The 3-body problem

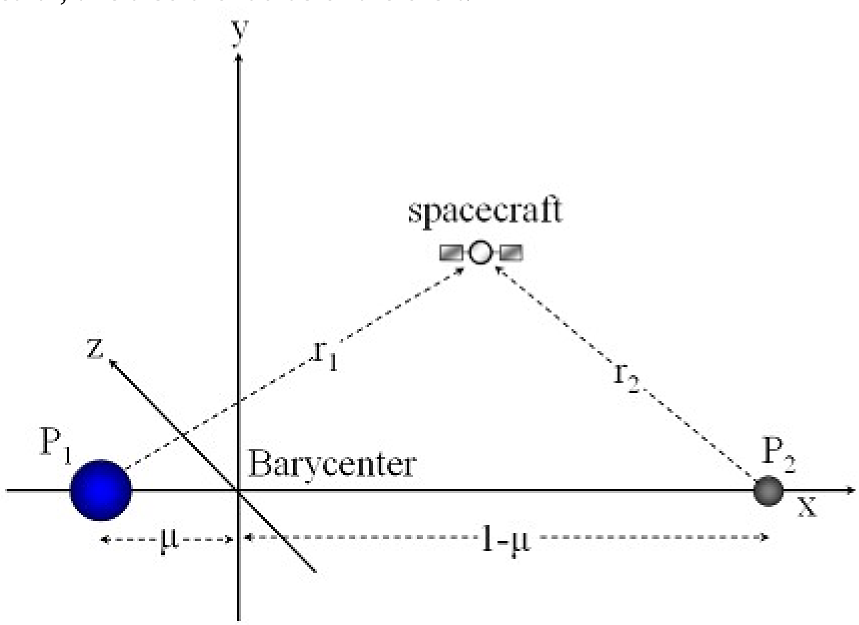
In general, the three body problem consists in determining the motion of 3 point masses which attract each other according to some kind of force for any initial conditions. We will focus here on the gravitational and hierarchical 3-body problem where the force is the Newtonian gravity and one body is significantly more massive than the others which orbit around it. The equation governing the motion of the three bodies in hierarchical form (the more massive body is at the center of the reference frame) is

$$\frac{d\boldsymbol{r_i}}{dt^2} = -\frac{G(m_0+m_i)}{r_i^3}\boldsymbol{r_i} + \sum_{j=1,\, j\neq i}^{3} Gm_j\left(\frac{\boldsymbol{r_j}-\boldsymbol{r_i}}{|\boldsymbol{r_j}-\boldsymbol{r_i}|^3} - \frac{\boldsymbol{r_j}}{r_j^3}\right)$$

If we concentrate on the scenario in which one of the 3 bodies has a mass negligible respect to the other 2, we can assume that its mass is 0 and deal with the *restricted 3-body problem*. The equation governing the motion of the third mass is in this case

$$\frac{d\boldsymbol{r}}{dt^2} = -\frac{Gm_0}{r_i^3}\boldsymbol{r_i} + \sum_{j=1}^{3} Gm_j\left(\frac{\boldsymbol{r_j}-\boldsymbol{r_i}}{|\boldsymbol{r_j}-\boldsymbol{r_i}|^3} - \frac{\boldsymbol{r_j}}{r_j^3}\right)$$

The problem is complicate enough not to be handled analytically. However, a simplified version of it can been described with analytical tools. If we assume that the orbits of the two lighter bodies occur on the same plane and the the orbit of the second mass m_2 is circular, then we deal with the *restricted, planar, circular 3-body problem*. To simplify the analytical treatment of the problem, we switch to *normalized units*. We set $G(m_1+m_2)=1$ and $a=1$ where *a* is the semimajor axis of body 2. Being the orbit circular, *a* is also the radius of the orbit.



