion of satellite galaxies and perturb Oort like clouds. At smaller scales tides are involved in the common envelope phase of binary star evolution. One of the stellar component expands beyond the orbit of its companion and orbital energy is converted into kinetic energy causing outflow of the envelope. Extrasolar planetary systems are thought to be strongly influenced by tides once a planet, after a phase of gravitational scattering, is injected in an eccentric orbit with the pericenter very close to the star. Tidal forces circularize its orbit to the pericenter distance resulting in a sort of planet migration. Planet satellites are also strongly affected by tides which often lead to a synchronization of the rotation with the orbital period and to semimajor axis migration. Here we will deal first with the tide excited on the planet by the satellite and then on the satellite by the planet. The two scenarios are distinct due to the different relevance of the rotational energy and momentum of the body perturbed by the tide respect to the orbital energy and momentum of the system.

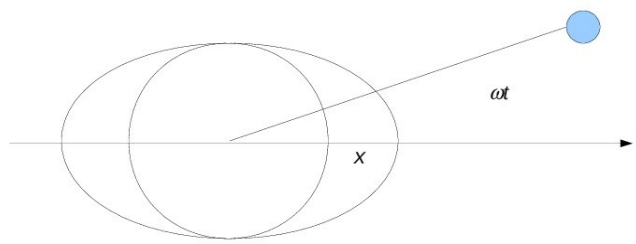
## Formation of a tidal bulge




Tidal forces distort a body experiencing differential gravitational forces due to the gravitational attraction of a satellite. When the size of a planet is comparable with the satellite-planet distance, the gravity gradient on the planet is significant (see figure showing the gravity differential field), different parts of the body experience different accelerations and its shape is distorted. Both the liquid and solid components of the planet move towards a new equilibrium shape. However, due to the higher cohesiveness of the solid part, the tides in the body of the planet are much smaller (of the order of cms) than in the liquid part (meters). The new equilibrium shape resembles an ellipsoid and the amount of mass departing from the approximate spherical shape of the planet in absence of tides is called the tidal *bulge*. If the orbit of the satellite is synchronous with the rotation of the planet, the gravity gradient is always in the same direction and the bulge is constant. However, in the more common configuration, the rotation of the planet is not synchronized with the orbital motion of the satellite and a phase shift develops in the tides due to the resistance of the planet to deformation.



**A simplified model to understand the tidal shift.**




The tidal bulge shift respect to the line connecting the centers of planet and satellite can be understood on the basis of a simplified 1-dimensional model of the system. Let's assume that the planet is rotating with a frequency W around its axis and the satellite is moving on a circular orbit around the planet with a frequency $n=\sqrt{\frac{G(M_{Earth}+M_{Moon})}{a}}$ . A body-fixed reference frame is attached to the planet, centered on the center of mass of the planet. The tidal deformation is computed only along the x-axis (1-dimensional problem). In figure we show an illustrative picture of the model. We compute the deformation along the x-axis due to the satellite gravitational attraction which, projected on the x-axis, is a periodic force of the form $F_0 \cos(\omega t)$ where $\omega=(n-\Omega)$ is the frequency of the motion of the satellite in the body fixed reference frame. It is given by the difference between the rotation rate of the planet and the orbital rate of the satellite. We assume that the planet behaves like an harmonic oscillator described by the following equation

$$m\frac{dx^2}{dt^2}=-Kx-\beta\frac{dx}{dt}+F_0\cos(\omega t)$$

where x is the shift from the equilibrium shape (radius *r*), *m* is the inertia of the planet against deformation, K is a positive restoring force. Finally, b is the dissipation term due to heat production during the deformation. This equation can be cast in the following form

$$\frac{dx^2}{dt^2}+\omega_0 x+\frac{1}{\tau}\frac{dx}{dt}=\frac{F_0}{m}\cos(\omega t)$$

with $\omega_0=\frac{K}{m}$ natural frequency of the planet and $\tau$ damping timescale. This is the equation of a damped, driven harmonic oscillator whose solution is:

$$x(t)=A(\omega)\cos(\omega t+\delta(\omega))$$

with

$$A(\omega)=\frac{F_0}{m}\left[(\omega_0^2-\omega^2)^2+\left(\frac{\omega}{\tau}\right)^2\right]^{1/2}$$

$$\sin(\delta(\omega))=-\frac{\omega}{\tau}\left[(\omega_0^2-\omega^2)^2+\left(\frac{\omega}{\tau}\right)^2\right]^{1/2}$$

The dissipative force is always positive (tidal friction produces heat) so the delay $\delta(\omega)$ is encompassed between

$$-\frac{\pi}{2} < \delta(\omega) \leq 0$$

The tide has its maximum along the x-axis *after* the Moon passage if $\omega = (n - \Omega)$ is positive, or *before* the Moon passage if w is negative. As a consequence, the tidal bulge on the planet is trailing the satellite position if $\Omega < n$ while it is leading if $\Omega > n$ . The actual scenario for the Earth Moon system is this last one with the rotation period of the Earth (24 hr) shorter than the orbital period of the Moon (27.32 d). As a consequence, the tidal bulge on the Earth is always ahead in time respect to the Moon position. However, the local tides are strongly influenced by coastline shapes, resonances etc... so the position of the Moon is only an indicator of the tide phase.

**The tidal dissipation function Q**

When dealing with tides, a tidal dissipation function *Q* is defined as the ratio between the gravitational energy accumulated in the tidal bulge due to its departing from a spherical shape, and the amount of energy which is dissipated while building up the bulge. It is given by:

$$Q = \frac{2\pi E_0}{\Delta E} = \frac{2\pi E_0}{-\int_0^T \dot{E}\, dt}$$

ΔE is the energy dissipated in one period of the satellite and $E_0$ is the maximum energy stored in the tidal bulge. The value of $E_0$ can be computed assuming that in 1 dimension the Earth acts like a spring with given k (related to its proper oscillation frequency):

$$E_0 = \int_0^A kx\, dx = \int_0^A \omega_0^2 m x\, dx = \frac{1}{2} m \omega_0^2 A^2$$

This is a potential energy against the recoil gravity force of the planet which tends to lead it back to a spherical shape. The energy dissipated is given by the work done by the viscous force acting when the planet is deformed:

$$dW = -\dot{x} \frac{m}{\tau} dx \;\Rightarrow\; \dot{E} = -\frac{dW}{dt} = -\frac{m}{\tau} \dot{x}^2 = -\frac{m}{\tau} (A\omega)^2 \sin(\omega t + \delta)^2$$

the averaged value of the energy derivative over one Moon orbit is given by:

$$\langle \dot{E} \rangle = -\frac{1}{2\pi} \frac{m}{\tau} (A\omega)^2 \int_0^{2\pi} \sin(x)^2 dx = -\frac{1}{2} \frac{m}{\tau} (\omega A)^2$$

As a consequene, the total energy dissipated in one tidal cycle is given by

$$\Delta E = -(-\dot{E} \frac{2\pi}{\omega}) = \frac{1}{2} \frac{m}{\tau} (\omega A)^2 \frac{2\pi}{\omega} = \frac{\pi m}{\omega \tau} A^2 \omega^2$$

At this point is is possible to compute Q

$$Q = \frac{\frac{1}{2} m \omega_0^2 A^2 2\pi}{m \frac{\pi}{\tau\omega} A^2} = \frac{\tau}{\omega} \omega_0^2$$

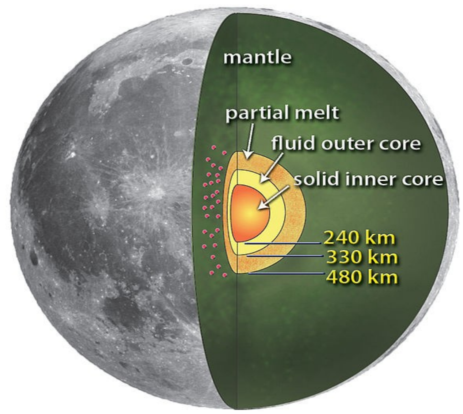
Assuming that $\omega_0 >> \omega$ i.e. the proper frequency of oscillation of the planet has a frequency which is much larger than the frequency of the orbital motion of the satellite, we get

