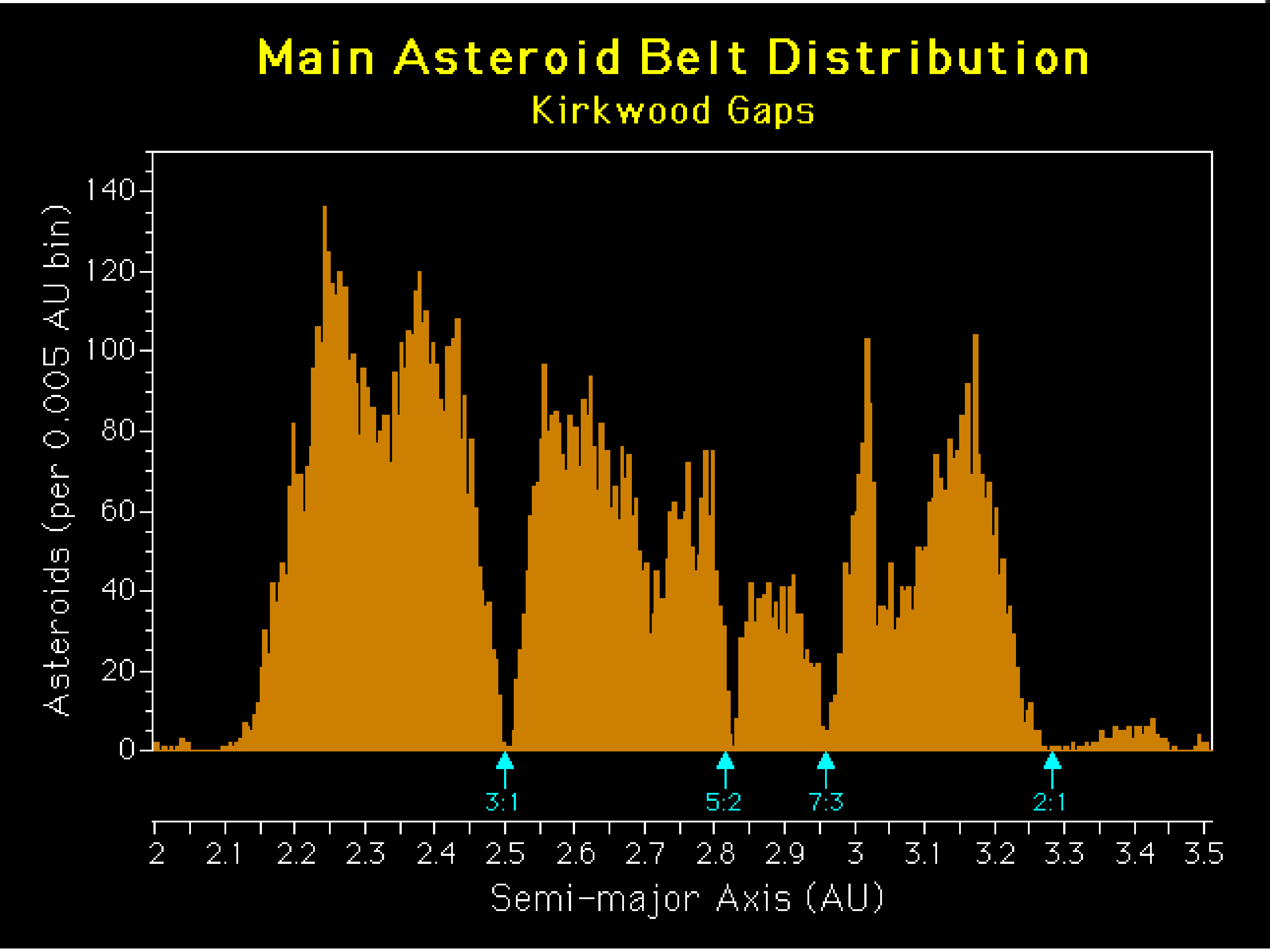
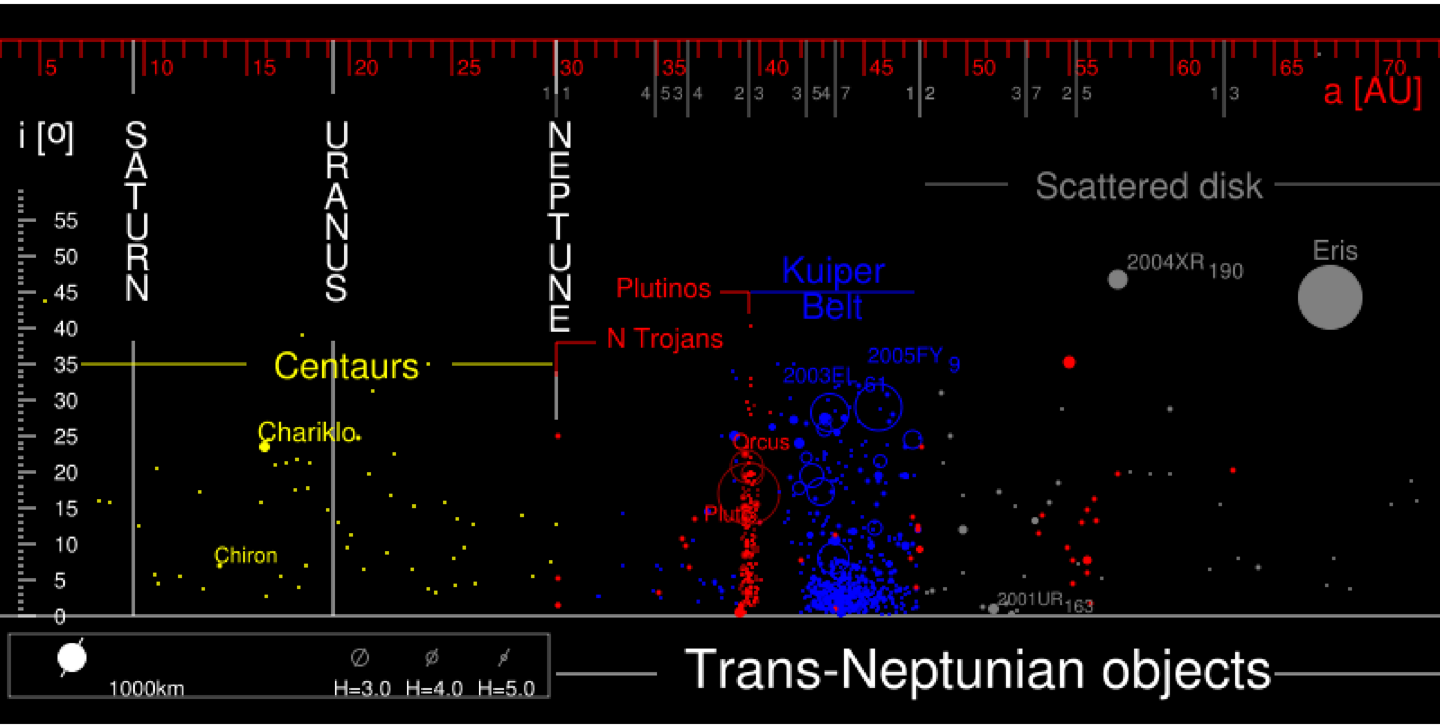
CHAPTER 10

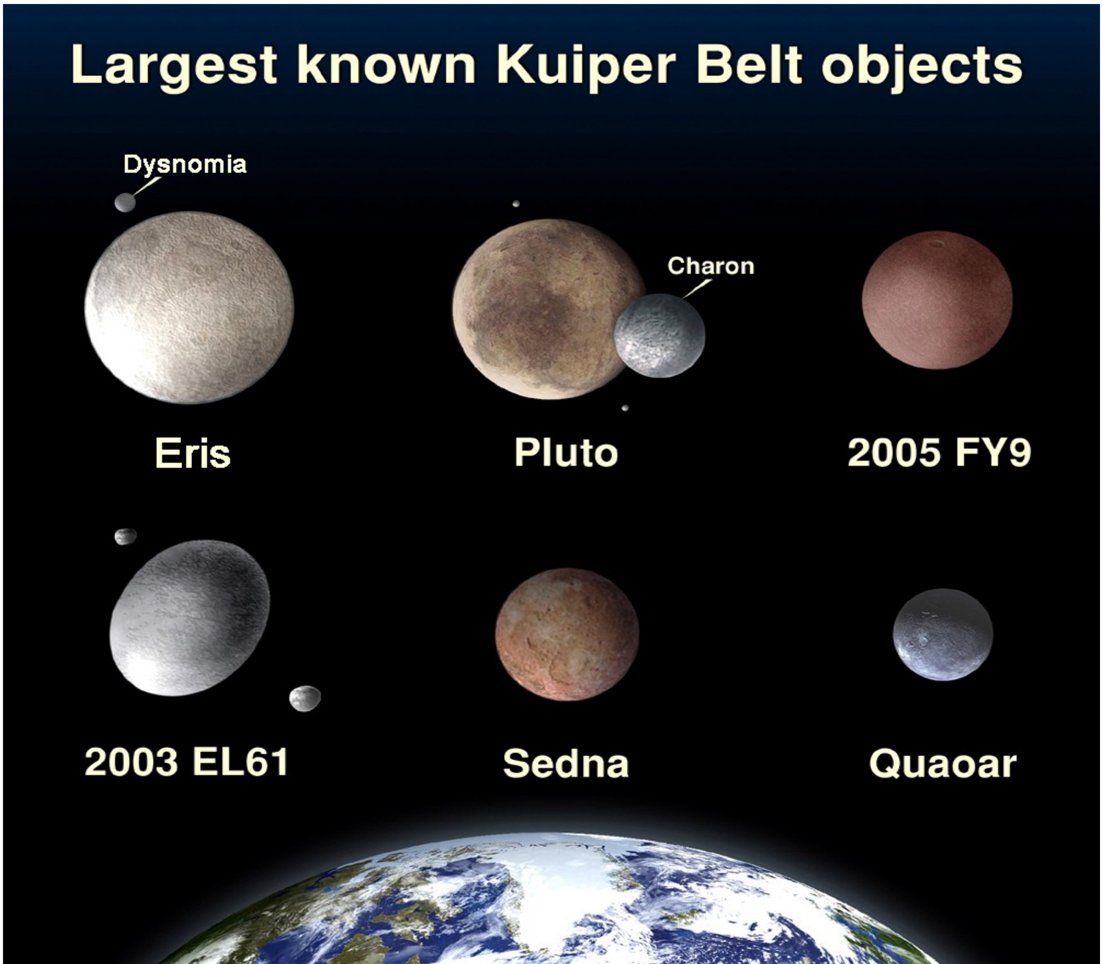
- Mean Motion resonance in the restricted 3-body problem
- Pendulum model for the resonant dynammics
- Resonance width
- Resonance superposition and chaos
- Esamples of numerical integrations
- Other sources of chaos in planetary systems: gravitational encounters

Kirkwood gaps: gaps in the distribution of asteroids in the Main Asteroid Belt. These gaps are due to chaotic motion within the mean motion resonances (superposition of resonances) with Jupiter.



