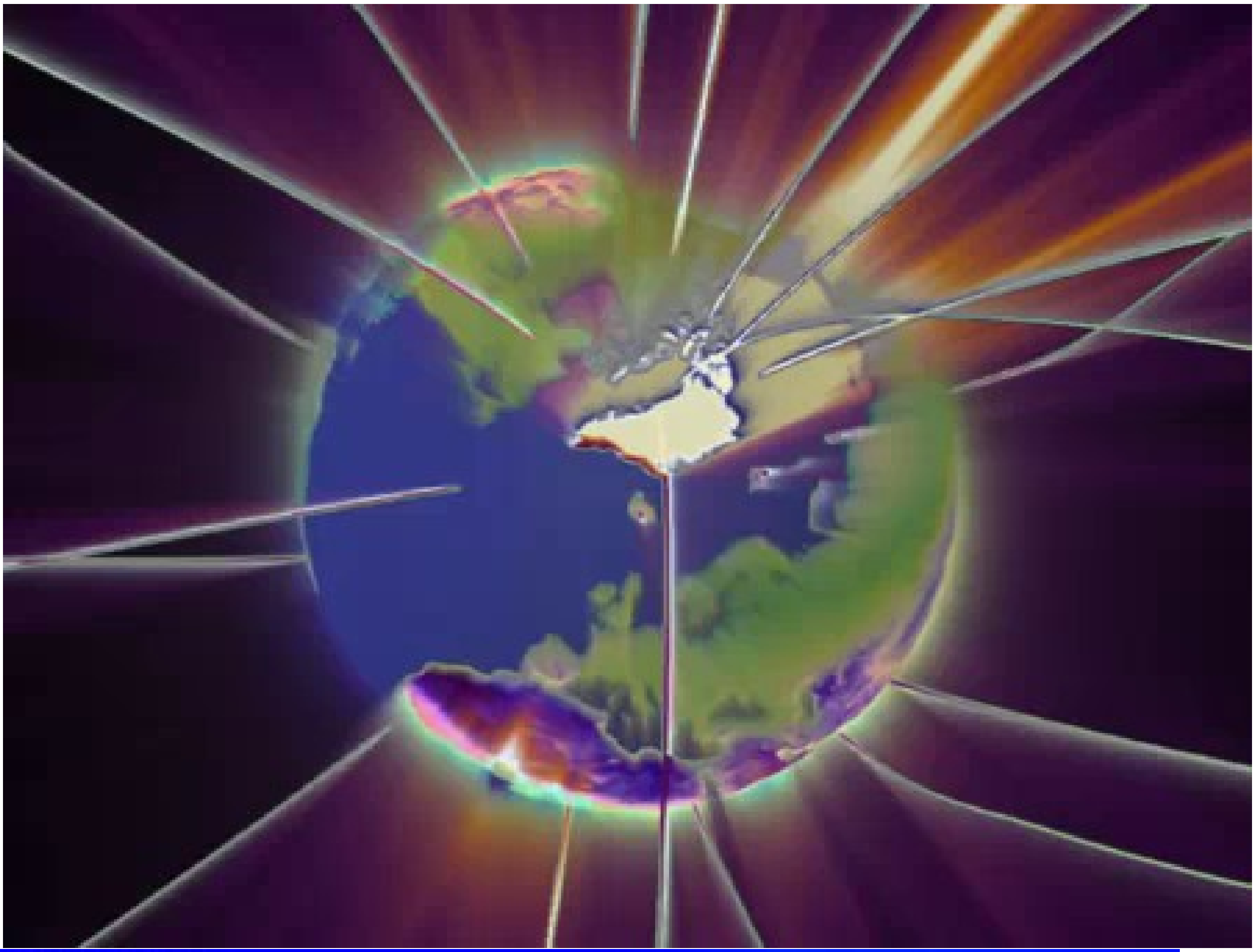