In these normalized units, we can define the mass ratio $\mu = \frac{m_2}{(m_1+m_2)}$ so that the mass of the central body is $\mu_1 = 1 - \mu$ while that of the secondary body is simply $\mu_2 = \mu$. The mean motion of body 2 around body 1 is given by $n = \sqrt{\frac{G(m_1 + m_2)}{a^3}} = 1$. The distance of body 1 to the center of mass of the system is $x_1 = \frac{m_2}{m_1 + m_2} = \mu$ while that of the second massive body is $x_2 = \frac{m_0}{m_1 + m_2} = 1 - \mu$. Since the orbit of m_2 is circular respect to m_1, it will be circular also respect to the center of mass of the system. We can describe the motion of the system in a reference frame rotating with frequency $n = 1$ around the barycenter. In this reference frame, the position of bodies 1 and 2 will be fixed along the x-axis and we can write in a simplified form the equations of the motion of the third massless particle

$$\ddot{\boldsymbol{r}} + 2(\boldsymbol{\omega} \times \boldsymbol{v}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = -\nabla V$$

where $\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ n \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$. The coordinate x,y lie on the orbital plane of body 2 respect to body 1. The second term on the left is the centrifugal force while the third is the Coriolis force, both due to the use of a rotating and then non-inertial reference frame. Re-writing the above equation in the coordinates we obtain the Hill's equation

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial V}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial V}{\partial x} \\ \ddot{z} &= \frac{\partial V}{\partial z} \end{aligned}$$

where $V(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{mu}{r_2}$ is the pseudo-potential which includes a centrifugal term in addition to the gravitational attraction of the two massive bodies. From now on, we concentrate on the *x,y* plan (*planar* problem) and neglect the motion along the z-axis. In this way the problem becomes bi-dimensional. Under this condition, $r_1 = \sqrt{(x+\mu)^2 + y^2}$ while $r_2 = \sqrt{(x-1+\mu)^2 + y^2}$ The system admits an integral of motion, the Jacoby constant C. If we multiply the Hill's equation by the corresponding velocity components we get

$$\begin{aligned} \dot{x}\ddot{x} - 2\dot{x}\dot{y} &= \dot{x}\frac{\partial V}{\partial x} \\ \dot{y}\ddot{y} + 2\dot{y}\dot{x} &= \dot{y}\frac{\partial V}{\partial x} \end{aligned}$$

We can add up the two equations in a single one obtaining

$$\dot{x}\ddot{x} + 2\dot{x}\dot{y} - 2\dot{x}\dot{y} + \dot{y}\ddot{y} = \dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\frac{\partial V}{\partial x} + \dot{y}\frac{\partial V}{\partial y}$$

This differential equation can be integrated respect to time giving

$$\int_{t_0}^{t_1} (\dot{x}\ddot{x} + \dot{y}\ddot{y})\,dt = \int_{t_0}^{t_1} \left(\dot{x}\frac{\partial V}{\partial x} + \dot{y}\frac{\partial V}{\partial y}\right)dt$$

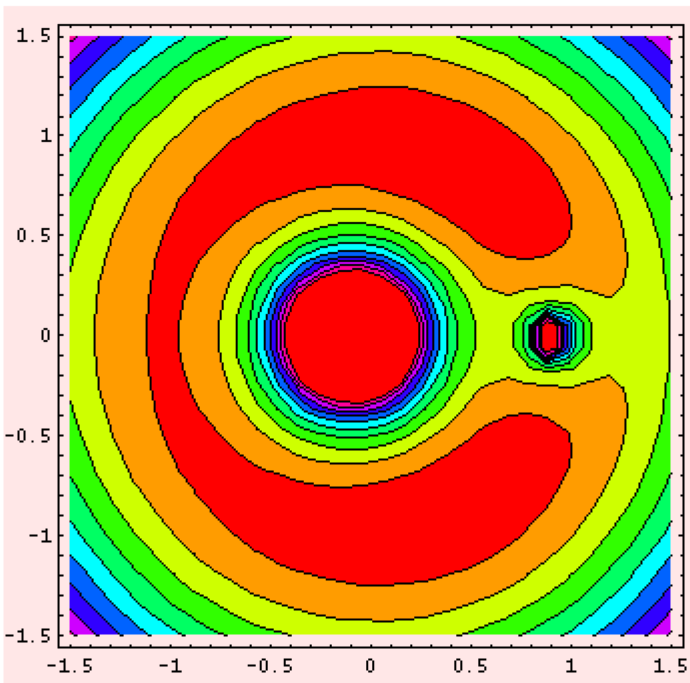
The integrals respect to time can be solved