Kuiper Belt: disk of planetesimals (comets) extending beyond the orbit of Neptune.





Among KBOs (or TNOs) there are bodies comparable in size with Pluto. Are they planets?

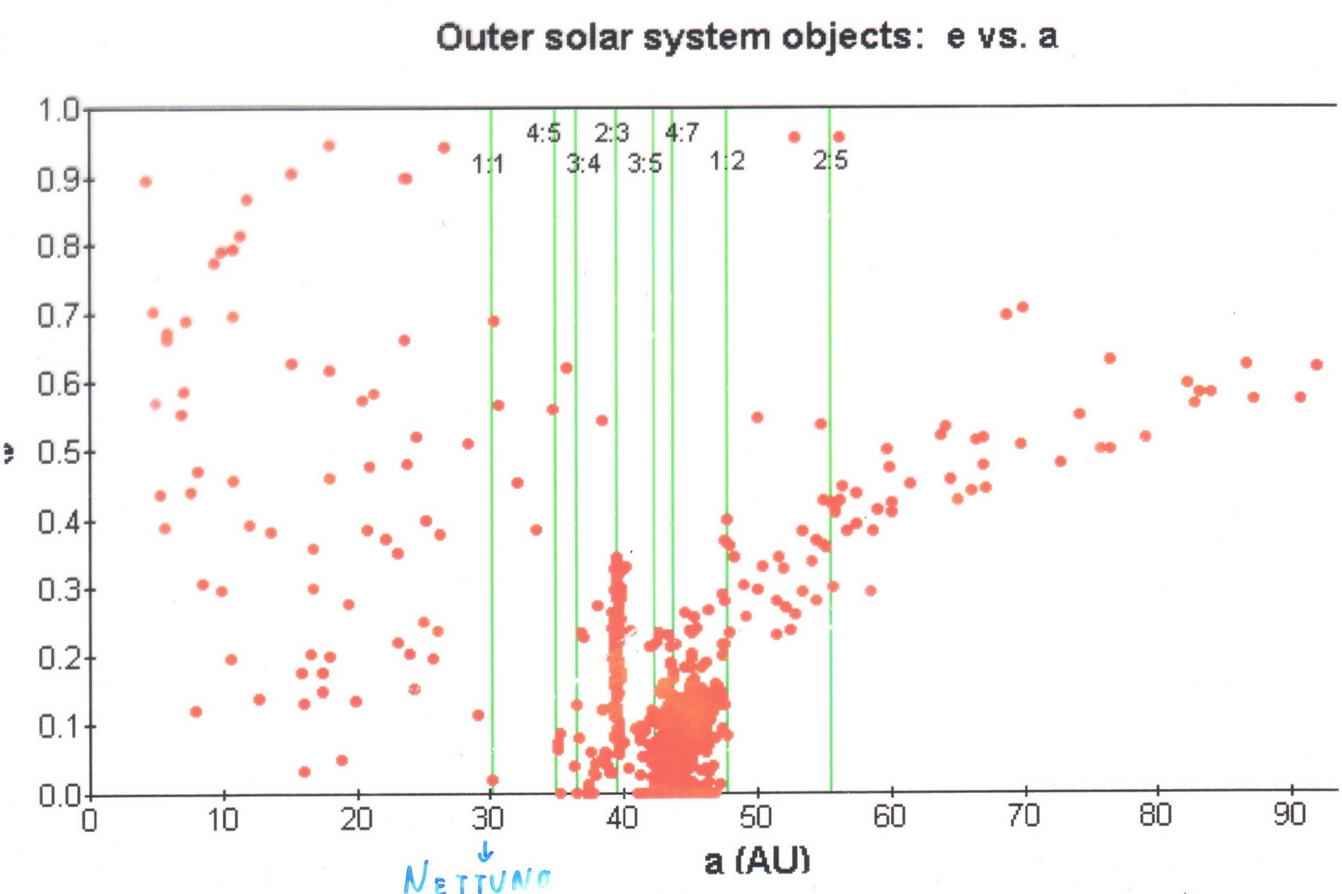
Definition of planet

- Orbit around the star
- Massive enough so that self gravity dominates in the shpe definition (almost spherical, hydrostatical equilibrium)
- It has cleared a ring around its orbit of left over planetesimals.

> If it fails to meet the third requirement, it is called **dwarf planet.**

	Ceres	Pluto	Eris
D	975×909 km	2306 km	2400 km
M	9.5×10^{20} kg	1.305×10^{22} kg	$\sim1.6\times10^{22}$ kg
a	2.766 AU	17.14 AU	67.69 AU
e	0.08	0.25	0.44
i	10.59^{o}	17.14^{o}	44.19^{o}

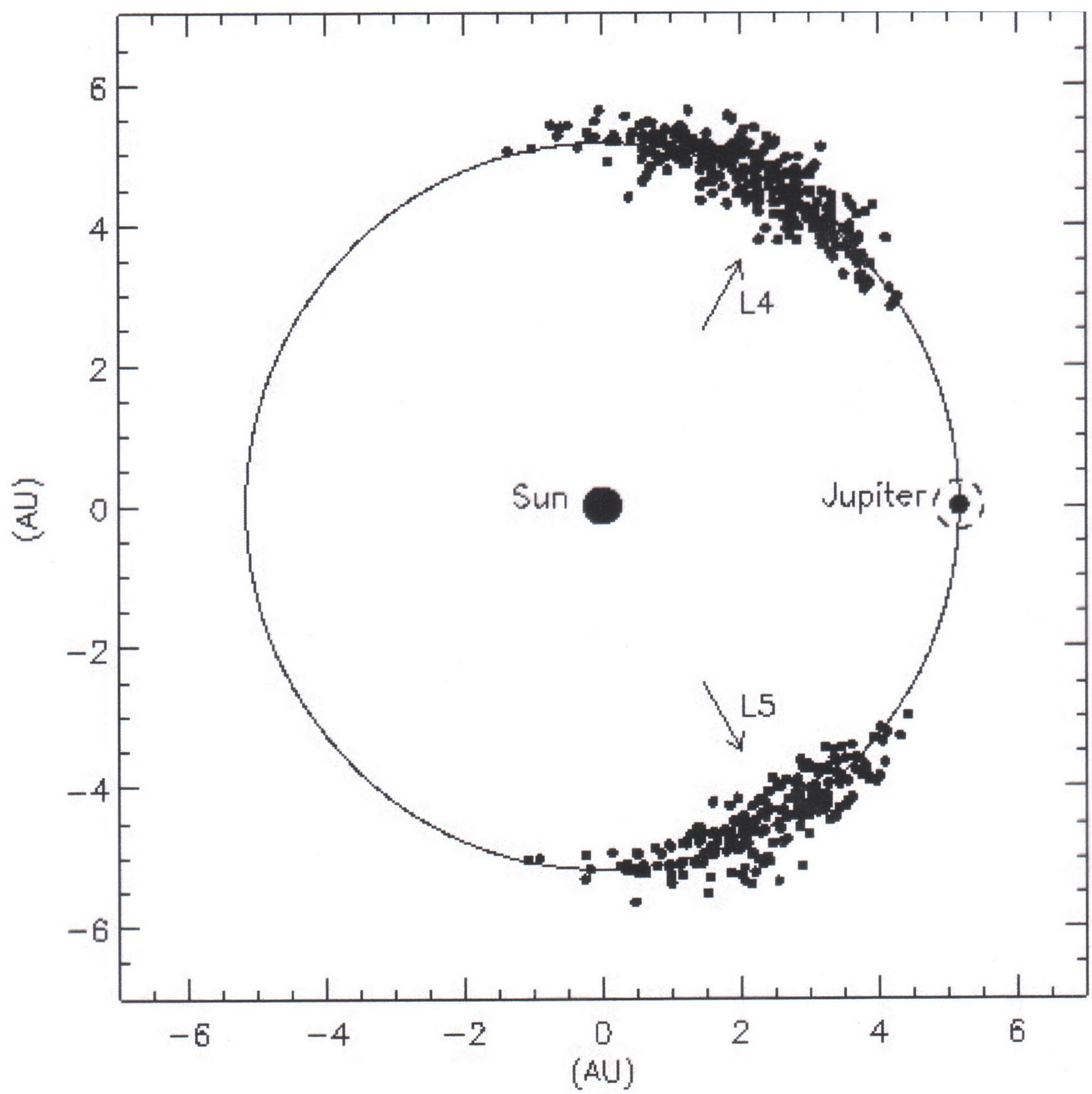
The Kuiper Belt is also sculpted by MMR. Plutinos are bodies in 2:3 resonance with Neptune. This resonance protects them from impacting the planet. For example, Pluto's orbit crosses that of Neptune (e ~ 0.25, I ~ 17.1°) but the resonance protects it from close encounters by phasing the orbital angles.



Troian asteroids are in a 1:1 resonance.

Jupiter: ~ 7000 oggetti

Mars: 7 **Neptune: 22**



Trojans of Venus, Uranus and Saturn are unstable due to the perturbations of the other planets.

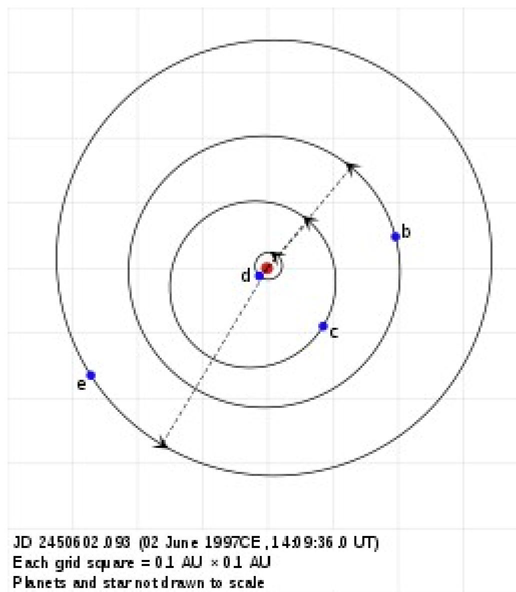
Exoplanets in resonance: the system in Gliese 876

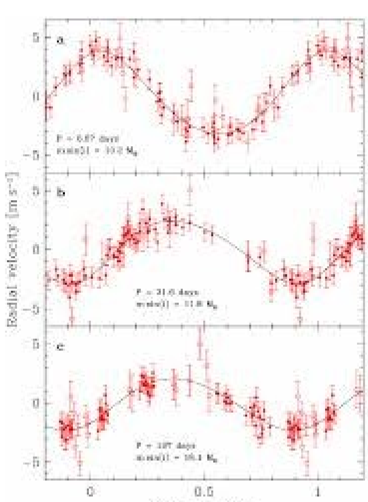
Companion (in order from star)	Mass	Semimajor axis (AU)	Orbital period (days)	Eccentricity	Inclination	Radius
d	6.83 ± 0.4 $M_{\oplus}$	0.02080665	1.937780	0.207 ± 0.055	—	—
c	0.7142 ± 0.004 M_J	0.129590 ± 0.000024	30.0081 ± 0.008	0.25591 ± 0.0009	—	—
b	2.2756 ± 0.0045 M_J	0.208317 ± 0.00002	61.1166 ± 0.0086	0.0324 ± 0.0013	—	—
e	14.6 ± 1.7 $M_{\oplus}$	0.3343 ± 0.0013	124.26 ± 0.70	0.055 ± 0.012	—	—

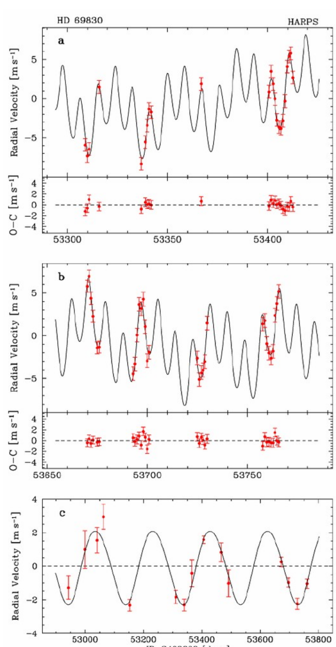
4:2:1 resonance among the 3 outer planets i.e. 2:1 between e and b and 2:1 between b and c.