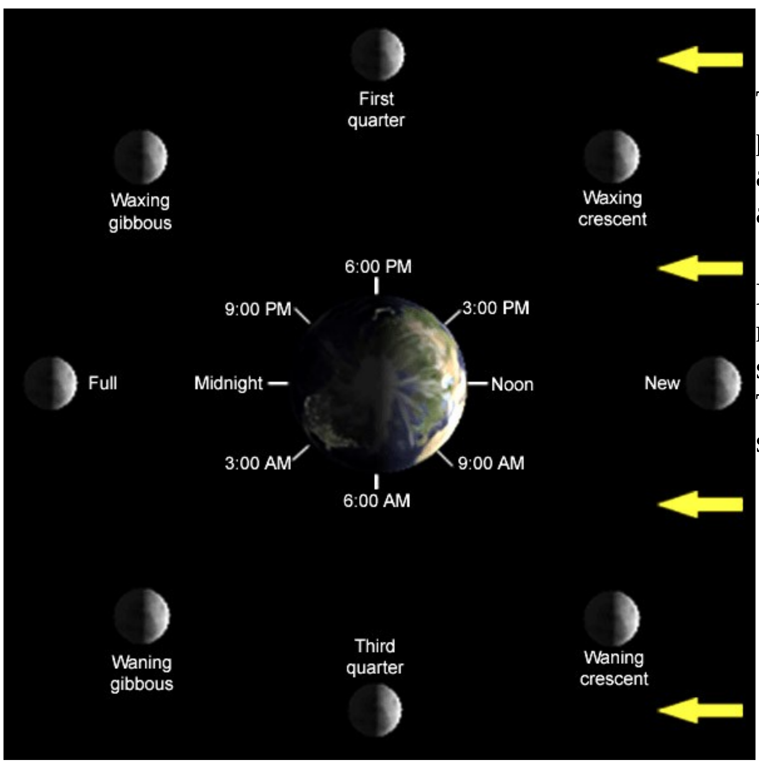
$$\sin\delta=-\frac{\omega}{\tau}\left[\left(\omega_0^2-\omega^2\right)^2+\left(\frac{\omega}{\tau}\right)^2\right]^{-\frac{1}{2}}\simeq-\frac{\omega}{\tau}\frac{1}{\omega_0^2}=-\frac{1}{Q}$$

Therefore there is a direct link between the tidal delay of the bulge and the tidal dissipation function Q. This shows that the factor Q determines the strength of the tide because it is proportional to the torque induced by the misalignment of the bulge respect to the line joining planet and satellite.

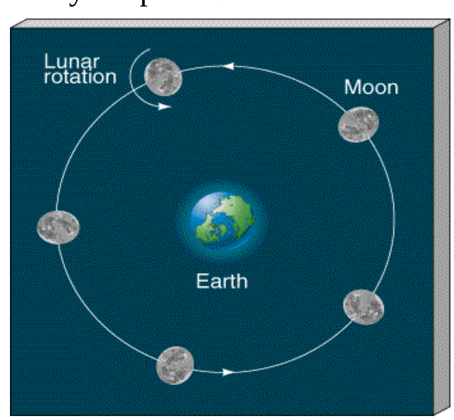
**The Earth-Moon system**

The Moon has a mass of $7.349\,10^{22}\,kg=0.012\,M_{Earth}$ and a radius of $R=1738\,km$ for a density of about $\rho\approx3.4\,g/cm^3$. Its orbit has a semimajor axis of $0.3844\,10^6\,km$ and an orbital period of $27.32\,d$. The inclination of the orbit is $i\approx5^o$ while the eccentricity is $e\approx0.055$.




The sidereal month is equal to the orbital period while the synodic month (computed as period between two subsequent alignment of Earth-Moon-Sun) is $T_s\approx29.5\,d$. The rotation rate of the Moon is synchronized with its orbital revolution rate so that the Moon always shows the same face to an Earth observer. This is an effect of the tide raised on the satellite by the planet.



**Tidal coupling in the Earth-Moon system**

The tidal interaction between the Earth and Moon has two important consequences: 1) the rotation rate of the Earth is decreasing 2) The orbit of the Moon is expanding. This is due to the exchange of angular momentum between rotational and orbital motion. This exchange is governed by the energy dissipation in form of heat due to frictional forces within the planet. Let's start from the expressions for the rotational and orbital energy and angular momentum of the system. The energy can be written as

$$E=\frac{1}{2}I_p\Omega^2-\frac{G\,m_p m_s}{2a}$$

We neglect the rotation energy of the satellite since we are dealing with the tide raised by the satellite on the planet. The first term on the right is the rotational energy of the planet and it depends on its moment of inertia $I_p$. The second term *i*s the orbital energy of the satellite. The time derivative of E must be negative (dissipation) and can be easily computed

$$\dot{E}=I_p\Omega\dot{\Omega}+G\frac{m_p m_s}{2a^2}\dot{a}=I_p\Omega\dot{\Omega}+\frac{1}{2}\frac{m_p m_s}{m_p+m_s}n^2 a\dot{a}<0$$

recalling that $G(m_p+m_s)=n^2a^3$ . Since the system is isolated, there are no external forces and the total angular momentum is conserved so that

$$L=I_p\Omega+\frac{m_p m_s}{m_p+m_s}h=I_p\Omega+\frac{m_p m_s}{m_p+m_s}n a^2$$

where h is the pseudo-angular momentum of the 2-body problem defined as $h=na^2\sqrt{(1-e^2)}$ . The eccentricity of Moon's orbit is small and we assume as a first approximation that $e=0$ so that $h=na^2$ . Deriving $L$ with respect to time we get

$$\dot{L}=I_p\dot{\Omega}+\frac{1}{2}\frac{m_p m_s}{m_p+m_s}n a\dot{a}=0$$

This expression follows after we derive the term *h* with respect to time

$$\frac{d}{dt}(na^2)=2a\frac{da}{dt}n+a^2\frac{dn}{dt}=2a\frac{da}{dt}n+a^2\frac{dn}{da}\frac{da}{dt}=2a\frac{da}{dt}n-\frac{3}{2}\frac{n}{a}a^2\frac{da}{dt}=\frac{1}{2}an\dot{a}$$

recalling that the orbital frequency is $n=\frac{\sqrt{G(m_p+m_s)}}{a^3}$ . From the derivative of the angular momentum we derive a relation between the rotation rate and the orbital parameter

$$I_p\dot{\Omega}=-\frac{1}{2}\frac{m_p m_s}{m_p+m_s}n a\dot{a}$$

which can be inserted into the equation for the time derivative of energy giving

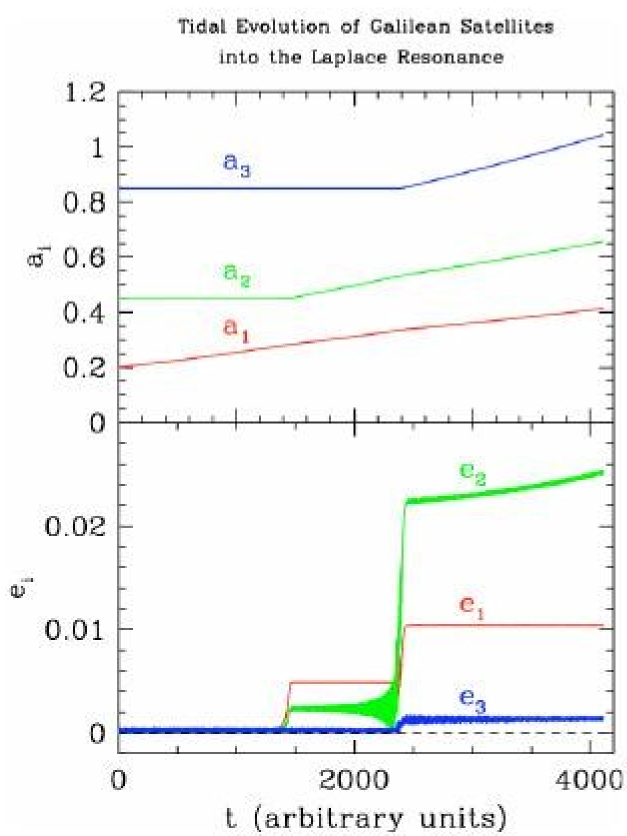
$$\dot{E} = -\frac{1}{2}\Omega\frac{m_p m_s}{m_p+m_s} n a \dot{a} + \frac{1}{2}\frac{m_p m_s}{m_p+m_s} n^2 a \dot{a} = -\frac{1}{2}\frac{m_p m_s}{m_p+m_s} n a \dot{a} (\Omega - n) < 0$$

This equation sets a relation between the sign of $\dot{a}$ and the difference between the two frequencies $(\Omega - n)$. If $n < \Omega$ i.e. the satellite is out of the synchronous orbit, then $\dot{a}$ must be positive and the satellite moves outwards. This is the case of the Moon and most of the Solar System satellites. If $n > \Omega$ then $\dot{a}$ is negative and the satellite slowly spirals towards the planet until an impact occurs. This is the case of Phobos, the inner one of Mars' satellites. For synchronized orbits, like that of Pluto and Charon, the equilibrium configuration is reached and $\dot{E} = 0$. Retrograde satellites, like Neptune's satellite Triton, are all doomed since for $n < 0$ the above equation predicts that $\dot{a}$ is always negative and the satellite will finally impact on the planet. From lase ranging, the change in the semimajor axis of the Moon is $\dot{a} \approx 3.8\, cm/y$. From the conservation of angular momentum, we derive that

$$\dot{\Omega} = \frac{1}{2 I_p}\frac{m_p m_s}{m_p+m_s} n a \dot{a} \approx -6\times 10^{-22} rad/s^2$$

The length of the day is then increasing of about 1 minute every 150 Myr.