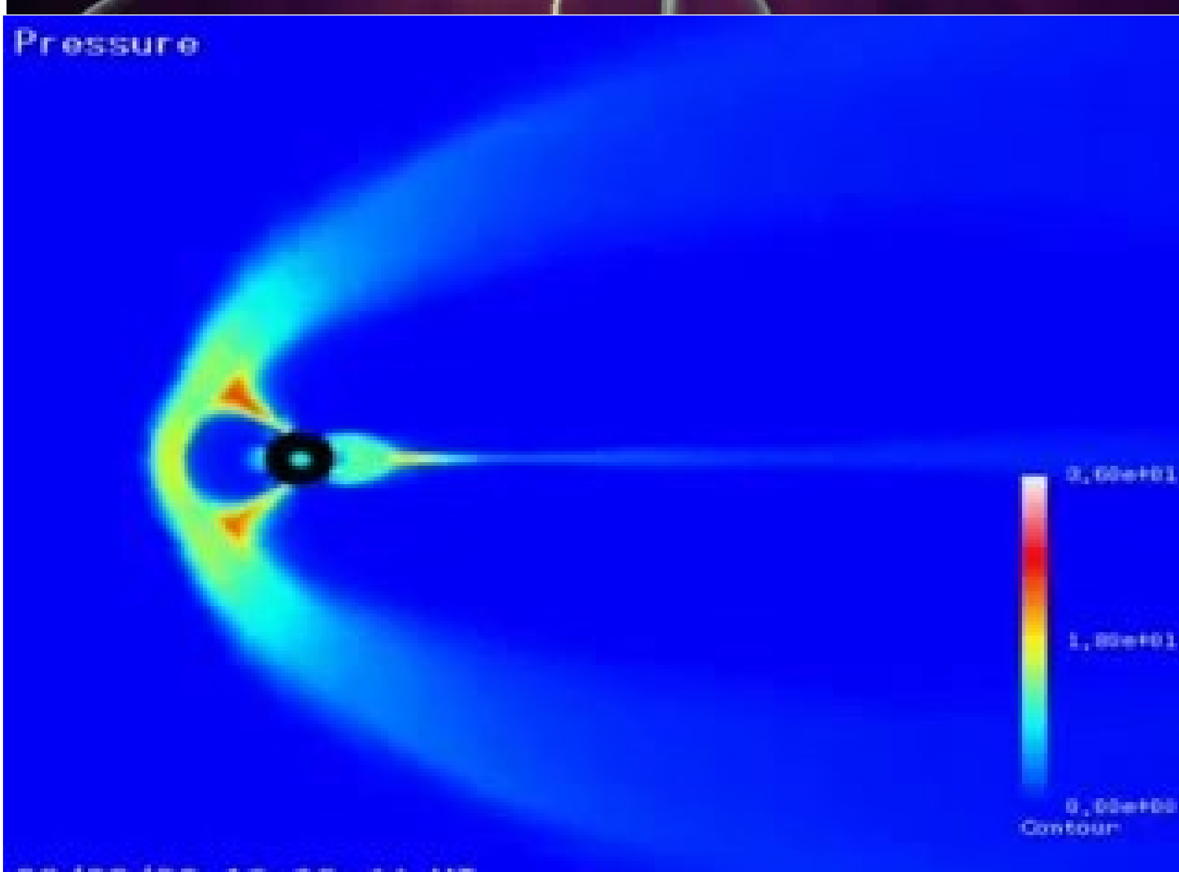