The star is a red dwarf with a mass of 0.32 solar masses.

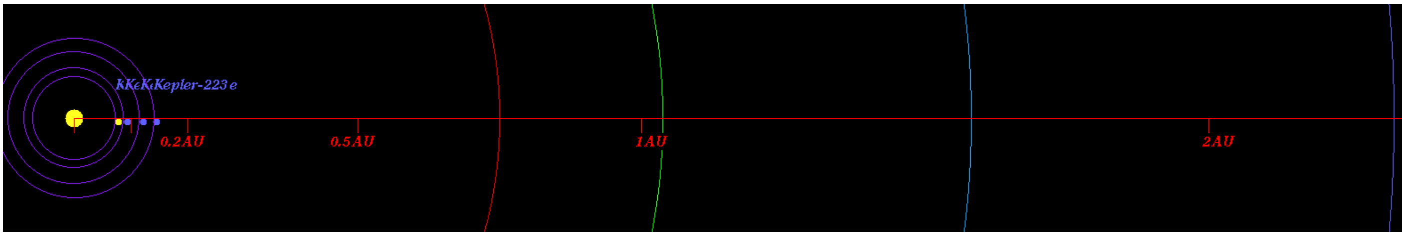
Multiple planet systems are detected from the radial velocity curve via progressive subtraction of each planet signal from the radial velocity curve (in figure the case of HD 69830).







Exoplanets in resonance: the system in Kepler 223

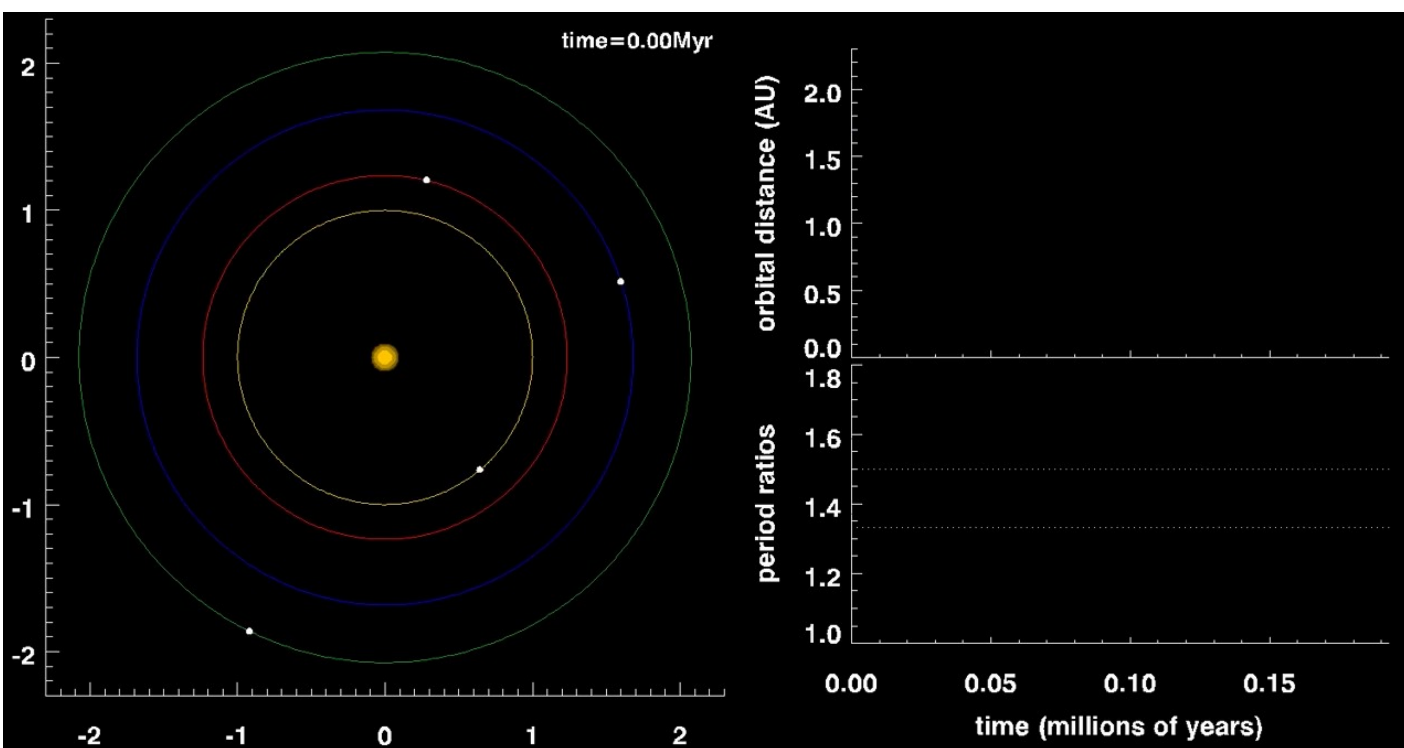


Planet	Mass (M_{Jup})	Radius (R_{Jup})	Period (day)	*a* (AU)	*e*	*i* (deg)
Kepler-223 e	—	0.214	19.721734	0.14	—	—
Kepler-223 d	—	0.266	14.788759	0.116	—	—
Kepler-223 c	—	0.178	9.848183	0.088	—	—
Kepler-223 b	—	0.151	7.384108	0.073	—	—

They are in a 8:6:4:3 mutual mean motion resonance.

$$\left(\frac{a_1}{a_2}\right)^{(3/2)} = \frac{i}{j}$$

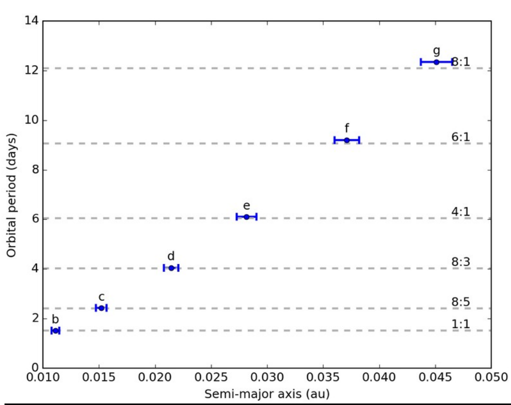
Convergent migration by interaction with the disk at the origin of the resonant configuration.



The ‘Trappist’ planetary system.

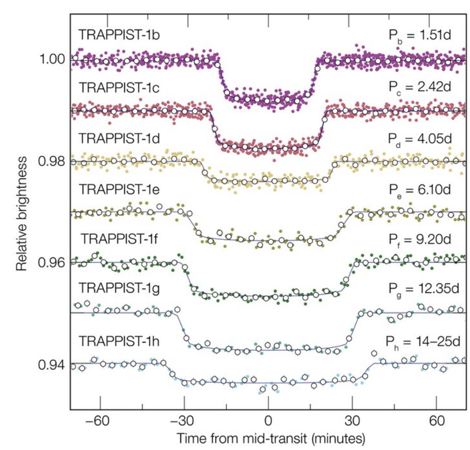
Companion (in order from star)	Mass[47]	Semimajor axis[47] (AU)	Orbital period[5] (days)	Eccentricity[47]	Inclination[5][38]	Radius[47]
b	$1.017^{+0.154}_{-0.143}\ M_\oplus$	0.01154775 (1.73 million km)	1.510 876 37 ± 0.000 000 39	0.006 22 ± 0.003 04	89.56 ± 0.23°	$1.121^{+0.031}_{-0.032}\ R_\oplus$
c	$1.156^{+0.142}_{-0.131}\ M_\oplus$	0.01581512 (2.37 million km)	2.421 807 46 ± 0.000 000 91	0.006 54 ± 0.001 88	89.70 ± 0.18°	$1.095^{+0.030}_{-0.031}\ R_\oplus$
d	$0.297^{+0.039}_{-0.035}\ M_\oplus$	0.02228038 (3.33 million km)	4.049 959 ± 0.000 078	0.008 37 ± 0.000 93	$89.89^{+0.08}_{-0.15}$°	$0.784^{+0.023}_{-0.023}\ R_\oplus$
e	$0.772^{+0.079}_{-0.075}\ M_\oplus$	0.02928285 (4.38 million km)	6.099 043 ± 0.000 015	0.005 10 ± 0.000 58	$89.736^{+0.053}_{-0.066}$°	$0.910^{+0.026}_{-0.027}\ R_\oplus$
f	$0.934^{+0.080}_{-0.078}\ M_\oplus$	0.03853361 (5.76 million km)	9.205 585 ± 0.000 016	0.010 07 ± 0.000 68	$89.719^{+0.026}_{-0.039}$°	$1.046^{+0.029}_{-0.030}\ R_\oplus$
g	$1.148^{+0.098}_{-0.095}\ M_\oplus$	0.04687692 (7.01 million km)	12.354 473 ± 0.000 018	0.002 08 ± 0.000 58	$89.721^{+0.019}_{-0.026}$°	$1.148^{+0.032}_{-0.033}\ R_\oplus$
h	$0.331^{+0.056}_{-0.049}\ M_\oplus$	0.06193488 (9.27 million km)	18.767 953 ± 0.000 080	0.005 67 ± 0.001 21	89.796 ± 0.023°	$0.773^{+0.026}_{-0.027}\ R_\oplus$

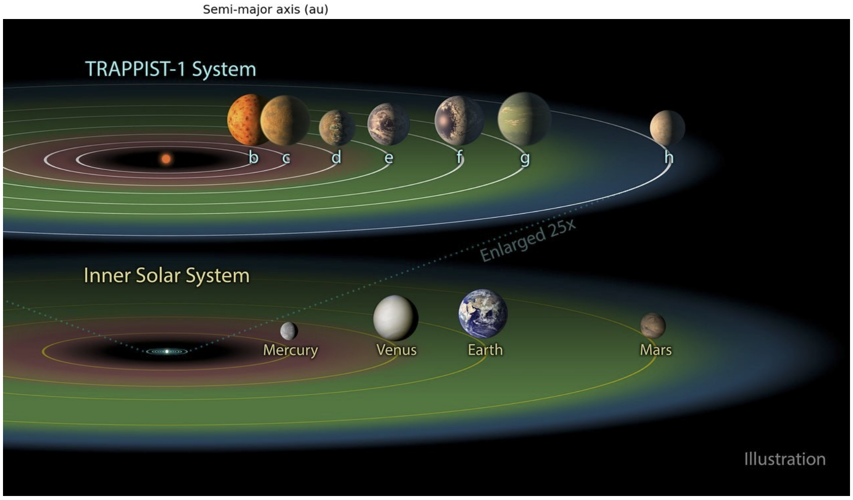
MM resonances between the planets.