$$\int_{t_0}^{t_1} (\dot{x}\ddot{x}+\dot{y}\ddot{y})dt=\frac{1}{2}(\dot{x}^2+\dot{y}^2)\ \Big|_{t_0}^{t_1}=\int_{t_0}^{t_1}(\frac{\partial x}{\partial t}\frac{\partial V}{\partial x}+\frac{\partial y}{\partial t}\frac{\partial V}{\partial y})dt=V(t_1)-V(t_0)=V(x_{1,}y_1)-V(x_{0,}y_0)$$

and the above equation reduces to

$$\frac{1}{2}(\dot{x}_1^2+\dot{y}_1^2)-V(x_{1,}y_1)=\frac{1}{2}(\dot{x}_0^2+\dot{y}_0^2)-V(x_{0,}y_0)=\frac{C}{2}$$

where C is the Jacoby constant. It is possible to construct curves in the plane on which the velocity vanishes. If such a zero-velocity curve is closed, then the particle cannot escape from the interior of the closed zero-velocity curve once it is placed there with the constant of the motion equal to the value used to construct the curve. The only way to escape is to use an external force (like a rocket engine for a spacecraft). The curves are obtained setting to 0 the velocity term $\frac{1}{2}(\dot{x}^2+\dot{y}^2)$ so they are determined uniquely by the pseudo-potential. In figure we show the curves for different values of the Jacoby constant.



The Jacoby has an additional meaning, it is the Hamiltonian of the system. The Hill's equation can be easily derived from the following Lagrangian

$$L=T+V=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+U(x,y)+U_r(x,Y)=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{(1-\mu)}{r_1}+\frac{\mu}{r_2}+\frac{1}{2}(x^2+y^2)+(x\dot{y}-\dot{x}y)$$

The system must obey the Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0$ which, after the insertion of the above expression for L, give back the Hill's equations. From the Lagrangian, we can derive the

Hamiltonian function

$$H=\sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$$

The generalized momenta are

$$p_x=\frac{\partial L}{\partial \dot{x}}=\dot{x}-y$$
$$p_y=\frac{\partial L}{\partial \dot{y}}=\dot{y}+x$$
$$p_z=\frac{\partial L}{\partial \dot{z}}=\dot{z}$$

The Hamiltonian then reads

$$H=\dot{x}^2-xy+\dot{y}^2+x\dot{y}+\dot{z}^2-\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{(1-\mu)}{r_1}+\frac{\mu}{r_2}+\frac{1}{2}(x^2+y^2)+(x\dot{y}-\dot{x}y)$$

This expression reduces to

$$H=\frac{1}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{(1-\mu)}{r_1}+\frac{\mu}{r_2}-\frac{1}{2}(x^2+y^2)$$

The Hamiltonian has then the same expression of the Jacoby constant (with a factor 2 difference). This is an alternative proof that C is constant since H is the energy of the system which, in absence of external and dissipative forces, is conserved. The equations of motion can be easily retrieved from the definition of Hamiltonian $H=\sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$ and the Hamilton equations

$$\dot{p}_x=\ddot{x}-\dot{y}=-\frac{\partial H}{\partial x}=\frac{\partial L}{\partial x}=\frac{\partial U}{\partial x}+x+\dot{y}$$
$$\dot{p}_y=\ddot{y}+\dot{x}=-\frac{\partial H}{\partial y}=\frac{\partial L}{\partial y}=\frac{\partial U}{\partial y}+y-\dot{x}$$
$$\dot{p}_z=\ddot{z}=-\frac{\partial H}{\partial z}=\frac{\partial L}{\partial z}=\frac{\partial U}{\partial z}$$