The tidal evolution could explain the Laplace resonance of the Jupiter's satellites Io, Europa and Ganymede. Their orbital periods are in a 1:2:4 commensurability which might have been the outcome of tidal evolution (Goldreich 1965, Yoder 1979, Yoder and Peale 1980) . An alternative explanation is that the resonance trapping was caused by migration due to interaction with the proto-satellite gaseous disk.



**Tides raised by the planet on the satellite.**

The tide raised by the planet on the satellite has two effects on the planet-satellite system, both related to the attempt by the satellite to reduce the energy dissipation. First of all the rotation rate of the satellite is synchronized with its orbital period so that the bulge is fixed and not changing with time. The second effect is the circularization of the orbit which prevents a phase shift of the bulge between perihelion and aphelion.

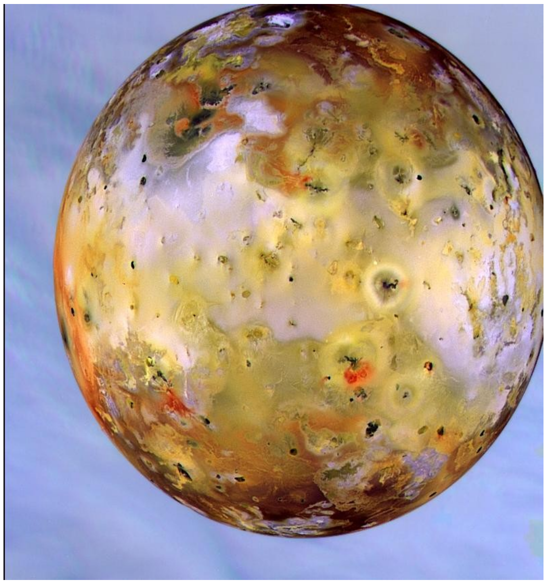
**Eccentricity damping**

To compute the rate of eccentricity damping due to tidal interaction we can follow the same approach used to compute the change in semi-major axis of the satellite due to the tide on the planet. We start from the conservation of angular momentum

$$L_0 = I_s \Omega - \frac{m_p m_s}{m_p + m_s} h = I_s \Omega - \frac{m_p m_s}{m_p + m_s} n a^2 \sqrt{(1-e^2)} = -\frac{m_p m_s}{m_p + m_s} n a^2 \sqrt{(1-e^2)}$$

Notice that we do not consider negligible the eccentricity of the orbit as in the case of the Moon. In addition, we neglect the term $I_s \Omega$ which is unimportant compared with the orbital angular momentum due to the small value of $I_s$ . If we solve the above equation respect to the eccentricity *e* we get

$$e^2 = 1 - L_0^2 \frac{m_p + m_s}{m_p m_s} \frac{1}{n^2 a^4} = 1 + \frac{2 E L_0^2}{G^2} \frac{m_p + m_s}{(m_p m_s)^3}$$



We recall here that the orbital energy is $E = \frac{-G m_p m_s}{2a}$ . If we derive the above equation with respect to time we obtain

$$\dot{e} = \frac{2 \dot{E} L_0^2}{2e} \frac{m_p + m_s}{G^2 (m_p m_s)^3} = -\frac{\dot{E}}{2eE}(1-e^2) \approx -\frac{\dot{E}}{2eE}$$

The energy is always dissipated so $\dot{E} < 0$ , the orbital energy $E < 0$ so, in conclusions, also $\dot{e} < 0$ . The negative derivative of the eccentricity implies that any initial eccentricity of the orbit of a satellite is damped by tidal interaction until the orbit is circularized and the tidal bulge is always oriented along the planet-satellite line.

If the circularization is prevented, like in the Laplace resonance, then the tide shifts during each orbit of the satellite and this causes heat production. Io, the closest satellite of Jupiter, responds to the tide heating with vulcanism. Many vulcanos are visible on its surface (image taken by Galileao spacecraft) and they are active. The black and bright red materials correspond to the most recent volcanic deposits, probably no more than a few years old. The vulcanic plumes produce a cloud of gases (in particular sodium) that are ejected from the planet. These gases are quickly ionized and trapped by the magnetic field of Jupiter into a plasma torus.

**Tidal interaction of planets and stars.**

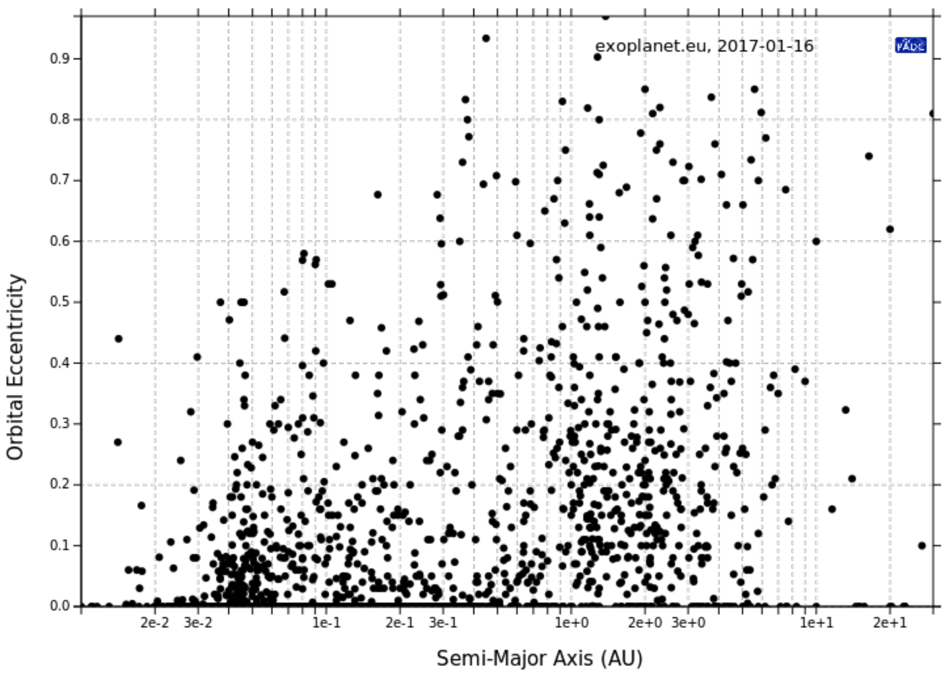
In exoplanetary systems, if the planet orbits close enough to the star, it can tidally interact with the star. There is a tide raised by the planet on the star, if it has a convective envelope, and a tide raised by the star on the planet. The important effect is a circularization of the planet orbit. In case of an eccentric planetary orbit, this has important consequences since it drags the planet closer to the star. From the conservation of angular momentum (only energy is dissipated)

$$L=\frac{m_p m_s}{m_p+m_s}\sqrt{(G M a(1-e^2))}=const$$

if the eccentricity decreases, so must do the semi-major axis to keep constant the product $L=a(1-e^2)$ . Taking into account that

$$a(1-e^2)=a(1+e)\cdot(1-e) \;\Rightarrow\; \sim 2a\cdot(1-e)$$

in the limit when the eccentricity of the planets is reduced to 0, the final semi-major axis will be equal to about twice the initial pericenter distance of the orbit $a\cdot(1-e)$ . This effect can be seen in the semi-major axis vs. eccentricity distribution of exoplanets shown in figure. For smaller values of *a* the eccentricity is reduced accordingly.