From TTVs estimate the masses

Transits (Hubble)





3 planets in the habitable zone (liquid water on surface), the green ring in the figure. The inner planet is at 0.0116 au, the outer at 0.062 au.

The star is a red dwarf with a mass=0.086 the solar mass and a temperature of ~ 2600 K

Hamiltonian approach to the 2-body problem

$$\frac{d\mathbf{r}^2}{dt^2} = -G(M_0+m_1)\frac{\mathbf{r}}{r^3}$$

$$\eta\frac{d\mathbf{r}^2}{dt^2} = -\eta G(M_0 m_1)\frac{\mathbf{r}}{r^3}$$

$$\mu = G(M_0+m_1)$$

$$\eta = \frac{M_0 m_1}{M_0+m_1}$$

Reduced mass

The conjugate momentum is: $\mathbf{p} = \eta\mathbf{v}$

$$H = \frac{1}{2}\frac{\mathbf{p}^2}{\bar{\nu}} - \frac{\mu\eta}{r}$$

$$\frac{d\mathbf{r}}{dt} = \nabla_{\mathbf{p}} H$$

$$\frac{d\mathbf{p}}{dt} = -\nabla_{\mathbf{r}} H$$

Hamilton equations

The orbital elements are not canonical variables. It is necessary to introduce the Delaunay variables.

$$L = \eta\sqrt{(\mu a)} \qquad l = M$$

$$G = L\sqrt{(1-e^2)} = \eta\sqrt{(\mu a(1-e^2))} \qquad g = \omega$$

$$H = G\cos i = \eta\sqrt{(\mu a(1-e^2))}\cos i \qquad h = \Omega$$

The Keplerian Hamiltonian of the 2-body problem is then:

$$H = -\frac{\mu\eta}{2L^2}$$

$$\frac{dl}{dt} = \frac{\partial H}{\partial L} = n$$

The Hamilton equations give back the linear evolution of the Mean Anomaly l= n t

Hamiltonian approach to the restricted 3-body problem

$M_0,\ m_1,\ m=0 \quad m_1 \ on\ circular\ constant\ orbit \quad i_1=i=0$

$$\frac{d\mathbf{r}^2}{dt^2}=-GM_0\frac{\mathbf{r}}{r^3}+Gm_1\left(\frac{\mathbf{r_1}-\mathbf{r}}{|\mathbf{r_1}-\mathbf{r}|^3}-\frac{\mathbf{r_1}}{|\mathbf{r_1}|^3}\right)$$

$$U(r)=-G\frac{M_0}{r}-Gm_1\left(\frac{1}{|\mathbf{r_1}-\mathbf{r}|}-\frac{\mathbf{r}\cdot\mathbf{r_1}}{|\mathbf{r_1}|^3}\right)$$

The Hamiltonian of the system is:

$$H=\frac{v^2}{2}-\frac{GM_0}{r}-Gm_1\left(\frac{1}{|\mathbf{r_1}-\mathbf{r}|}-\frac{\mathbf{r}\cdot\mathbf{r_1}}{|\mathbf{r_1}|^3}\right)=\frac{v^2}{2}-\frac{GM_0}{r}-R(\mathbf{r},\mathbf{r_1})$$

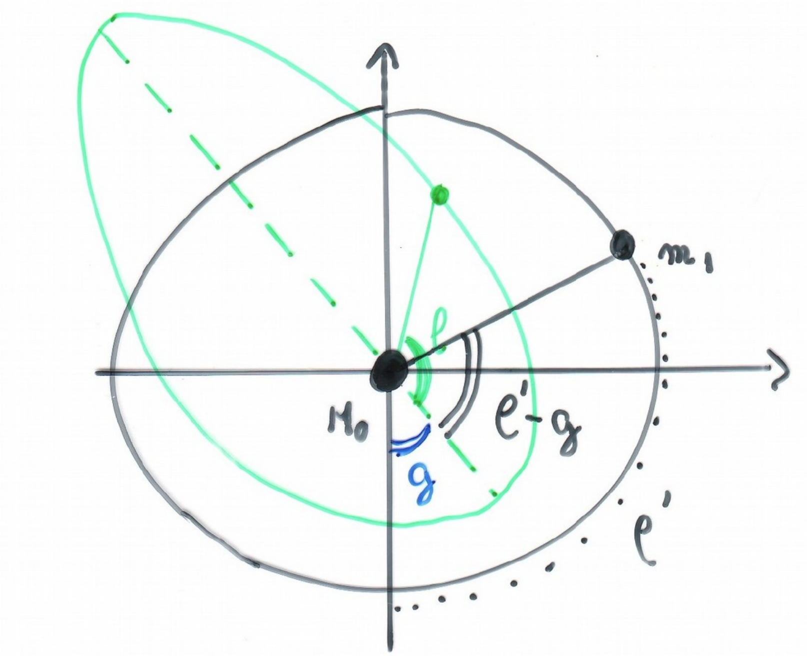
Delaunays' variables (the reduced mass has disappeared) :

$$L=\sqrt{(GM_0a)} \qquad l=M$$
$$G=L\sqrt{(1-e^2)}=\sqrt{(GM_0a(1-e^2))} \qquad g=\omega$$
$$H=G\cos i=\sqrt{(GM_0a(1-e^2))}\cos i \qquad h=\Omega$$

$$H=-\frac{G^2M_0^2}{2L^2}-R(L.G,H,l,g,h,t)$$

$$R(L.G,H,l,g,h,t)=Gm_1\sum_{i=0}^{\infty}\sum_{J=-\infty}^{\infty}K^{i,j}(G,L)\cos(il+j(l'-g))$$

Since it is the planar 3-body problem, h does not appear in the Hamiltonian and then H is a constant of motion and it can be neglected in the Fourier development.



1) There is a hidden dependence on time since l' = n' t (for the planet)
2) The first sum in the Fourier expansion goes from 0 to infinity to avoid doubling the terms

$$\cos(x+y)=\cos(-x-y)$$
$$\cos(x-y)=\cos(-x+y)$$

Resonance condition:

$$i\dot{l}(L,G)+j(\dot{l}'-\dot{g}(L,G))=0$$

If it is satisfied, repetition of mutual geometrical configurations, enhanced gravitational interaction.

$$i=p+q$$
$$j=-p$$

$$(p+q)\dot{l}(L,G)+.p(\dot{l}'-\dot{g}(L,G))=0$$

$$H=-\frac{G^2M_0^2}{2L^2}-G\,m_1K^{0,0}-Gm_1K^{i,j}\cos\psi$$

Only the dominating resonant term is kept in the Hamiltonian (the system is in a i,j resonance). The angle ψ is the critical argument of the resonance and it librates.

$$\psi=i\,l+j(l'-g)$$

Poincare's resonant canonical variables.

$$\Phi = [(p+q)G - pL]/q \qquad \phi = l - (l' - g)$$
$$\Psi = (L - G)/q \qquad \psi = (p+q)l - p(l' - g)$$

They are canonical because they can be computed from a generating function from the Delauney's variables.

$$f(t, l, g, \Psi, \Phi) = [(p+q)l - p(l' - g)]\Psi + (l + g - l')\Phi = \psi\Psi + \phi\Phi$$

$$\psi = \frac{\partial f}{\partial \Psi} \quad \phi = \frac{\partial f}{\partial \Phi} \quad L = \frac{\partial f}{\partial l} \quad G = \frac{\partial f}{\partial g}$$

The Hamiltonian in the new variables reads:

$$K = H + \frac{\partial f}{\partial t} H = -\frac{G^2 M_0^2}{2(\Phi + (p+q)\Psi)^2} - pn'\Psi - n'\Phi$$
$$- Gm_1 K^{0,0}(\Psi, \Phi) - Gm_1 K^{p+q,p}(\Psi, \Phi)\cos\psi - R$$

$$\frac{\partial f}{\partial l'}\frac{\partial l'}{\partial t} = -pn'\Psi - n'\Phi$$

$-Gm_1 K^{0,0}$ *is neglected because small*

First order resonances; q= ± 1

$$j = p + q$$
$$j + 1 = p$$

For example 2:1, 3:2 , 4:3 ...etc..

If q=-1, the minor body is inside the orbit of the planet.
1 n_a = 2 n_p i.e. 2 T_a = 1 T_p

(see paper by Winter and Murray, Astronomy & Astrophysics, 318, 290-304, 1997)

$$K = -\frac{G^2 M_0^2}{2(\Phi + j\Psi)^2} - (j+1)n'\Psi - n'\Phi + Gm_1 K^{p+q,p}(\Psi,\Phi)\cos\psi$$

$$\Phi = -\sqrt{G M_0 a}\left(1 - \sqrt{(1-e^2)}\right)$$

$$\Psi = -\sqrt{G M_0 a}\left(j\sqrt{(1-e^2)} - j - 1\right)$$

Condition for the onset of resonance:

$$\dot{\psi} = \frac{\partial K}{\partial \Psi} = -\frac{G^2 M_0^2 j}{(\Phi + j\Psi)^3} - (j+1)n' = 0$$

The term $Gm_1 K^{p+q,p}(\Psi,\Phi)\cos\psi$ **averages to 0**
This condition is equivalent to:

$$j - (j+1)\frac{n'}{n} = 0 \quad \Rightarrow \quad \frac{n}{n'} = \frac{a_p^{3/2}}{a^{3/2}} = \frac{(j+1)}{j}$$