Lagrangian equilibrium points

The stationary points of the system in the x,y plane are obtained imposing the condition that both the velocities and accelerations are 0. By inspecting the equations of motion

$$\dot{x}\ddot{x}-2\dot{x}\dot{y}=\dot{x}\frac{\partial V}{\partial x}$$
$$\dot{y}\ddot{y}+2\dot{y}\dot{x}=\dot{y}\frac{\partial V}{\partial x}$$

we find that the necessary condition is that $\nabla V=0$. From the definition of V we get

$$\nabla V = \boldsymbol{r} - (1-\mu)\frac{\boldsymbol{r_1}}{r_1^3} + \mu\frac{\boldsymbol{r_2}}{r_2^3}$$

recalling that

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \boldsymbol{r_1} = \begin{pmatrix} x+\mu \\ y \end{pmatrix} \quad \boldsymbol{r_2}\begin{pmatrix} x-1+\mu \\ y \end{pmatrix}$$

we have the following indentity

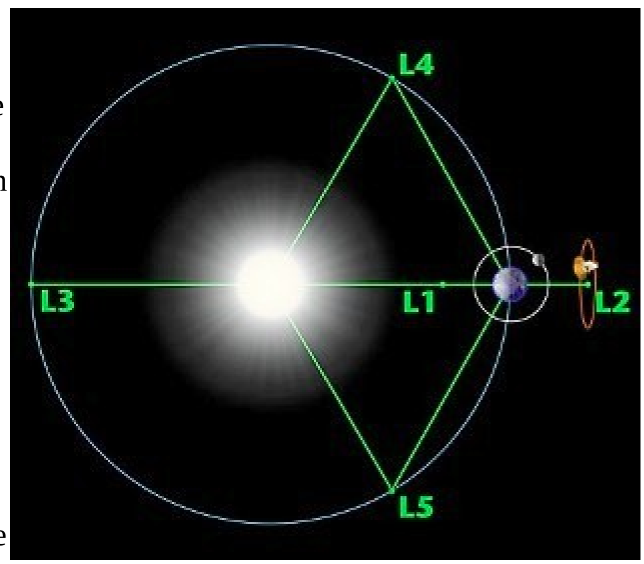
$$\boldsymbol{r} = (1-\mu)\boldsymbol{r_1} + \mu\,\boldsymbol{r_2}$$

Inserting this relation in the equation for the equilibrium, the equation for finding the stationary points becomes

$$\boldsymbol{r} = (1-\mu)\boldsymbol{r_1} + \mu\,\boldsymbol{r_2} = (1-\mu)\frac{\boldsymbol{r_1}}{r_1^3} + \mu\frac{\boldsymbol{r_2}}{r_2^3}$$

Grouping the terms in $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ we obtain

$$(1-\mu)\boldsymbol{r_1}\left(1-\frac{1}{r_1^3}\right) + \mu\,\boldsymbol{r_2}\left(1-\frac{1}{r_2^3}\right) = 0$$



Two easy solutions are given by $\boldsymbol{r_1} = \boldsymbol{r_2} = 1$. These are the equilateral Lagrangian points L_4 and L_5, *stable* equilibrium points located at the vertices of an equilateral triangle as in figure. Their coordinates are

$$L_4 = \begin{pmatrix} \frac{1}{2}-\mu \\ \frac{\sqrt{3}}{2} \end{pmatrix} \quad L_5 = \begin{pmatrix} \frac{1}{2}-\mu \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$

To go back to physical units, the coordinates must be multiplied by the semimajor axis of the perturbing planet. The other 3 stationary *unstable* points are again solutions of the stability equation, all lie on the x-axis and are labeled L_1, L_2 and L_3. Their coordinates are given as series expansion in the following way