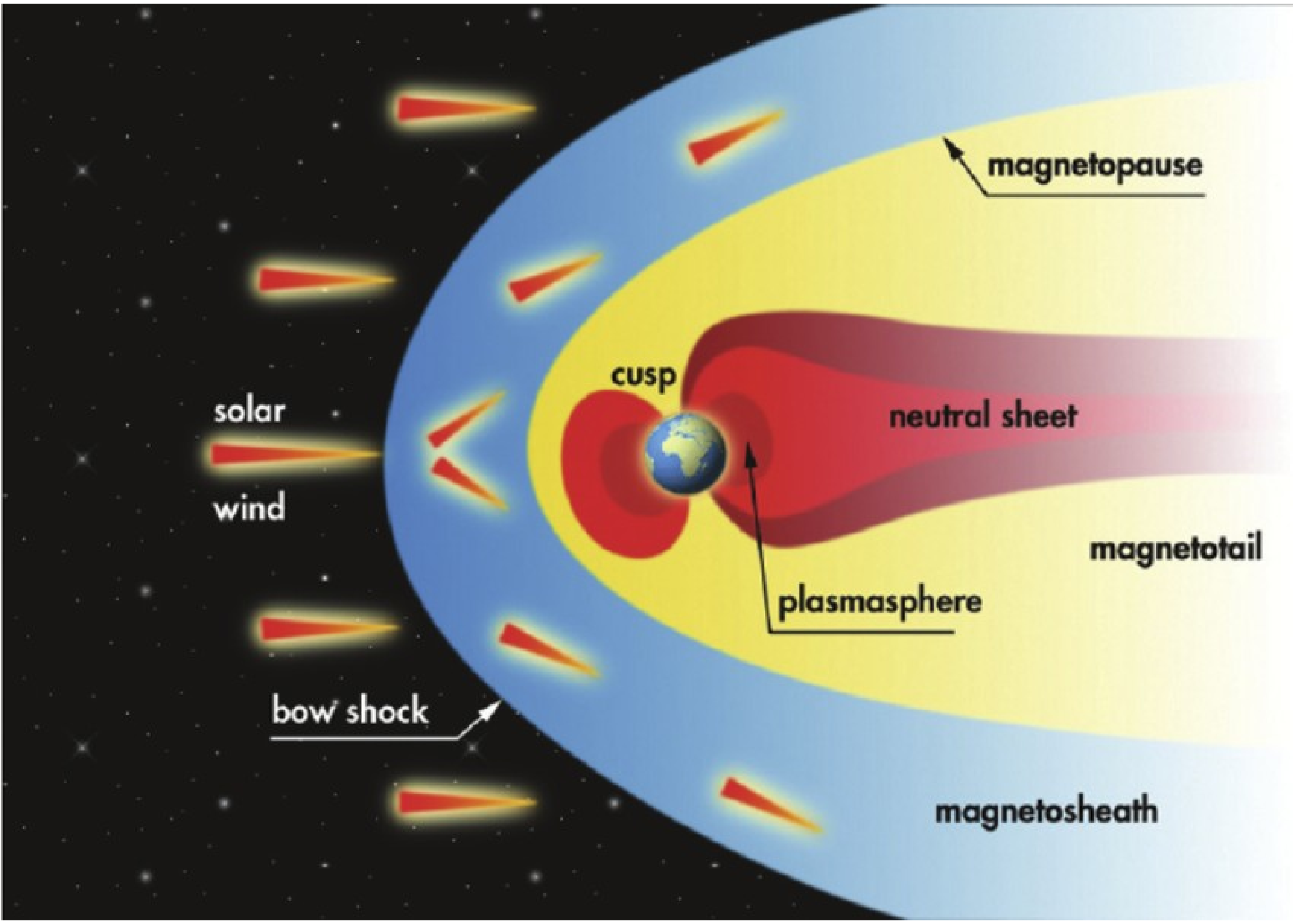
MAGNETOSFERA