Pendulum model of the resonant dynamics

Series development (Taylor) of the hamiltonian K around the resonant value of $\Psi = \Psi_r$

$$K = -\frac{G^2 M_0^2}{2\left(\Phi + j\Psi_r\right)^2} + \frac{G^2 M_0^2 j}{\left(\Phi + j\Psi_r\right)^3}\left(\Psi - \Psi_r\right) - \frac{3 G^2 M_0^2 j^2}{\left(\Phi + j\Psi_r\right)^4}\left(\Psi - \Psi_r\right)^2 + \ldots - (j+1)n'\Psi_r - (j+1)n'\left(\Psi - \Psi_r\right) - n'\Phi$$

$$K = A_0 + A_1\left(\Psi - \Psi_r\right) + A_2\left(\Psi - \Psi_r\right)^2 + B_0 \cos\psi$$

$$A_0 = -\frac{G^2 M_0^2}{2\left(\Phi + j\Psi_r\right)^2} - (j+1)n'\Psi_r - n'\Phi$$

$$A_1 = -\frac{G^2 M_0^2 j}{\left(\Phi + j\Psi_r\right)^3} - (j+1)n'$$

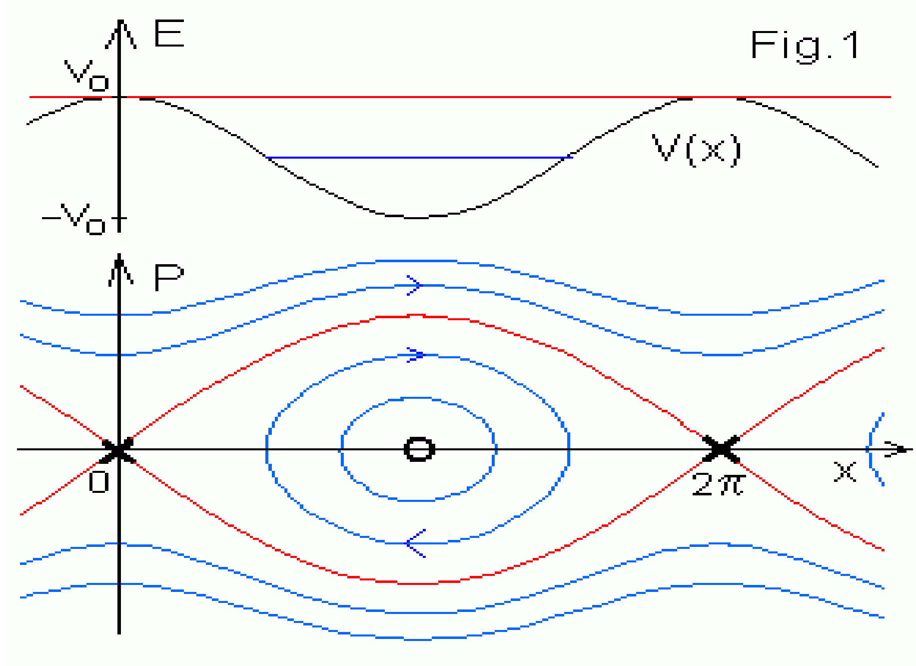
← **resonance condition $A_1 = 0$**

$$A_2 = -\frac{3 G^2 M_0^2 j^2}{2\left(\Phi + j\Psi_r\right)^4}$$

n.b. Φ is constant because K does not depend on φ

$$K = A_0 + A_2\left(\Psi - \Psi_r\right)^2 + B_0 \cos\psi$$

Phase space portrait: the red curve is the separatrix. The level curves are given by the following equation:



$$\Psi = \Psi_r \pm \left(\frac{K - A_0 - B_0 \cos \psi}{A_2} \right)^{1/2}$$

K is the energy and it changes depending on the level curve

The separatrix crosses the point $(0,\Psi_r)$ and we can compute the value of K in that point:

$$\Psi(\psi=0) = \Psi_r \qquad K = A_0 + B_0$$

$$\Psi_{sep}(\psi) = \Psi_r \pm \left(\frac{B_0}{A_2}(1 - \cos\psi) \right)^{1/2} =$$

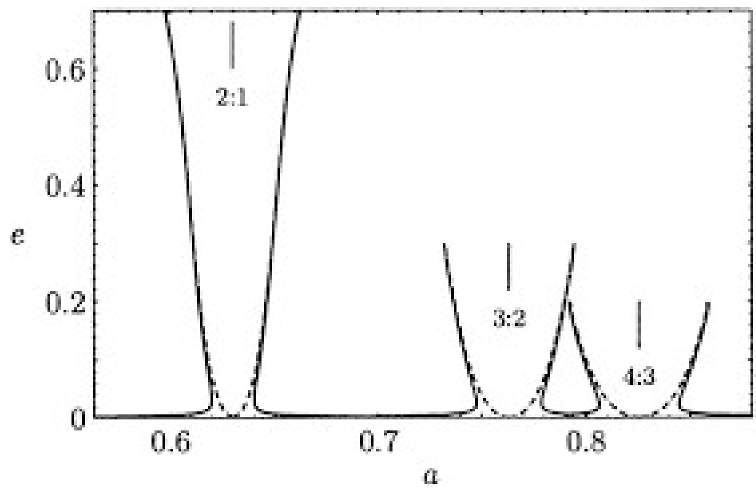
$$= \Psi_r \pm \sqrt{\frac{2B_0}{A_2}} \sin\frac{\psi}{2}$$

$$\sin\left(\frac{\psi}{2}\right) = \sqrt{\frac{1-\cos\psi}{2}}$$

The resonance half-width is evaluated at the point $(\Delta\Psi, \pi/2)$

$$\Delta\Psi = \sqrt{\frac{2B_0}{A_2}}$$

Resonance width with a planet located at 1 au (semi-major axes normalized to that of the planet). There is a dependence on the eccentricity of the minor body (see definition of Ψ)

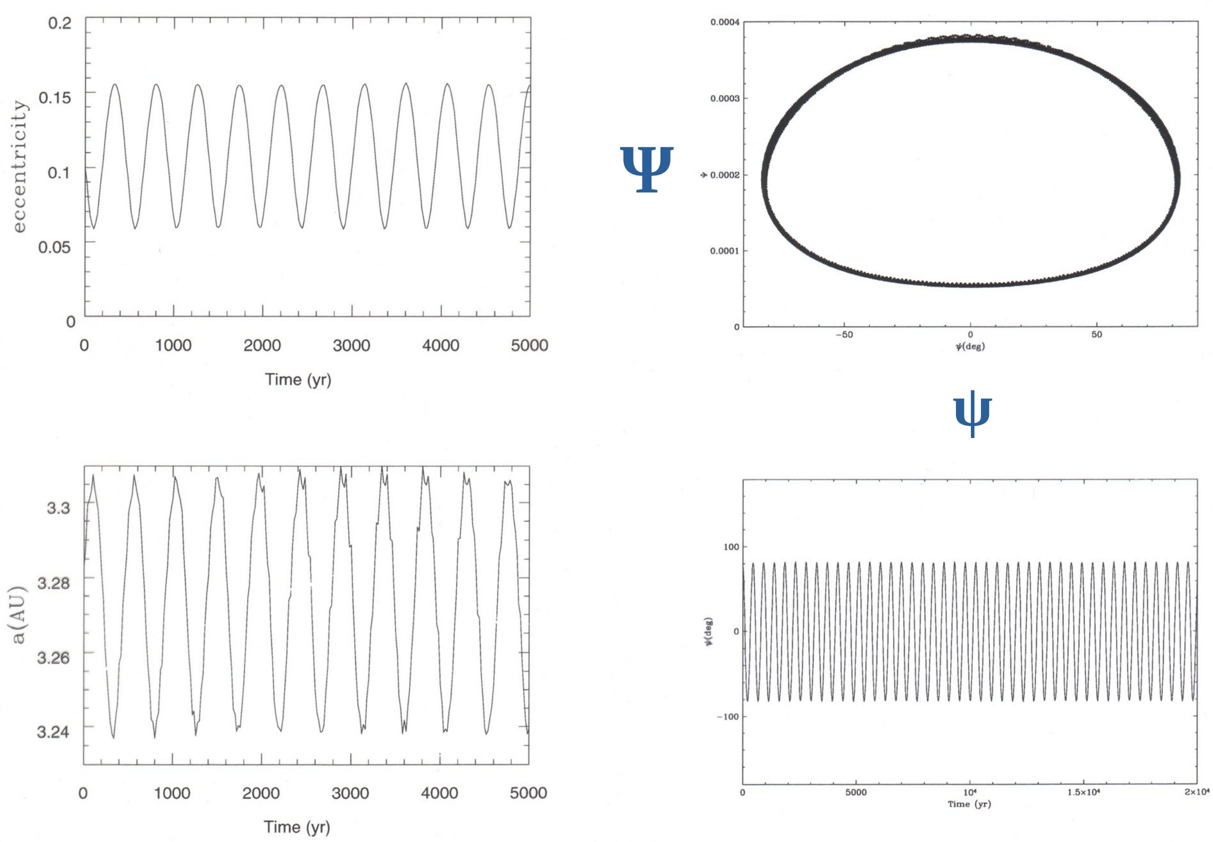


$$\delta n = \pm\sqrt{12 n^2 e \frac{m_1}{M_0} \frac{a}{a'} f\left(\frac{a}{a'}\right)}$$

$$\delta a = -\frac{2a}{3n}\delta n$$

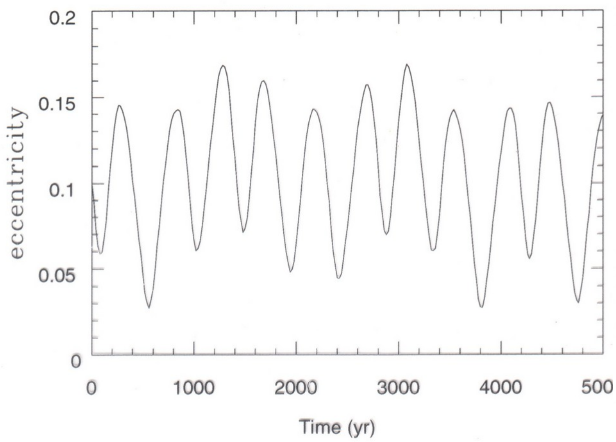
(see Murray & Dermott, Solar System Dynamics, pg. 338)

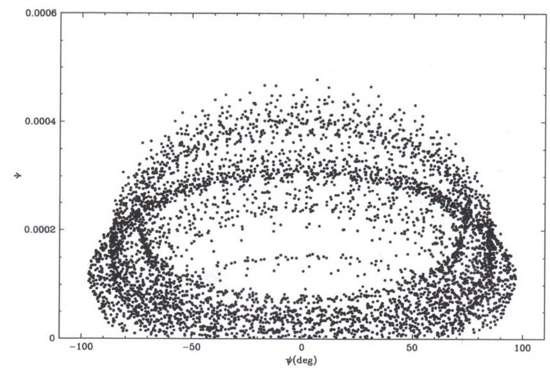
Numerical examples of the orbital elment evolution of a minor body in a 2:1 resonance with a planet (Jupiter)

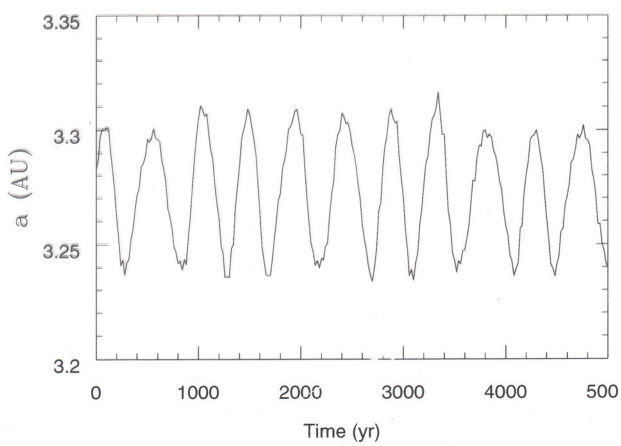


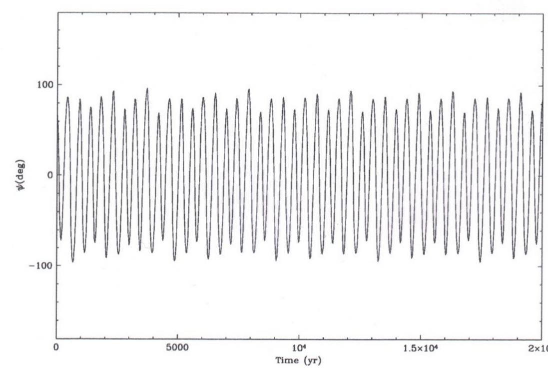
The eccentricity of the planet is set to 0 so the restricted 3-body problem is a good approximation.