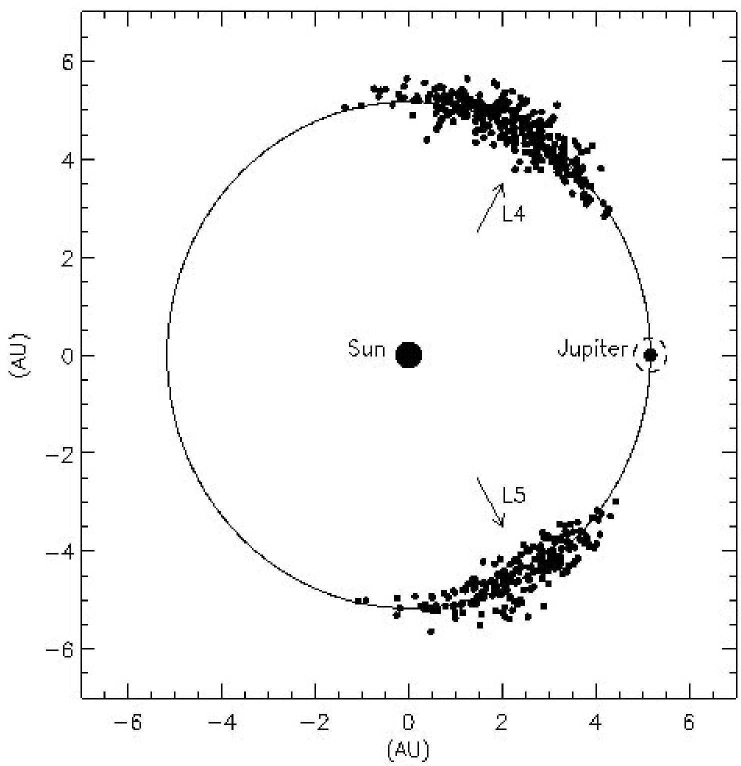
$$L_1 = \begin{pmatrix} 1-\mu+\left(\frac{\mu}{3}\right)^{\frac{1}{3}}+.. \\ 0 \end{pmatrix} \quad L_2 = \begin{pmatrix} 1-\mu-\left(\frac{\mu}{3}\right)^{\frac{1}{3}}+.. \\ 0 \end{pmatrix} \quad L_3 = \begin{pmatrix} -1-\frac{5}{12}\mu+.. \\ 0 \end{pmatrix}$$

L_1, L_2 are located on both sides of the perturbing body (the planet) while L_3 is located on the other side of the massive body (the star) .

Trojan asteroids

Around the equilateral L_4 and L_5 points, stable periodic orbits are possible. A consistent population of asteroids have been found around the two Lagrangian points of Jupiter while a potentially comparable population is inferred for Neptune. Mars is known to posses a few Trojans while recently a Trojan asteroid for the Earth has been found. More than 4000 Jupiter Trojans are presently known and they are named after Greek and Trojan heroes. The largest body, 624 Hektor, is an irregular body with dimensions 370 km×195 km×195 km. Two main theories have been proposed so far to explain the formation and evolution of the Trojans. Trojans might have been local planetesimals trapped in tadpole (Trojan type) orbits in the final phase of Jupiter growth when its mass increased from a few Earth masses to its present value (~318 Earth masses) on a very short timescale by gas infall. In alternative, the so called 'Nice model' predicts that Trojans were bodies formed in the outer part of the solar system, trapped by Jupiter during its evolution through a 2:1 resonance with Saturn.

A special class of orbits which surround both the L_4 and L_5 points and, in addition, the L_3 point are called *horseshoe* orbits. As shown in figure, the body moves back and forth from the planet with a trajectory (in the rotating reference frame) that encloses the three Lagrangian points. This kind of trajectory is unstable on the long term and it is a gate towards Trojan type orbits. This kind of orbit is shown in light blue in figure with the body moving from A, B etc... These kind of trajectories are relevant in planet migration since gaseous mass from disk crosses the planet orbit from outer regions of the disk into inner regions through these kind of trajectories. They can cause significant effects on the migrating planet orbit changing the direction of migration from outward inward and viceversa.