MAGNETOSPHERE OF EARTH: estimate of dimensions.



An approximate radius of Earth's magnetosphere is computed from the balancing of the wind pressure and magnetic pressure.

Computation of pressure for a normal gas

$$\Delta p = -2 m v_x$$

$$F = \frac{\Delta p}{\Delta t} = -\frac{2mv}{2L} v_x = -\frac{m}{L} v_x^2 \quad single \;\; particle$$

$$F = N \frac{m}{L} v_x^2 \quad all \;\; particles$$

$$P = N \frac{m}{L^3} \frac{1}{3} v^2 = \frac{1}{3} \rho V^2$$

For the solar wind v ~ vx i.e. the flux is unidirectional (radial).

$$P \sim \rho V^2$$

The magnetic pressure has the same expression as the energy density:

$$B = \frac{1}{2} \frac{\boldsymbol{B}^2}{\mu_0}$$

$$\rho v^2 \sim \frac{1}{2}\frac{\boldsymbol{B}^2}{\mu_0}$$

At the equator

$$B(r) = \frac{M_B}{r^3} = \frac{7.9 \times 10^{25}}{r^3} \quad Gauss$$

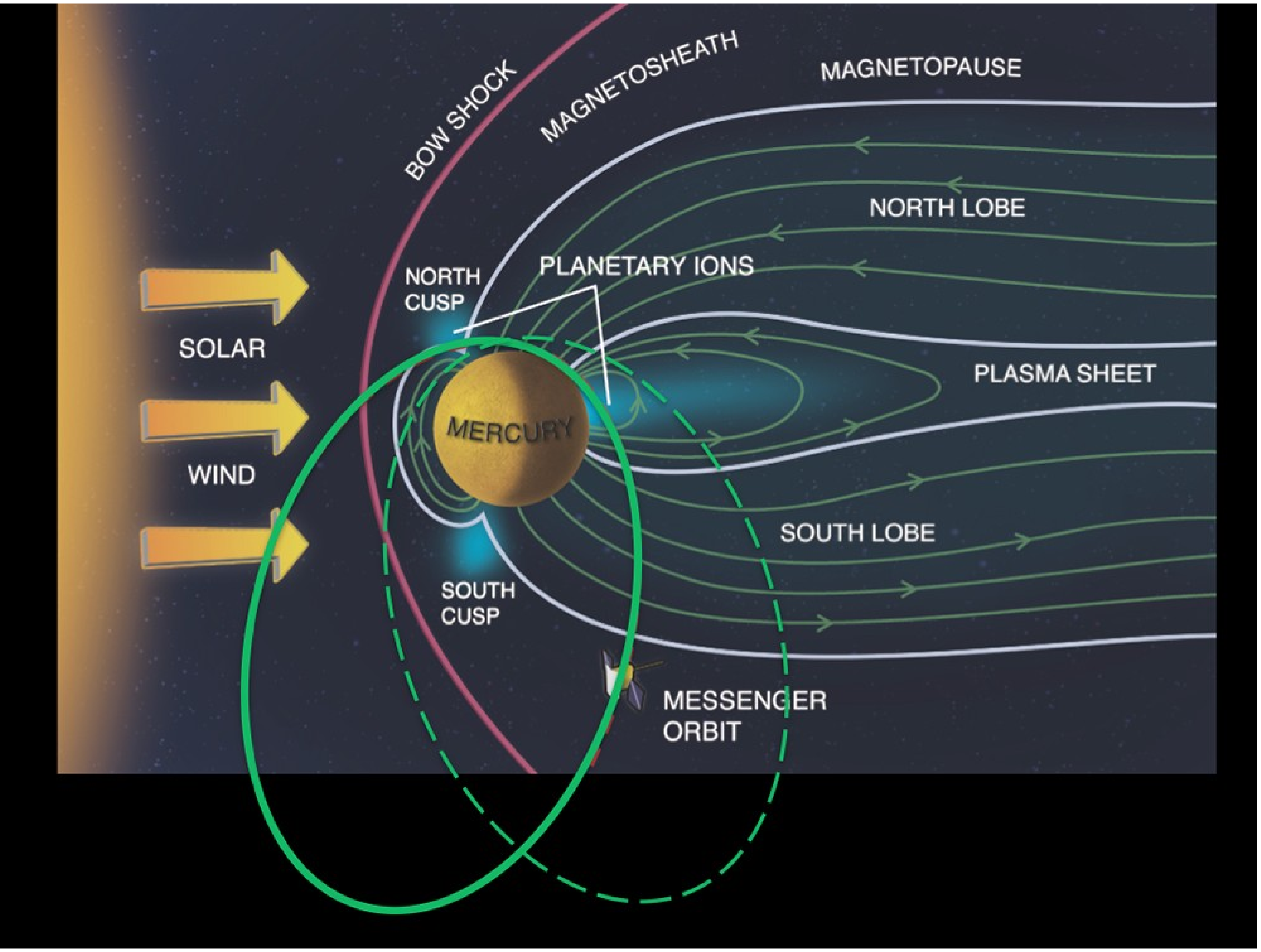
$$\rho v^2 \sim \frac{M_B^2}{2\mu_0 r^6}$$

$$R_M \sim \left(\frac{M_B^2}{2\mu_0 \rho v^2}\right)^{1/6}$$

$$for \quad \rho \sim 5 \quad p^+/cm^3 \qquad v \sim 300 \quad km/s \;\Rightarrow$$

$$R_M \sim 10 \quad R_E$$

Mercury’s magnetic field is 20% off the center of the planet. It is also strongly affected by the solar wind.