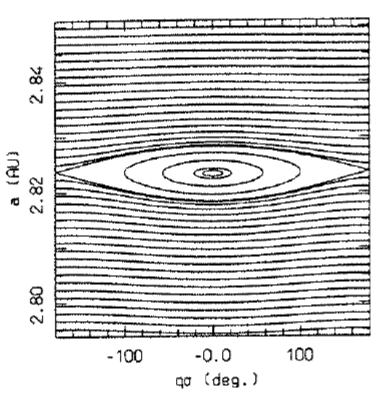
If the eccentricity of Jupiter is set to its average value, 0.05, the value of Ψ_r is not constant anymore. Additional perturbative terms come into play and the evolution is more complex.

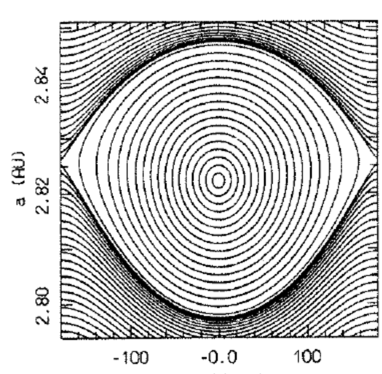




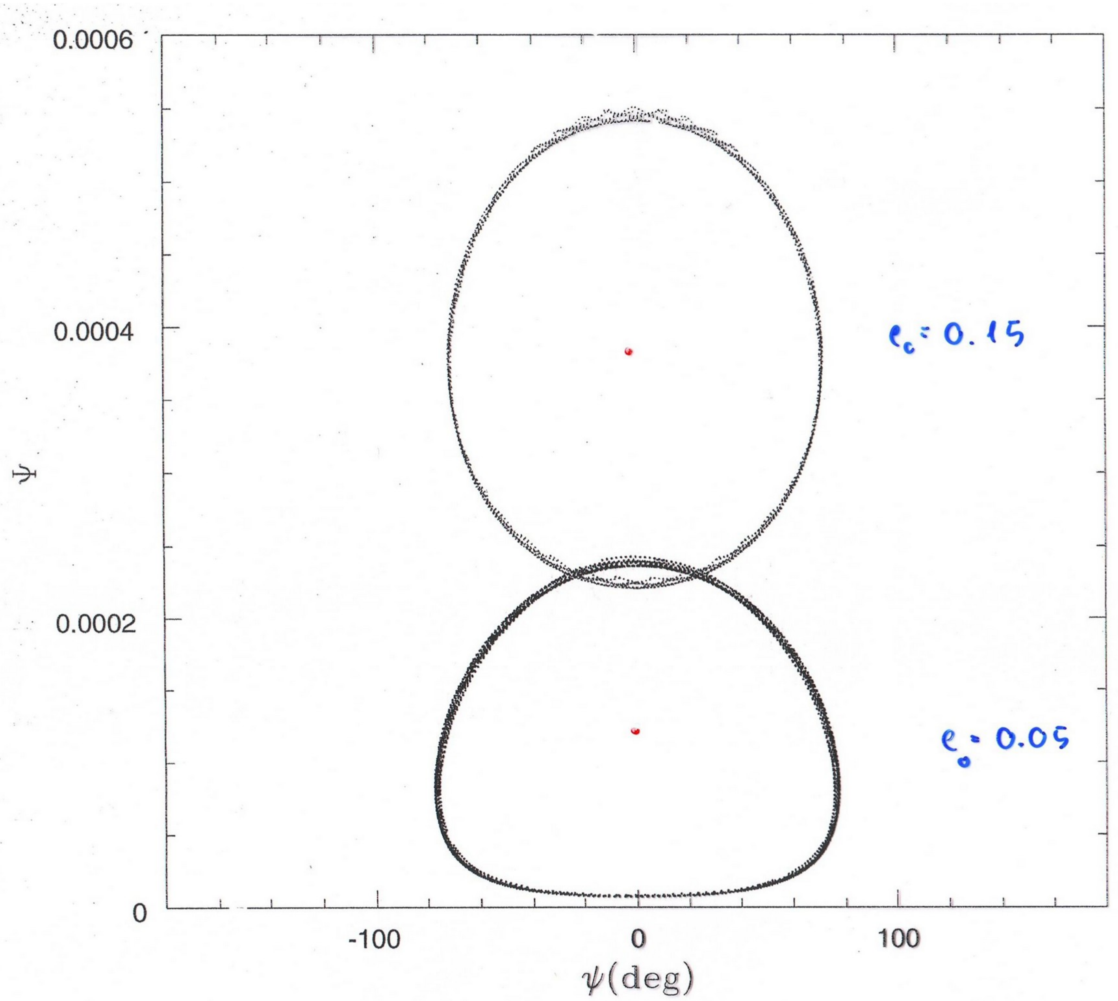








Widening of the resonant region due to an increased eccentricity of Jupiter (from Morbidelli's book "Modern Celestial Mechanics).



Ψ_{res} depend on the eccentricity of the minor body i.e. $(1\text{-}e^2)^{1/2}$ and the libration center shifts depending on the minor body eccentricity.

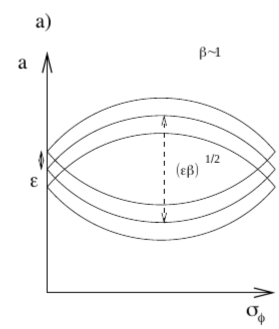
Restricted 3.body problem with inclined orbits.

$$R(L.G,H,l,g,h,t)=$$
$$=G m_1 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{j'=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} K^{i,j}(G,L)$$
$$\cos\left(il+i'l'+j\tilde{\omega}+j'\tilde{\omega}'+m\Omega+m'\Omega'\right)$$

Example: the 3:1 resonance:

$$\cos(3l-l'-2\tilde{\omega})$$
$$\cos(3l-l'-2\tilde{\omega}')$$
$$\cos(3l-l'-2\Omega)$$
$$\cos(3l-l'-2\Omega')$$
......

Any of this term has a slightly different resonance center (Ψ_r)



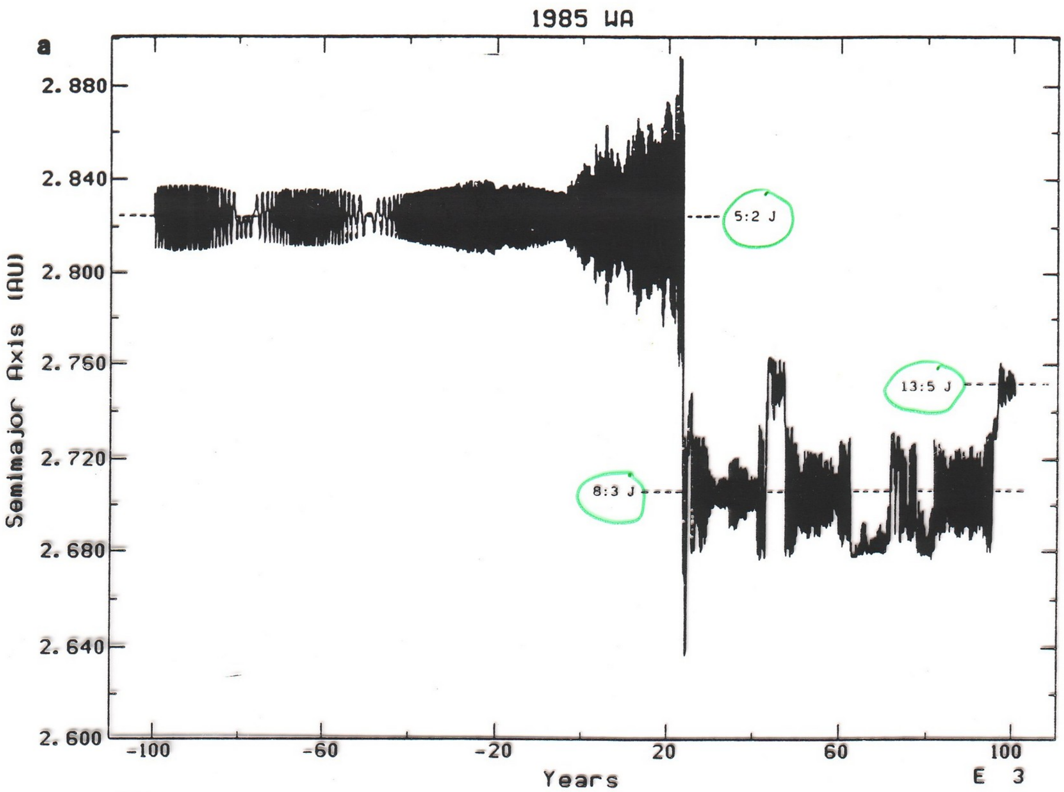
At pg. 261 of Murray & Dermott, Solar System dynamics, there is the disturbing function of the 3:1 resonance. The A coefficients depend only on the semi-major axes and masses.

$$\begin{aligned}\frac{a'}{\mu'}\langle\mathcal{R}\rangle &= A_0 + A_1 e^2 + A_2 s^2 + A_3 e e' \cos(\varpi' - \varpi) + A_4 s s' \cos(\Omega' - \Omega) \\ &+ A_5 e^2 \cos(3\lambda' - \lambda - 2\varpi) + A_6 e e' \cos(3\lambda' - \lambda - \varpi' - \varpi) \\ &+ A_7 e'^2 \cos(3\lambda' - \lambda - 2\varpi') + A_8 s^2 \cos(3\lambda' - \lambda - 2\Omega) \\ &+ A_9 s s' \cos(3\lambda' - \lambda - \Omega' - \Omega) + A_{10} s'^2 \cos(3\lambda' - \lambda - 2\Omega'), \qquad (6.194)\end{aligned}$$

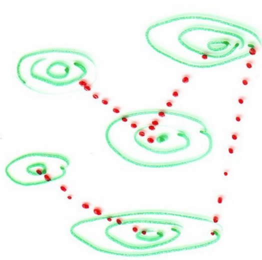
Origin of chaos (Chiricov)