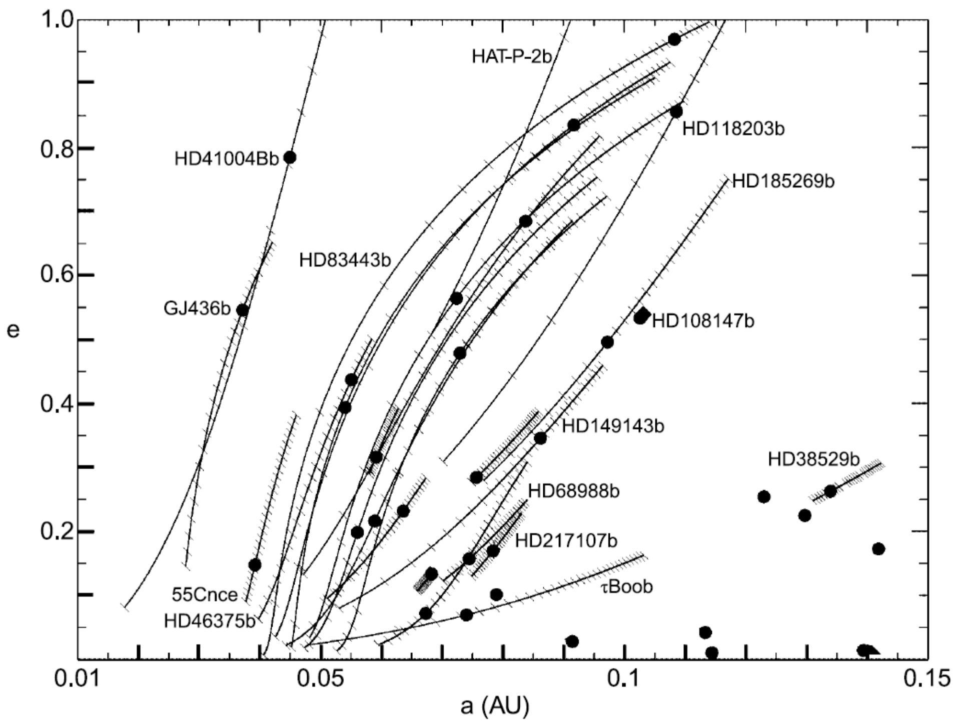

In the case of planet-star tidally interactions, there are two different possible kinds of tidal interaction: The equilibrium tide which is characterized by the formation of an equilibrium bulge. In this case the dissipation of energy is caused by the time variation of the tidal bulge. This kind of tidal interaction is typical of bodies in low eccentricity orbits and large rocky component. Within the star or in presence of giant planets with an extended gaseous envelope, dynamical tides can form. The body is assumed to be an oscillator with a number of modes that are excited when the companion passes at the pericenter. The modes are damped when the companion is at the apoastron. These tides occur mainly for bodies in highly eccentric orbits.

In the case of planet-star involved in a statics tidal interaction, it is possible to track back the original eccentricity of the planet by a backward integration of the tidal equations.

$$\frac{1}{e}\frac{de}{dt}=-\left[\frac{63}{4}(GM_s^3)^{1/2}\frac{R_p^5}{Q_p M_p}+\frac{171}{16}\left(\frac{G}{M_s}\right)^{1/2}\frac{R_s^5 M_p}{Q_s}\right]a^{-13/2}$$

$$\frac{1}{a}\frac{da}{dt}=-\left[\frac{63}{2}(GM_s^3)^{1/2}\frac{R_p^5}{Q_p M_p}e^2+\frac{9}{2}\left(\frac{G}{M_s}\right)^{1/2}\frac{R_s^5 M_p}{Q_s}\right]a^{-13/2}$$

In these equations $M_s$ and $Q_s$ are for the star while $M_p$, $R_p$ and $Q_p$ for the planet (see Jackson et al., ApJ 678, 1396, 2008). By integrating back in time these equations for a smple of exoplanets using $Q_s=10^{5.5}$ and $Q_p=10^{6.5}$ the bottom plot is obtained.




Solid curves show the orbital evolution from the present orbits (lower left end of each curve, filled circles) backward in time (toward the upper right). On each trajectory, tick marks are spaced every500 Myr to indicate the rate of tidal evolution. Tidal integrations were performed for 15 Gyr for all planets. The original orbits were significantly more eccentric, possibly the outcome of a period of planet-planet scattering.

**Evolution of a satellite spin**

Tides can periodically change the shape of a body leading to the onset of a torque that causes an exchange of angular momentum between the satellite and the central body. The same occurs if the body is intrinsically aspherical like some planet satellites (see for eample Hyperion). In this case there is a constant torque which may significantly affect the rotation rate of the satellite itself.
The potential of an irregularly shaped body on a point mass P can be approximated by the McCullogh potential (Murray & Dermott "Solar System Dynamics")

$$V=-\frac{G m_s}{r}-\frac{G}{2r^5}f(A,B,C;x,y,z)$$
$$f(A,B,c;x,y,z)=(B+C-2A)x^2+(C+A-2B)y^2+(A+B-2C)z^2$$

where A,B,C are the principal inertia moments. The force per unit mass of the planet exerted on P by the irregular object is then

$$F_x=-\frac{\partial V}{\partial x}=-\frac{G m_s x}{r^3}+\frac{G(B+C-2A)x}{r^5}-\frac{5Gx}{2r^7}f(A,B,C;x,y,z)$$
$$F_y=-\frac{\partial V}{\partial y}=-\frac{G m_s y}{r^3}+\frac{G(A+C-2B)y}{r^5}-\frac{5Gy}{2r^7}f(A,B,C;x,y,z)$$
$$F_z=-\frac{\partial V}{\partial z}=-\frac{G m_s z}{r^3}+\frac{G(A+B-2C)z}{r^5}-\frac{5Gz}{2r^7}f(A,B,C;x,y,z)$$

while the torque $\boldsymbol{N}=\boldsymbol{r}\times\boldsymbol{F}$ is

$$N_x=zF_y-yF_z=-z\frac{G m_s y}{r^3}+y\frac{G m_s z}{r^3}-3G(C-B)\frac{yz}{r^5}+$$
$$-\frac{5Gzy}{2r^7}F(A,B,C;x,y,z)+\frac{5Gzy}{2r^7}F(A,B,C;x,y,z)$$
$$N_y\ldots\ldots\ldots\ldots$$
$$N_z\ldots\ldots\ldots\ldots$$

The first 2 and last 2 terms cancel out and the torque becomes:

$$N_x=3G(C-B)\frac{yz}{r^5}$$
$$N_y=3G(A-C)\frac{zx}{r^5}$$
$$N_z=3G(B-A)\frac{xy}{r^5}$$

This is the torque by the irregular body on the point mass P. Since the system is dynamically isolated, the point mass P exerts a torque back on the body equal in modulus but opposite in direction. As a consequence, the evolution of the spin of the satellite with time, governed by the Eulero equation

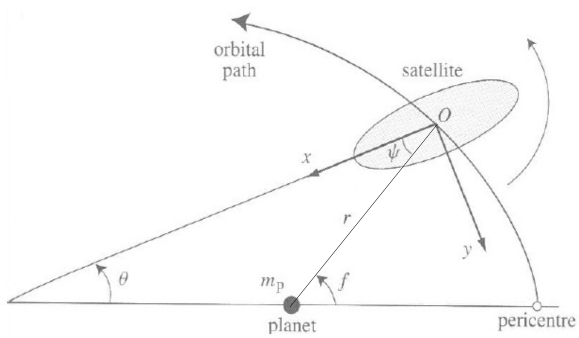
$$\frac{d\boldsymbol{J}}{dt}+\boldsymbol{\omega}\times\boldsymbol{J}=N$$ ( $\boldsymbol{J}$ is the angula momentum of the satellite), is

$$A\dot{\omega}_x+(C-B)\omega_y\omega_z=N_x$$
$$B\dot{\omega}_y+(A-C)\omega_z\omega_x=N_y$$
$$C\dot{\omega}_z+(B-A)\omega_x\omega_y=N_z$$