Trojan capture mechanisms

At present, about 7000 Jupiter Trojans, 29 Neptune Trojans, whose population might overcome that of Jupiter Trojans, and 9 Mars Trojans are known. The Trojan orbits of Saturn and Uranus and venus are known to be unstable (Marzari et al., 2002, 2003, 2005). There are static and dynamic mechanisms for the capture in Trojan orbits. The static mechanisms include 1) capture in horseshoe orbits because of orbital decay due to gas drag (Yoder, 1979), 2) trapping of fragments of collisions occurring close to the Trojan region (Schoemaker et al. 1989) and 3) capture by the widening of the Trojan region during the mass growth of the planet (Marzari & Scholl, 1998). Dynamic trapping can occur 1) during the migration of the planets if there is mean motion resonance superposition and chaotic evolution (Morbidelli et al., 2005) 2) during a steep jump in the semi-major axis of the planet due to a planet-planet scattering event (Nesvorny' et al., 2013) and 3) by the mass growth of the planet during its migration by tidal interaction with the circumstellar disk (Pirani et al., 2019).   These last two mechanisms explain also why there are more L4 Trojans than L5. However, while the model with the planet jump can produce either more L4 or more L5, the mechanism related to migration is not symmetric and inward migration leads to a larger L4 population.
To understand the capture of Trojans during migration or mass growth, an hamiltonian model can be developed. The Trojan motion can be described in a simplified way by the following equation:

$$\ddot{\phi}+\frac{27}{4}\mu n_P^2\phi=0$$

where n_P is the mean motion of the planet and φ the difference between the longitude of the planet and that of the body. This equation can be re-written in Hamiltonian form:

$$H=\frac{1}{2}a_P^2\dot{\phi}^2+\frac{1}{2}\omega a_P\phi^2$$

where the libration frequency of the angle φ is given by

$$\omega^2=\frac{27}{4}\mu n_P^2 \qquad \mu=\frac{m_P}{M_s+m_P}$$

It is the typical hamiltonian of a harmonic oscillator and the time evolution of the angle φ is then

$$\phi(t)=\frac{A}{2}\cos(\omega t+\alpha)$$

where A is the libration amplitude. When we change either the mass of the planet or its semi-major axis, we can compute how the libration amplitude varies. We introduce an adiabatic invariant

$J=\int^{period} p\,dq$ where $p=a_P\dot{\phi}$ $q=a_P\phi$ so that the hamiltonian is $H=\frac{1}{2}p^2+\frac{1}{2}\omega\frac{q^2}{a_P}$.

It is an adiabatic invariant if in one libration period the properties of the planet i.e. a_p and m_p and the amplitude and frequency of the Trojan motion A and ω do not significantly change. The integral becomes:

$$J=\int_0^{2\pi} a_P \dot{\phi}\, a_P\, d\phi=\int_0^{T} a_P^2 \frac{A^2}{4}\omega^2 \sin^2(\omega t+\alpha)\,dt$$

$$J=a_P^2 \frac{A^2}{4}\omega^2 \int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t+\alpha)\,dt$$

since $\int \sin^2(ax)\,dx=\frac{x}{2}-\frac{1}{4a}\sin(2ax)$ then the integral becomes

$$J=a_P^2 \frac{A^2}{4}\omega^2 \frac{2\pi}{2\omega}=\frac{\pi}{4}A^2 a_P^2 \omega=$$
$$\frac{\pi}{4}A^2 a_P^2 \sqrt{\left(\frac{27}{4}\frac{m_P}{m_P+M_s}\right)}\sqrt{\left(\frac{G(m_P+M_s)}{a_P^3}\right)}=$$
$$\frac{\pi}{4}A^2 \sqrt{\left(\frac{27}{4}G\right)}\, m_P^{1/2} a_P^{1/2}$$