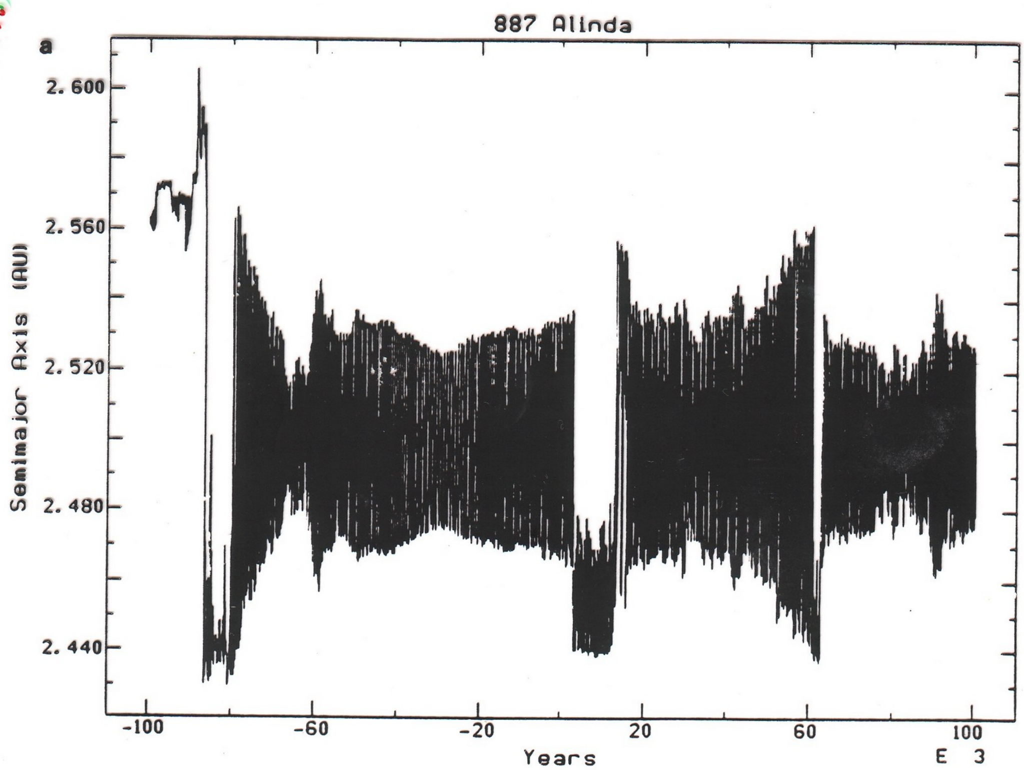
- Superposition of mean motion resonances and secular resonances (resonance with the fundamental frequencies of the N-body planetary system)
- Superposition of different terms of the same resonance.
- Superposition of different MMRs like close to a planet (external region of the asteroid belt near Jupiter).

NEO asteroids perturbed by resonances with Jupiter and Earth.



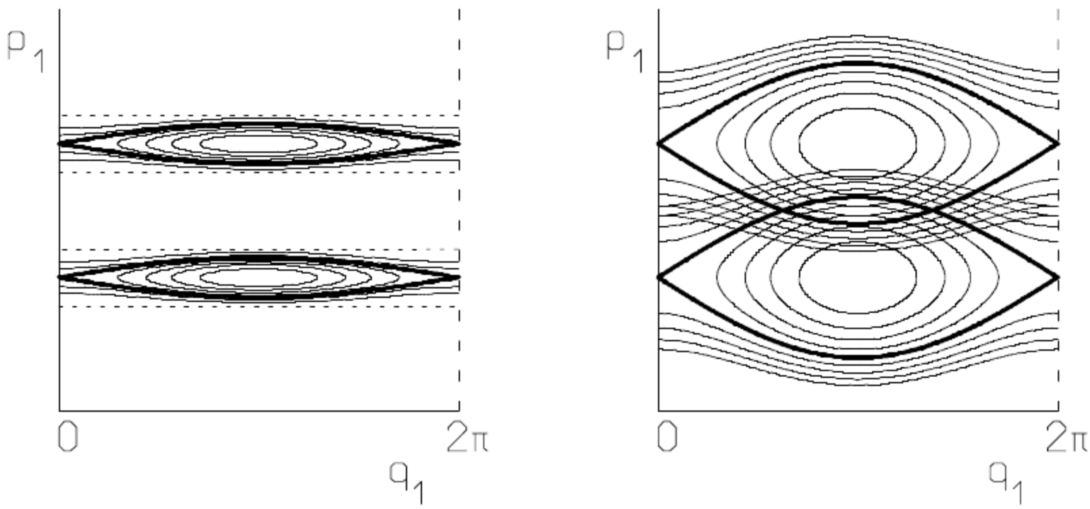
1) Different MMRs

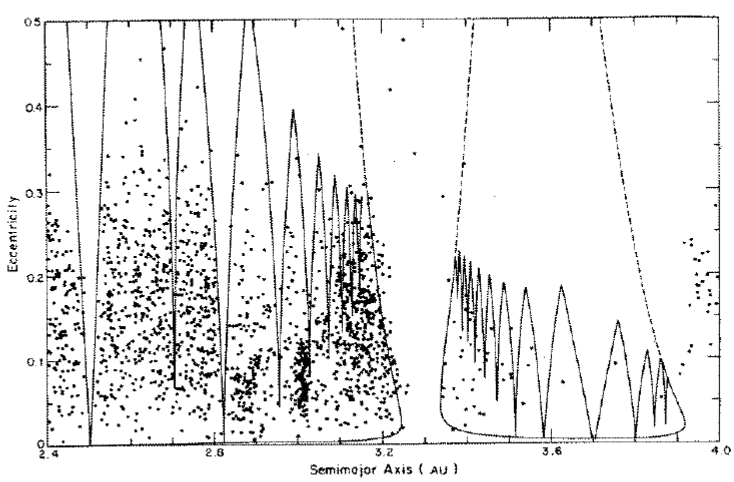




2) Different terms of the 3:1 MMR

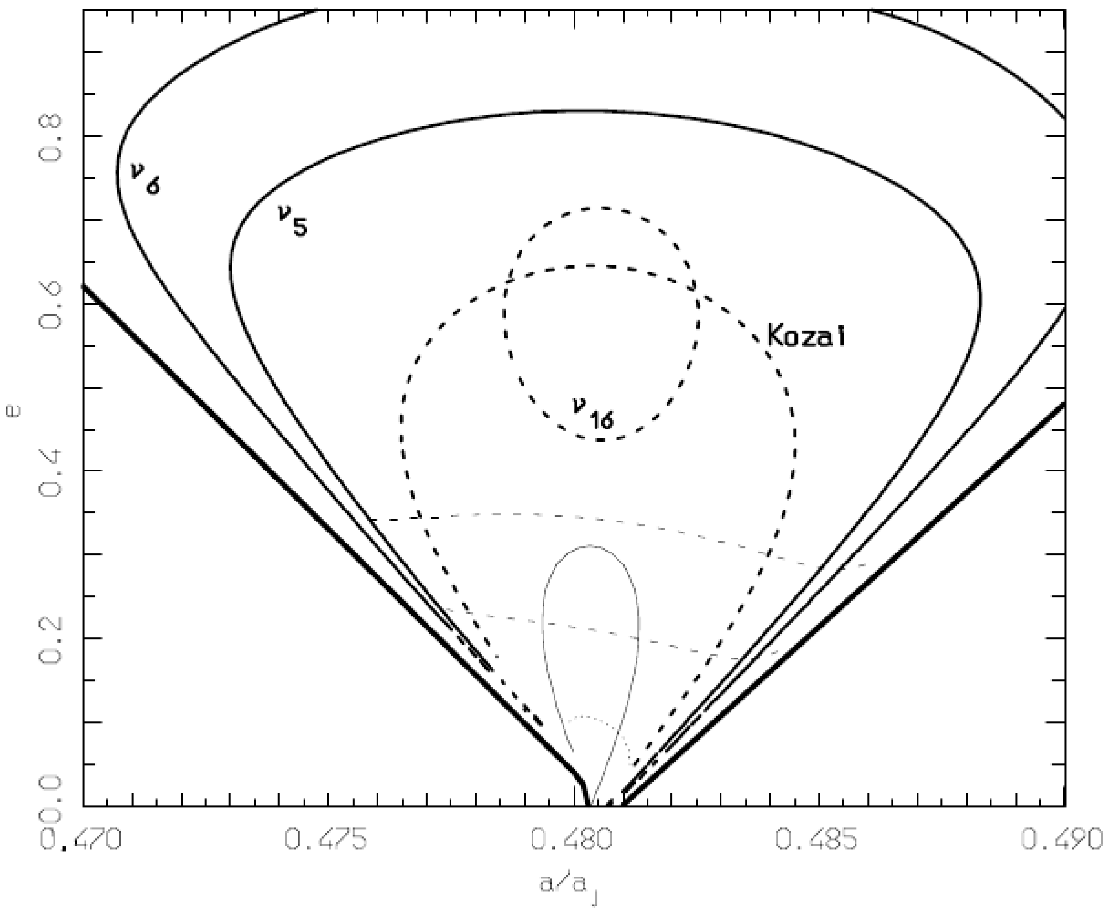
Superposition of MMR and the asteroid belt.





Taken from Morbidelli's book

Secular resonances within the 3:1 MMR. The secular resonances are with the f5 and f6 fundamental frequencies of the solar system. There is also a Kozai-type resonance.



The situation is more complex when two massive bodies are in resonance (planets). For first order resonances:

$$H = -\frac{G M m_1}{2 a_1} - \frac{G M m_2}{2 a_2} - \frac{G m_1 m_2}{a_2}\left(f_{res}^1 e_1 \cos(k\lambda_2 - (k-1)\lambda_1 - \tilde{\omega}_1) + f_{res}^2 e_2 \cos(k\lambda_2 - (k-1)\lambda_1 - \tilde{\omega}_2)\right)$$

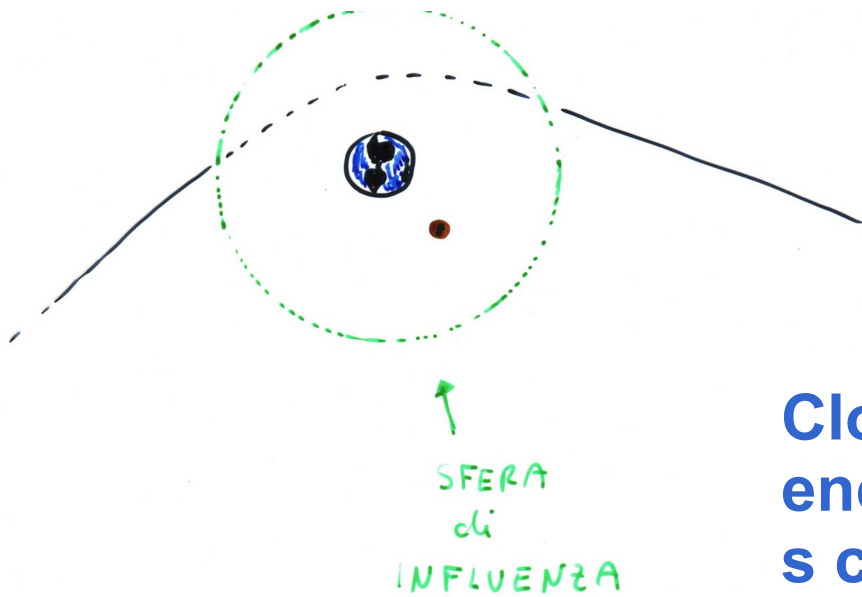
There are 2 potentially librating critical angles. With the Poincarè action-angle variables defined as:

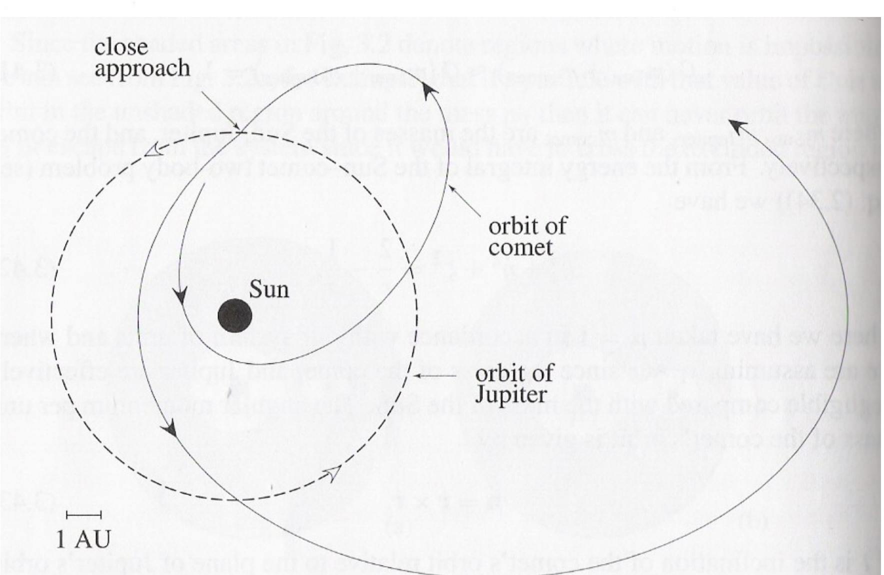
$$\Lambda = \mu\sqrt{G(M+m)a} \qquad \lambda = M + \tilde{\omega}$$
$$\mathrm{T} = \Lambda\left(1 - \sqrt{1-e^2}\right) \qquad \gamma = -\tilde{\omega}$$

$$H_K = -\frac{G^2 M^2 m_1^3}{2\Lambda_1} - \frac{G^2 M^2 m_2^3}{2\Lambda_2}$$
$$H_r = -\frac{G^2 M m_1 m_2^3}{\Lambda_2^{3/2}}\left(f_{res}^1 \sqrt{\frac{2\mathrm{T}_1}{\Lambda_1}} \cos(k\lambda_2 - (k-1)\lambda_1 + \gamma_1) + f_{res}^2 \sqrt{\frac{2\mathrm{T}_2}{\Lambda_2}} \cos(k\lambda_2 - (k-1)\lambda_1 + \gamma_2)\right)$$

Which can be transformed to an integrable single degree of freedom Hamiltonian (see Batygin & Morbidelli, A&A 556, A28, 2013).

Additional source of chaos for NEO asteroids and Short Period Comets are close encounters with planets (terrestrial planets for NEOs, Jupiter for SPC).





Close encounters cause large impulsive variations of the orbital elements which cumulate.

Close encounters can be simple parabolic encounters or more complex if there is a quasi-satellite capture.

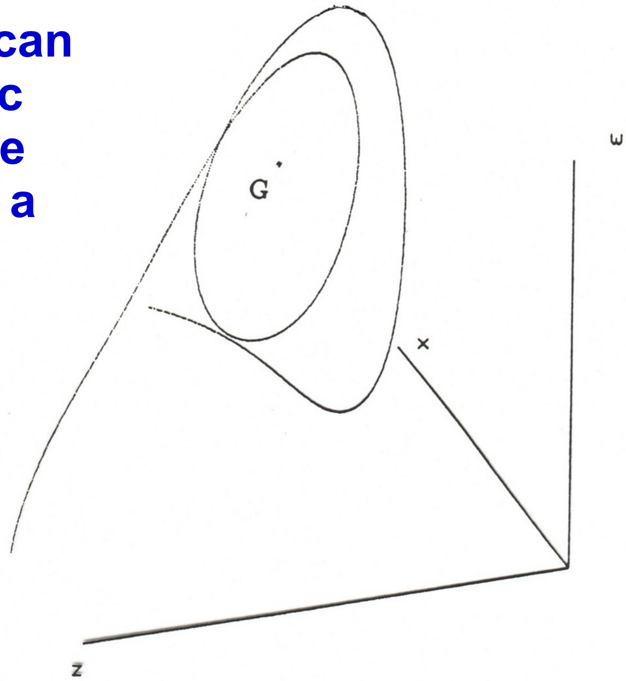


Fig. 6: Traiettoria dell'asteroide 1980 VA rispetto a Giove (punto G) durante un close appproach. Il piano x, z é il piano orbitale del pianeta. Si nota la temporanea cattura in orbita attorno a Giove.

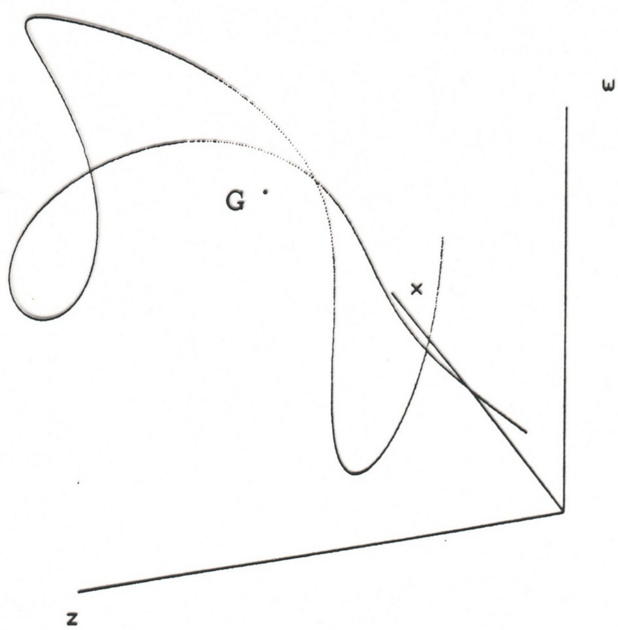
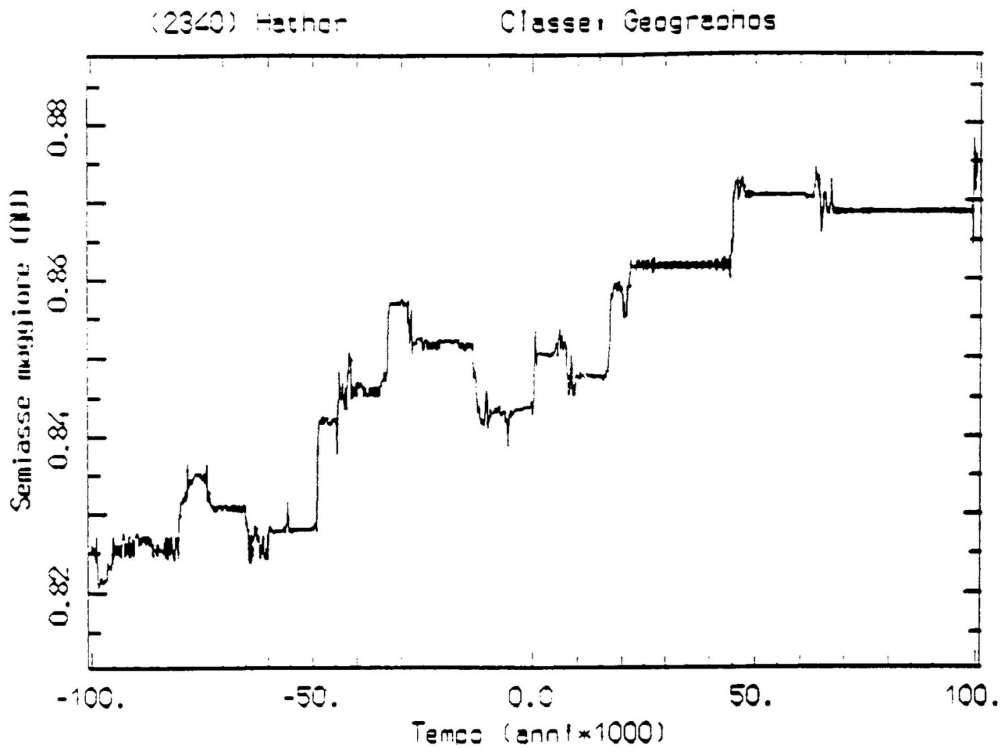
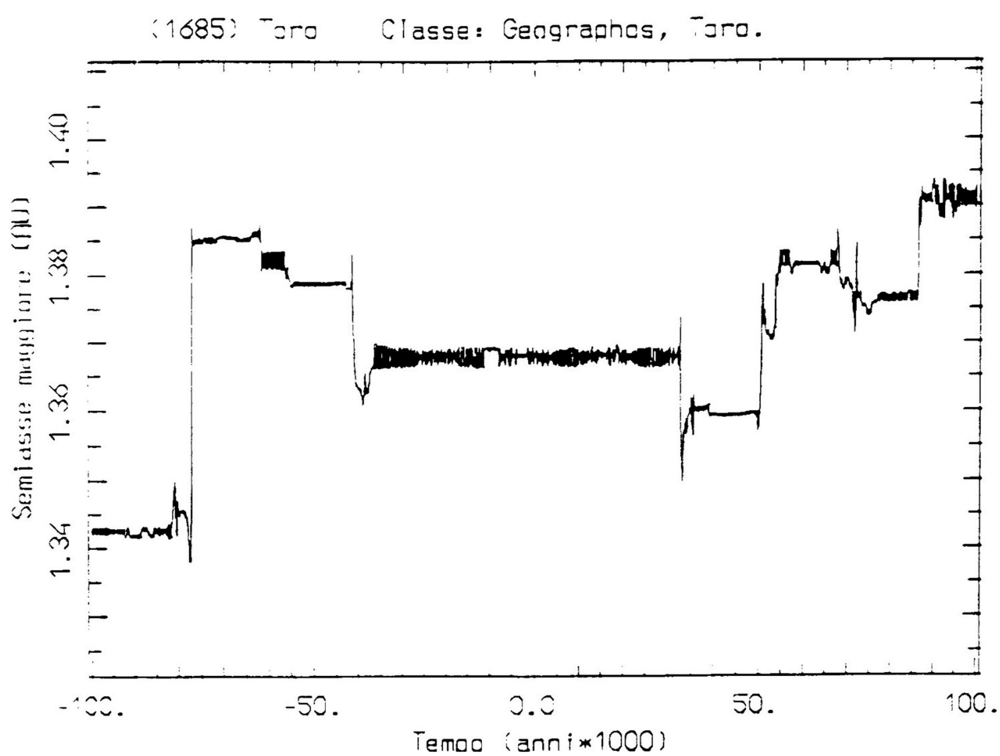


Fig. 7: Close approach di difficile interpretazione tra 1980 VA e Giove.

Example of NEO orbits perturbed by resonances and close encounters. The evolution is highly chaotic and unpredictable on a short timescale.





Jupiter Family Comet orbital evolution. The repeated encounters with Jupiter eject the comet out of the solar system on a hyperbolic orbit on a timescale of the order of 10^5 year.