In the simplified scenario where the angular momentum $\boldsymbol{J}$ is perpendicular to the orbital plane of the satellite, $\omega_x=\omega_y=0$ , the system of equations reduces to a single equation

$$C\dot{\omega}_z=N_z=3G(B-A)\frac{x\,y}{r^5}$$

We need now an angle that describes the rotational motion of the satellite but that can be easily related to the orbital motion. In the figure, a body-fixed reference frame (x,y) is attached to the irregular satellite. The angle that describes the rotation of the satellite is $\theta$ , computed respect to the apsidal line of the satellite orbit. The rotation rate along the z-axis is then $\omega_z=\dot{\theta}$ . The value of the coordinates x and y are related to the radial distance r and the the angle $\psi$ by the relations




$$\frac{x}{r}=\cos\psi$$
$$\frac{y}{r}=\sin\psi$$

The equation that determines the evolution of the rotation rate of the satellite is finally

$$C\ddot{\theta}-\frac{3}{2}G\,m_p\frac{(B-A)}{r^3}\sin 2\psi=0$$

**Spin-orbit resonance**

The above equation describing the evolution of the rotation angle $\theta$ has a dependence on the orbital angle $\psi$ t that can lead to resonances between the rotation period and the orbital period. To inspect the evolution of the system while in resonance, we introduce the new angle $\gamma=\theta-p\,M$ where *p* is a ratio between two integers (*i/j*) and *M* is the mean anomaly of the orbit. We notice that $\ddot{\gamma}=\ddot{\theta}$ since the mean anomaly M has a fixed frequency *n*. In addition, from the figure we can show that $\psi=f-\theta$ . In conclusion, the previous equation can be cast in the following form close to a resonance of the kind $j\dot{\theta}=in$ or, for periods, $iT_r=jT_o$ with $T_r$ period of rotation and $T_o$ orbital period

$$\ddot{\gamma}+\frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3\sin(2\gamma+2pM-2f)=0$$

This equation has the unpleasant problem of too many variables depending on time, like *r* and *f*. Our goal is now to express both these variables as a function of the mean anomaly *M*. This can be done expanding both $\frac{a}{r}$ and *f in* series of M.

$$\left(\frac{a}{r}\right)^3=1+3e\cos M+\frac{3}{2}e^2(1+3\cos 2\mathrm{M})+...$$
$$\sin f=\left(1-\frac{7}{8}e^2\right)\sin M+e\sin 2M+\frac{9}{8}e^2\sin 3M...$$
$$\cos f=\left(1-\frac{9}{8}e^2\right)\cos M+e(\cos 2\mathrm{M}-1)+\frac{9}{8}e^2\cos 3\mathrm{M}...$$

Using trigonometric identities it is possible to write

$$\sin(2\gamma+2pM-2f)=\sin 2\gamma(\cos(2\mathrm{pM})\cos(2\mathrm{f})+\sin(2\mathrm{pM})\sin 2\mathrm{f})+....$$

Substituting and grouping the different terms we get to the simplified equation

$$\ddot{\gamma}+\frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left((S_1+S_2)\sin 2\gamma+(S_3-S_4)\cos 2\gamma\right)=0$$

where the terms $S_i$ are series in *e* and *M*. Close to a resonance the critical angle, $\gamma$ i.e. the difference between the pM and $\theta$ , changes slowly. As a consequence, $\dot{\gamma}\ll n$ and we can average the term on the right over an orbital period. In other words, the evolution of the resonant angle $\gamma$ occurs on a timescale much longer than an orbital period, so that the periodic variation of the perturbing term over an orbital period can be averaged

$$\bar{S}_i=\frac{1}{2\pi}\int_0^{2\pi}S_i(p,e)\,dM$$

Finally, we get a differential equation on the single angle $\gamma$ which is the only variable depending on time

$$\ddot{\gamma}+\frac{3}{2}n^2\left(\frac{B-A}{C}\right)H(p,e)\sin 2\gamma=0$$

Notice that the terms $S_3$ and $S_4$ do not contribute to $H(p,e)$ . The above equation is the classical pendulum equation. The function $H(p,e)$ is specific for each resonance. Here below we report the most relevant terms

$$H(1,e)=1-\frac{5}{2}e^2+\frac{13}{16}e^4$$
$$H(-1,e)=\frac{1}{24}e^4$$
$$H(\frac{1}{2},e)=-\frac{1}{2}e+\frac{1}{16}e^3$$
$$H(\frac{3}{2},e)=\frac{7}{2}e-\frac{123}{16}e^3$$

The first term works for synchronous orbits, the second for retrograde 1:1 resonance, the third corresponds to a 2:1 resonance and the fourth to a 3:2 resonance. The above equation can be cast in the following form

$$\ddot{\gamma}=-\frac{1}{2}\omega_0^2\sin 2\gamma\cdot sign(H(p,e))$$

where

$$\omega_0=n\left[3\left(\frac{B-A}{C}\right)|H(p,e)|\right]^{\frac{1}{2}}$$

We can now describe the coupled evolution of the rotation rate of a satellite affected by tides. The tidal interaction tends to slow down the spin of a satellite towards synchronization. However, while approaching this state, it can encounter one or more resonances. The capture in resonance depends on the balance between the restoring force of the resonance and the dissipative force due to the tide. Let's call $\bar{N}_M<0$ the average tidal torque exerted by the tide until the synchronization and circularization of the orbit is achieved. To compute the value of $\bar{N}_M$ is a difficult task and we will give an approximate estimate as

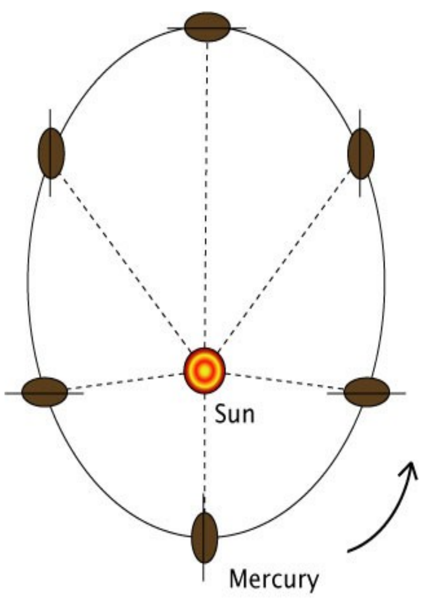
$$\bar{N}_M=-D\left(\frac{a}{r}\right)^6\cdot sign(\dot{\Omega}-n)$$

where

$$D=\frac{3}{2}\frac{k}{Q}\frac{n^4}{G}R_s^5$$

with k Love number, which gives an estimate of how much the body distorts, Q is a dissipation function given by $Q=\frac{2\pi E_0}{\Delta E}$ where $\Delta E$ is the energy dissipated in the tidal cycle and $E_0$ the energy stored in the tidal bulge. Adding this term to the equation for the rotation rate evolution, we get

$$\ddot{\gamma}=-\frac{1}{2}\omega_0^2\sin 2\gamma\cdot sign(H(p,e))+\frac{\bar{N}_M}{C}$$



The condition for resonance trapping is then that the dissipative term due to the tide is smaller than the resonance strength i.e.

$$\left|\frac{\bar{N}_M}{C}\right| < \frac{1}{2}\omega_0^2$$

This has probably happened to planet Mercury which is locked into a 3:2 spin-orbit resonance where it rotates three times on its axis for every two orbits around the sun. As shown in figure, it completes 1.5 turns every 1 orbit. The rotational period is in fact $T_{rot} \sim 58.65\,d$ while the orbital period is $T_{orb} \sim 87.97\,d$. The resonance halted in the past the tidal evolution towards synchronization. In the table we report the critical values of the ratio $\left(\frac{B-A}{C}\right)$, called the asphericity parameter, needed to allow resonance trapping and halt the rotational evolution towards synchronization for both Mercury and the Moon (the corresponsing values for $H(p,e)$ have been computed assuming an eccentricity $e_M \sim 0.206$ for Mercury and $e_L \sim 0.055$ for the Moon).

| $P=\frac{i}{j}$ | $\left(\frac{B-A}{C}\right)_M$ | $\left(\frac{B-A}{C}\right)_L$ |
|---|---|---|
| 3 | $2\times10^{-8}$ | $7\times10^{-5}$ |
| $5/2$ | $7\times10^{-9}$ | $7\times10^{-6}$ |
| 2 | $3\times10^{-9}$ | $8\times10^{-7}$ |
| $3/2$ | $2\times10^{-9}$ | $10^{-7}$ |
| 1 | $10^{-9}$ | $2\times10^{-8}$ |

The present values of $\left(\frac{B-A}{C}\right)$ for both Mercury and the Moon are of the order of $3\times10^{-4}$, much larger than the critical values given in table 1 for resonance trapping. It means that Mercury must have had a primordial rotation rate encompassed between $p=2$ and $p=3/2$ and the tidal evolution dragged the rotation rate down to this last commensurability. In addition, the primordial spin of the Moon could not have been too far from synchronism.