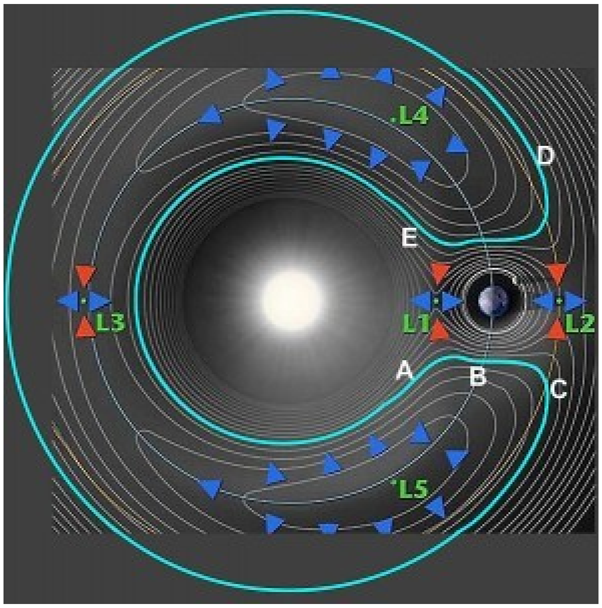
The conservation of J in presence of adiabatic variations of a_p and m_p leads to

$$\frac{A_f}{A_i}=\left(\frac{a_{P,i}}{a_{P,f}}\right)^{(1/4)}\left(\frac{m_{P,i}}{m_{P,f}}\right)^{(1/4)}$$

This equation implies that either if the planet migrates inward or it grows in mass, the libration amplitude decreases (from Fleming & Hamilton, Icarus 148, 479, 2000) . As a consequence, when a body is in the Trojan region and the semi-major axis or mass of the planet decreases/increases, the libration amplitude diminishes leading to a stronger trapping into the Trojan region.

Hill's sphere

An additional definition following the computation of the stationary Lagrangian points is that of Hill's sphere. In figure, it is the approximately circular shape encompassed between L_1 and L_2 surrounding the planet. According to the definition of 0 velocity curves, bodies that move within this region cannot escape, they are under the influence of the planet gravity field. The approximate radius of the Hill's sphere is given by the difference between the x coordinate of L_1 and L_2 along the x-axis.

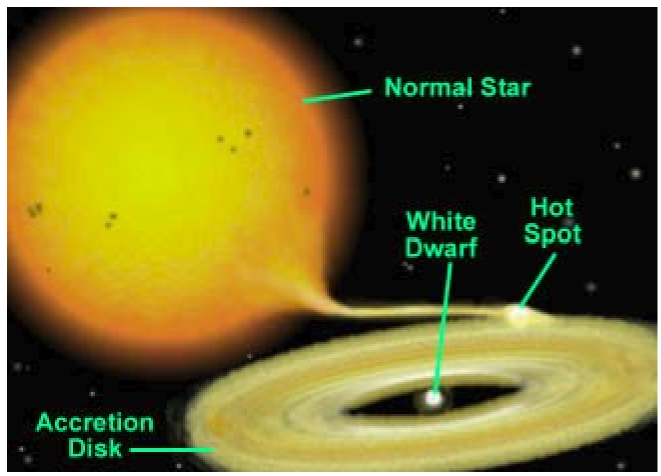


$$r_{Hill}=\left(\frac{\mu}{3}\right)^{\frac{1}{3}}$$

which, translated to physical units, becomes

$$r_{Hill}=\left(\frac{m_2}{3(m_1+m_2)}\right)^{\frac{1}{3}}a$$

Ad example, the $r_{Hill}\approx 0.355\,AU$ for Jupiter while it is $r_{Hill}=0.010\,AU$ for the Earth. The Hill's radius is relevant in asteroid satellite search since the smaller body must orbit within the Hill's sphere of the asteroids. It is also important in the explanation of the evolution of cataclysmic variables. They are binary star systems having a white dwarf (primary star) with a normal star companion. Their orbit is small with periods ranging from 1 to 10 hours. The companion star transfers mass to its compact companion and this interaction gives rise to a rich range of behaviour, of which the most noticeable are the outbursts of luminosity that give the class its name. Because of conservation of angular momentum, the infalling gas from the companion can't plunge directly onto the surface of the white dwarf. In systems where the white dwarf doesn't have an appreciable magnetic field, the infalling gas forms an accretion disk with the white dwarf at its center. The gas in the disk spirals down towards the white dwarf, radiating its gravitational potential energy away as it goes. The temperatures on this disk ranges from 5000 to 10000 K and can be as high as 10^5-10^8 K in the inner border, emitting in the UV and X. The resulting accretion process gives significant changes in brightness with outbursts. From this the name of cataclysmic variables. The transfer of mass begins when the secondary star fills its Hill's sphere and material begins to escape through L_1 into the primary star with disk.