**Hamiltonian approach**

In the previous approach to the spin-orbit dynamics, we deal with a different equation depending on the closeness of the system to a resonance. A more comprehensive approach with a single equation that describe the relevant dynamics in any scenario is the Hamiltonian approach. The Hamiltonian function describing the evolution of the satellite spin is

$$H = \frac{\Theta^2}{2} - \frac{3}{4}n^2\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3 \cos 2(\theta - f)$$

where $\Theta = \dot{\theta}$ is the momentum conjugate to the angle $\theta$. It is easy to show that the Hamilton equations lead to the same equation for the angle $\theta$. In particular

$$\dot{\theta}=\frac{\partial H}{\partial \Theta}=\Theta$$
$$\dot{\Theta}=-\frac{\partial H}{\partial \theta}=\frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3\sin 2(\theta-f)=\ddot{\theta}$$

We can develop in Fourier series the perturbative term since it is periodic. Both the angle $\theta$ and $f$ change periodically with time, the first with the rotational period of the satellite, the second with the orbital period. The Hamiltonian then reads

$$H=\frac{\Theta^2}{2}-\frac{3}{4}n^2\left(\frac{B-A}{C}\right)\sum_r K_r(e)\cos(2\theta-r\,M)=\frac{\Theta^2}{2}+R$$

where r is an integer. For the Earth-Moon system, Celletti (1990) derived the following development

$$R=\left(-\frac{e}{2}+\frac{e^3}{32}\right)\cos(2\theta-M)$$
$$+\left(1-\frac{5}{2}e^2+\frac{13}{32}e^4\right)\cos(2\theta-2M)$$
$$+\left(\frac{7}{2}e-\frac{123}{16}e^3\right)\cos(2\theta-3M)$$
$$+\left(\frac{17}{2}e^2-\frac{115}{6}e^4\right)\cos(2\theta-4M)$$
$$+\left(\frac{845}{48}e^3-\frac{32525}{768}e^5\right)\cos(2\theta-5M)....$$

When the system is close to a resonance, the corresponding term in the development evolves slowly with time and perturbs the system. All the other terms change rapidly and average to 0. Close to a resonance, in the Hamiltonian we can consider *only the corresponding term* and neglect all the others. We can compare the previous approach with the Hamiltonian one and we will notice that the terms are equal.

| p | r |
|---|---|
| 3 | 6 |
| 5/2 | 5 |
| 2 | 1 |
| 3/2 | 3 |
| ... | ... |

In the proximity of a resonance we can then use the approximate Hamiltonian

$$H=\frac{\Theta^2}{2}-\nu_0^2 H(p,e)\cos(2\psi)$$

where $\psi$ is the resonant angle and $\nu_0^2 = 1/4\,\omega_0^2$ .
To compute the *resonance width*, it is necessary to perform a canonical transformation to a new set of action-angle variables. We define:

$$\Psi = \dot{\theta} - \frac{r}{2} n$$

$$\psi = \theta - \frac{r}{2} M$$

This is a canonical transformation since

$$det \begin{pmatrix} \frac{\partial \Psi}{\partial \Theta} & \frac{\partial \Psi}{\partial \theta} \\ \frac{\partial \psi}{\partial \Theta} & \frac{\partial \psi}{\partial \theta} \end{pmatrix} = 1$$

and it is time dependent because M depends on time t through $M = nt$

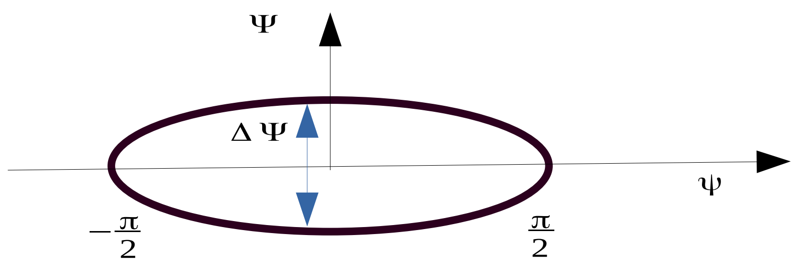
The new Hamiltonian becomes

$$K = \frac{\Psi^2}{2} - \nu_0^2 H(p,e)\cos(2\psi)$$

Since the energy K is constant, the equation for the separatrix can be easily derived. Expressing the action in function of the angle we get:

$$\Psi(\psi) = \pm\sqrt{2K + 2\nu_0^2 H(p,e)\cos(2\psi)}$$

The exact resonance where $\Psi$ is equal to 0 occurs when $\psi = \pm\pi/2$ so we can determine the value of $K$




$$K = \nu_0^2 H(p,e)$$

and the separatrix equation becomes

$$\Psi = \pm\sqrt{2\nu_0^2 H(p,e)(1+\cos(2\psi))}$$

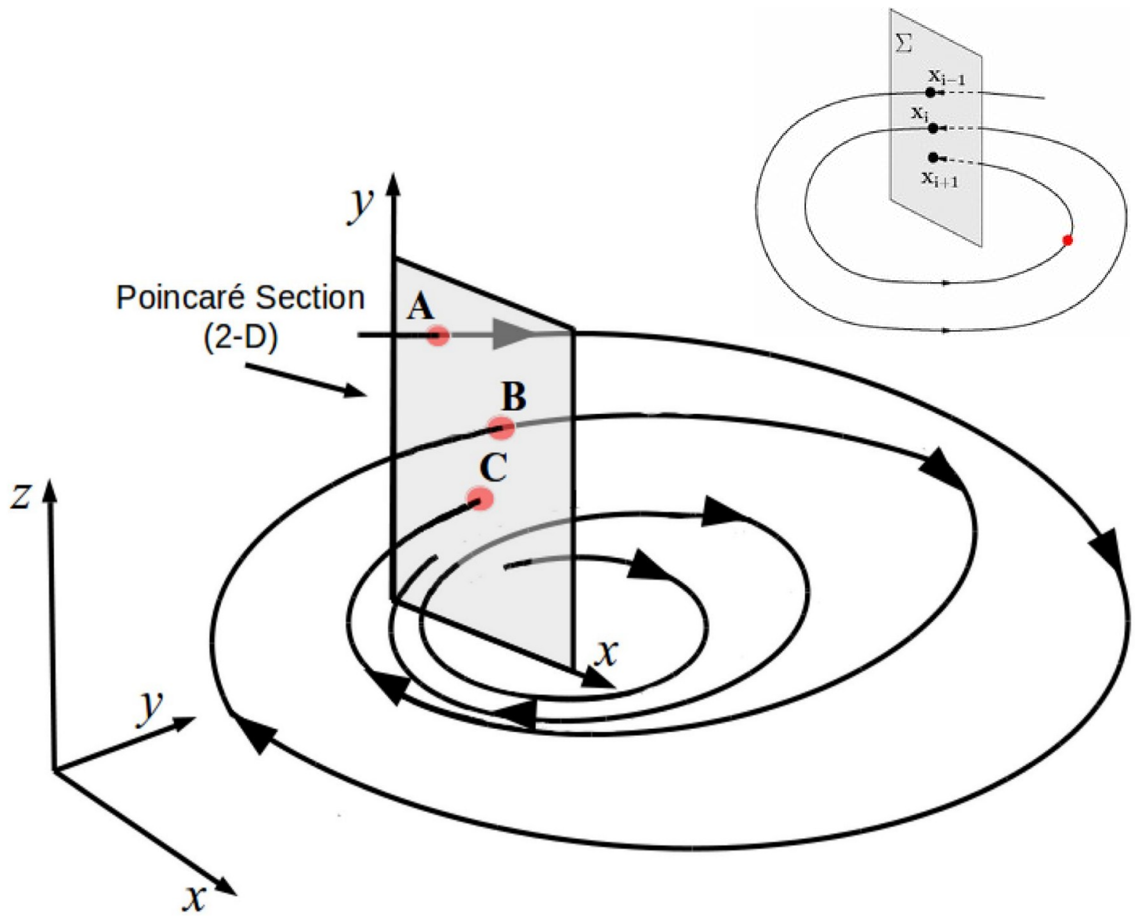
when $\psi = 0, \pi/2$ we have the maxima that give the resonance amplitude as

$$\Delta\Psi = 2\nu_0\sqrt{H(p,e)} = \omega_0\sqrt{H(p,e)}$$

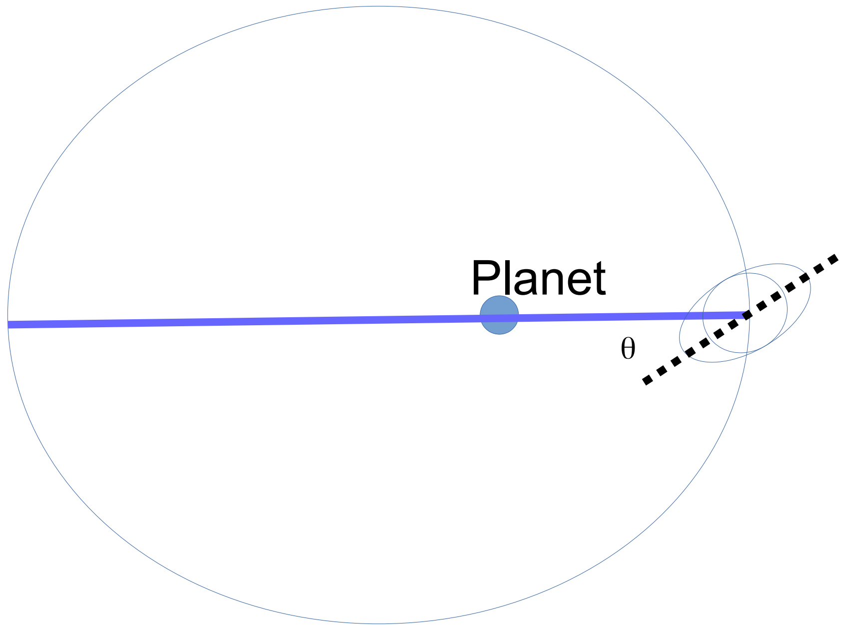
The resonance amplitude depends both on the resonance type and on the eccentricity of the orbit.

**Resonance overlapping and chaos: the Chirikov criterion**

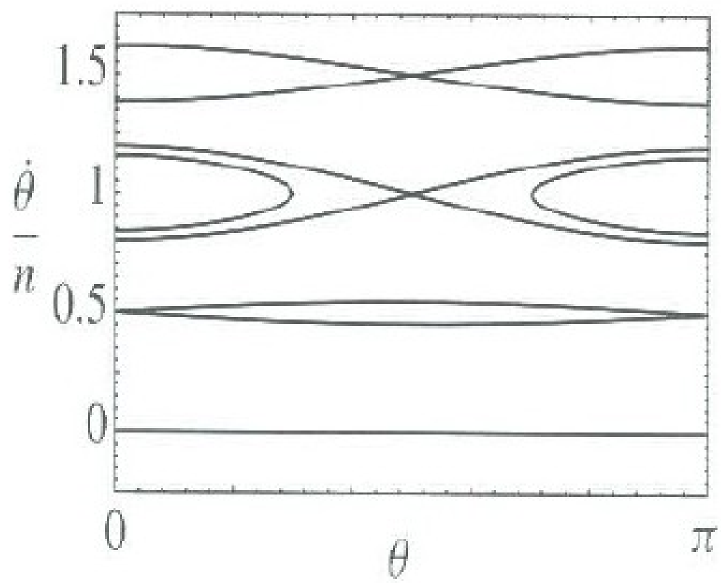
The resonance overlap criterion is based on the hypothesis that the onset of chaos in a dynamical system is related to the superposition of resonances. When the resonance amplitude becomes too large, close resonances overlap and the system wanders chaotically from one to the other. To plot the evolution of a system close to resonances, usually Poincare's maps are used. Such maps are obtained by plotting in 2D the crossings, at subsequent evolutionary times, of the trajectory with a fixed plane, as in figure.




In the case of the rotation rate of a satellite, the rotation angle $\theta$ and its speed normalized to the mean motion of the satellite $\dot{\theta}/n$ are plotted anytime the satellite passes at the pericenter of its orbit.




In the figure below we show the phase space when the resonances are well separated. The first resonance is the 3:2, then the 1:1 and, at the bottom, the 1:2. If we now increase the value of either $\omega_0$ (a larger aspherical coefficient) or of $H(p,e)$ (by increasing the eccentricity), the separatrix move closer and finally intersect, as show in the figure.



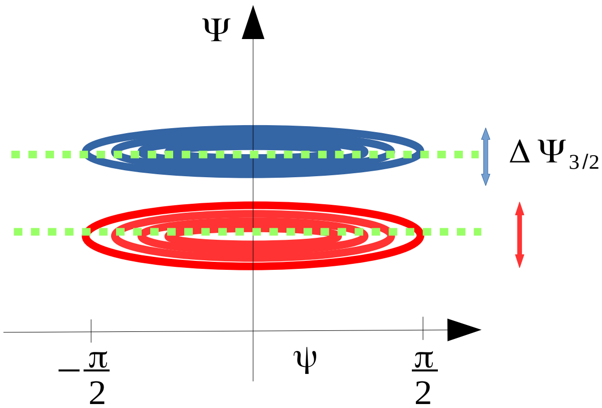

When the system is in the resonance overlap regime, chaotic motion develop and it is not possible to predict the state of the system after some time. We can analytically predict when resonance overlap occurs by using the expression for the resonance width. Ad example, if we concentrate on the 3:1 and 1:1 resonances, the condition for the separatrix to cross is

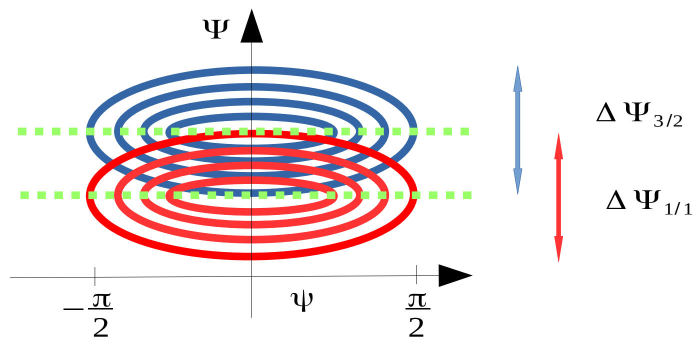
$$\omega_0\sqrt{H(1,e)}+\omega_0\sqrt{H(3/2,e)}=(3/2-1)n=1/2n$$

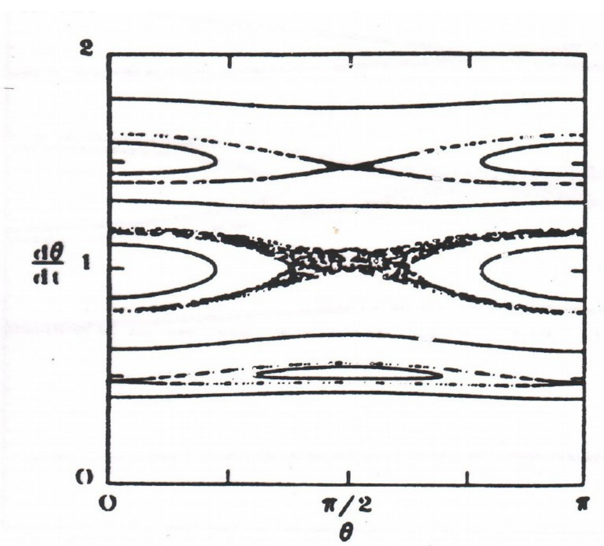
remember here that $\Theta=\dot{\theta}$ and the resonance is between $\dot{\theta}$ and *n* that must be commensurable according to a ratio between integers. In this case the hamiltonian has two terms which may interfere:

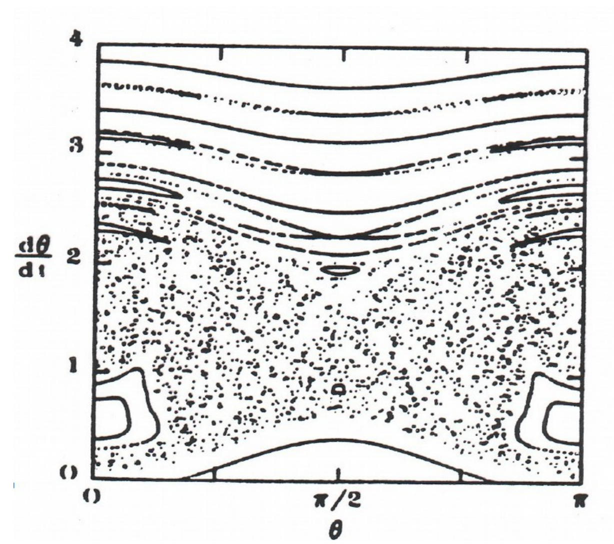
$$H_r = \frac{\Theta^2}{2} - \varpi_0^2 H(1, e)\cos(2\theta - 2M) - \varpi_0^2 H(3/2, e)\cos(2\theta - 3M)$$

In figure it is shown how the phase space changes when we go from a regime where the resonances are well separated to the resonance overlapping regime.