The Tisserand invariant

This invariant is used to distinguish between asteroids and short period comets. It is expressed in heliocentric orbital elements. To derive it, we have to move from the reference frame centered on the barycenter and rotating with frequency *n* to a non-rotating reference frame centered on the star. We neglect the difference between the location of the barycenter respect to the center of the sun and we focus on the computation of the velocity in the fixed reference frame. The body velocity is then given by $\boldsymbol{v}=\boldsymbol{V}-\boldsymbol{v}_{SR}$ where **v** is the velocity in the velocity in the rotating reference frame, **V** is the velocity in the fixed one centered on the star, and $\boldsymbol{v}_{SR}$ is the instantaneous velocity of the rotating reference frame. This velocity is given as $\boldsymbol{v}_{SR}=\begin{pmatrix}-nY\\ nX\end{pmatrix}$ where *X,Y* are the coordinates of the massless body in the fixed reference frame. The relation between the velocity coordinates in the two reference frames are

$$\dot{x}=\dot{X}+nY$$
$$\dot{y}=\dot{Y}-nX$$
$$\dot{z}=\dot{Z}$$

Adding up the three equations we get

$$(\dot{x}^2+\dot{y}^2+\dot{z}^2)=(\dot{X}^2+\dot{Y}^2+\dot{Z}^2)-2n(X\dot{Y}-\dot{X}Y)+n^2(X^2+Y^2)$$

The above equation can be re-written as

$$\frac{1}{2}v^2=\frac{1}{2}V^2-nh_z+\frac{n^2}{2}(X^2+Y^2)=\frac{1}{2}V^2-nh_z+\frac{n^2}{2}(x^2+y^2)$$

where h_z is the z-component of the orbital angular momentum. To derive the last equation we recall also that since the rotating and fixed reference frame are assumed to have the same origin $X^2+Y^2=x^2+y^2$. Recalling the definition of the Jacoby constant we obtain

$$\frac{1}{2}v^2-\frac{n^2}{2}(x^2+y^2)=C+\frac{Gm_1}{r_1}+\frac{Gm_2}{r_2}$$

that gives

$$\frac{1}{2}V^2-nh_z=C+\frac{Gm_1}{r_1}+\frac{Gm_2}{r_2}$$

This relation holds in the fixed reference frame and it is a consequence of the invariance of C in the rotating reference frame. If we use the relations $\frac{1}{2}V^2=Gm_1\left(\frac{1}{r_1}-\frac{1}{2a}\right)$ and $h_z=\sqrt{Gm_1a(1-e^2)}\cos i$ from the 2-body dynamics, we can rearrange the above equation in the following form

$$C+\frac{Gm_2}{r_2}=-Gm_1\left(\frac{1}{2a}+n\sqrt{Gm_1a(1-e^2)}\cos i\right)=-Gm_1\left(\frac{1}{2a}+\sqrt{\frac{a(1-e^2)}{a_p^3}}\cos i\right)$$

since *n* is the mean motion of the planet with semimajor axis a_p. We can neglect the term $\frac{Gm_2}{r_2}$ which is small compared to those depending on Gm_1 so finally we obtain

$$\frac{a_p}{a}+2\sqrt{\frac{a(1-e^2)}{a_p}}\cos i=T$$

T is called the Tisserand invariant and it allows to recognize the orbital elements of a body after it had a close encounter with a planet (like Jupiter for short period comets). While the orbital elements will undergo significant changes due to the hyperbolic passage close to the planet, the value of *T* will be the same.