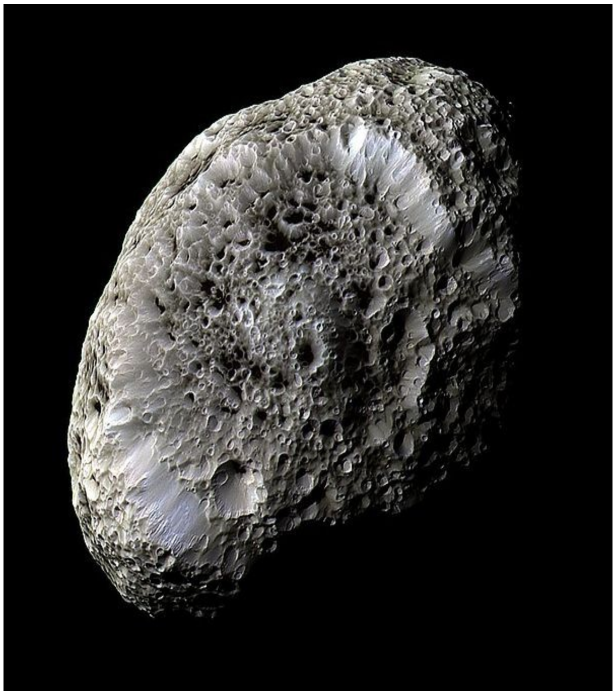



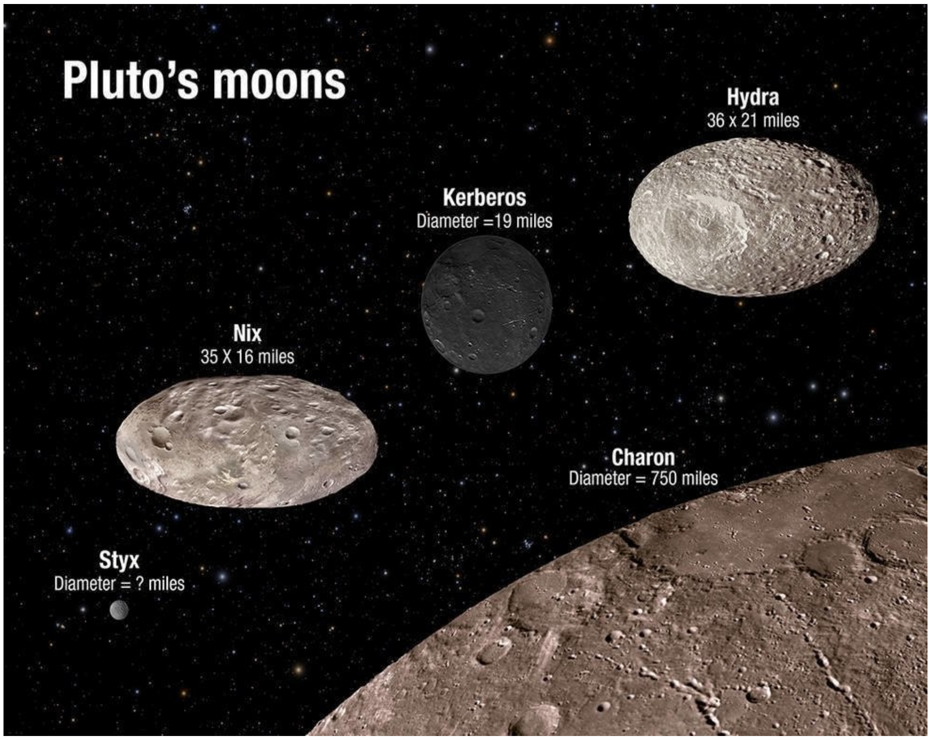

The figure on the left shows a phase space portrait for the case of a satellite on a low eccentricity orbit with a small value of the (B-A)/C. The resonances are well separated and the motion is quasi periodic. For higher values of eccentricity and (B-A)/C resonances overlap and a large chaotic zone appears where before the motion was periodic.

Some satellites are presently highly irregular in shape possibly due to a past collisional history. They are Miranda, Mimas and Hyperion. Hyperion (show in the picture) is highly irregular and its spin is evolving chaotically. The criterion for resonance overlap gives a limiting value of $\omega_0^c \sim 0.31$ while the physical value for Hyperion is $\omega_0 \sim 0.8$ . Mimas and Miranda may have been chaotic in the past. Another satellite which might be chaotic is Nereid, Neptune satellite, due to its large eccentricity.

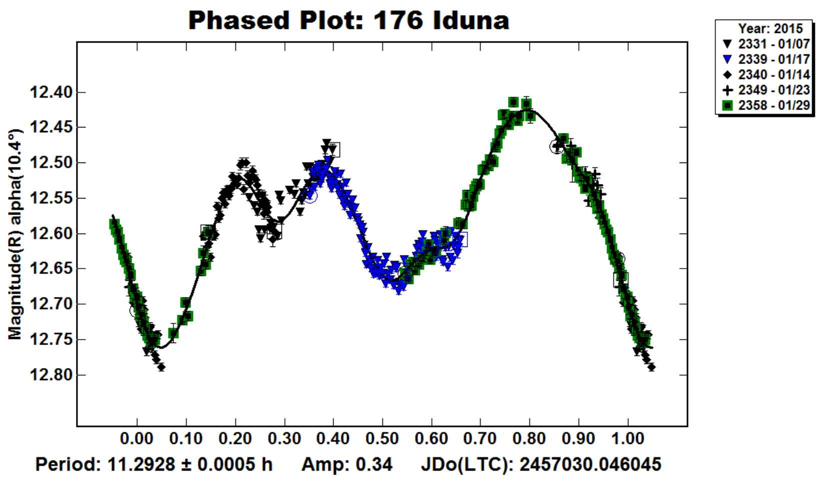
Also Nereid, satellite of Neptune (400 km of diameter) which has an eccentricity  e = 0.751 could be chaotic.



Another example of chaotic evolution of the spin is the chaotic tumbling. The orientation of the spin axis does not perform a regular precession motion but it evolves in an unpredictable way. An example is Nix, satellite of Pluto.



The tumbling is deduced from the evolution of the lightcurves. Possibly, also Styx, Kerberos and Hydra are chaotic. Also asteroids are often found in a tumbling state and it is testified by the irregular light curve. The asteroid does not rotate around a fixed axis and the reflecting area (and then the measured reflected sun light) changes with time.




## Appendix A: canonical transformations

Definitions: A coordinate transformation from *(p,q)* to *(w,z)*, where p and w are the momenta, is canonical if 1) it preserves the Hamiltonian form of the equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad H(p,q) = k(w,z) \qquad \dot{z}_i = \frac{\partial K}{\partial w_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{w}_i = -\frac{\partial K}{\partial z_i}$$

2) it preserves the area, in other words the Jacobian of the transformation has determinant equal to 1

$$det\begin{pmatrix} \frac{\partial w}{\partial p} & \frac{\partial w}{\partial q} \\ \frac{\partial z}{\partial p} & \frac{\partial z}{\partial q} \end{pmatrix} = 1$$

3) A generatrix function of mixed variables exists so that, ad example

$$p = \frac{\partial F(q,w)}{\partial q} \qquad z = \frac{\partial F(q,w)}{\partial w}$$

A simple example is $F = k\,q\,w$ so that $w = p/k$ and $z = kq$ .
Another example is the harmonic oscillator whose equations of motion can be solved by exploiting canonical transformations. Its Hamiltonian is:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

A suited canonical transformation can be generated by the following function in mixed variables:

$$F(q, z) = \frac{m}{2}\omega q^2 cotg\, z$$

The new variables *w,z* are related to *p,q* in the following way:

$$p = \frac{\partial F}{\partial q} = m\,\omega\, q\, cotg\, z \qquad q = \pm\sqrt{\frac{2w}{m\omega}}\sin z$$

$$w = -\frac{\partial F}{\partial z} = \frac{m\omega q^2}{2\sin^2 z} \qquad p = \pm\sqrt{2m\omega w}\cos z$$

The new Hamiltonian, obtained by substituting p,q with the new variables w,z is then:

$$K(w, z) = H(p, q) = \omega w \cos^2 + \omega w \sin^2 z = \omega w$$

In the new coordinates, the Hamilton equations become:

$$\dot{z} = \frac{\partial K}{\partial w} = \omega$$

$$\dot{w} = -\frac{\partial K}{\partial z} = 0$$

In these coordinate system, the momentum w is a constant of motion so the equations can be easily solved giving:

$$z = \omega t + z_0$$

$$w = w_0$$

Going back to the old coordinates, we find a solution to the motion in the p,q coordinates.

$$q(t) = \pm\sqrt{\frac{2w_0}{m\,\omega}}\sin(\omega t + z_0)$$

$$p(t) = \pm\sqrt{2m\,\omega w}\cos(\omega t + z_0)$$

Canonical transformations are then useful when trying to solve the equations of motion.