MHD: Ideal magnetohydrodynamics

Euler equations:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \boldsymbol{u}) = 0$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\,\boldsymbol{u}\right) = -\nabla P + \rho \boldsymbol{f} + \boldsymbol{j}\times\boldsymbol{B}$$

Lorentz force

Generalized Ohm' law: in the fluid reference frame

$$\boldsymbol{J} = \sigma \boldsymbol{E}'$$

In the fixed reference frame, taking into account that E' = E + v x B

$$\frac{1}{\sigma}\boldsymbol{J} = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$$

In ideal conditions, there is perfect conductivity i.e. $\sigma \rightarrow \infty$ so that the above equation becomes:

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$$

Inserting this equation into Maxwell's equation

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad\Rightarrow\quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$

Induction equation

From Maxwell's equations we can derive an expression for the current density:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

In non-relativistic conditions the second term can be neglected and we get for the current density

$$\boldsymbol{J} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B}$$

The equations for the ideal MHD are then:

$$\frac{\partial \rho}{\partial t} + \nabla (\rho \boldsymbol{u}) = 0$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \nabla \boldsymbol{u}\right) = -\nabla P + \rho \boldsymbol{f} + \boldsymbol{j} \times \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

$$\boldsymbol{J} = \frac{1}{\mu_0} \nabla \times \boldsymbol{B}$$

$$P = K \rho^{\gamma}$$

Magnetic pressure from MHD

The Lorentz force term can be changed according to the induction equation as:

$$-\boldsymbol{j}\times\boldsymbol{B}=\boldsymbol{B}\times\boldsymbol{J}=\frac{1}{\mu_0}\boldsymbol{B}\times(\nabla\times\boldsymbol{B})$$

Vector calculus identities:

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{C})=\boldsymbol{A}\cdot\nabla\,\boldsymbol{C}-(\boldsymbol{A}\cdot\nabla)\boldsymbol{C}$$

$$\frac{1}{2}\nabla(\boldsymbol{A}\cdot\boldsymbol{A})=\boldsymbol{A}\cdot\nabla\,\boldsymbol{A}$$

$$\frac{1}{\mu_0}\boldsymbol{B}\times(\nabla\times\boldsymbol{B})=\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}-\frac{1}{2\mu_0}\nabla B^2$$

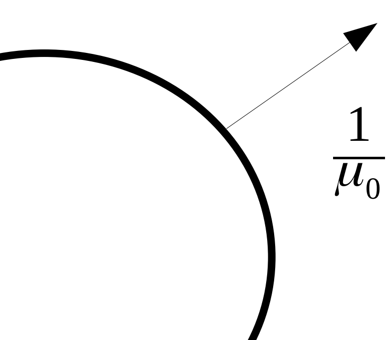
There is a **magnetic pressure** term given by:

$$-\nabla\left(\frac{B^2}{2\mu_0}\right)$$

Which contributes to the plasma kinetic pressure . The total pressure term becomes:

$$-\nabla\left(P+\frac{B^2}{2\mu_0}\right)$$

The additional term is anti-parallel to the curvature radius of the local magnetic field line



$$\frac{1}{\mu_0}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$$

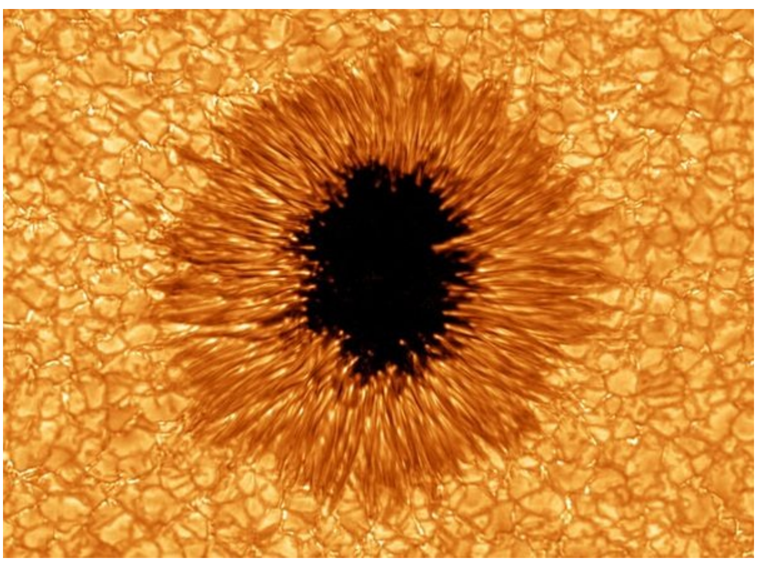
Comparing the relevance of the kinetic pressure with the magnetic one leads to the definition of the coefficient β

$$\beta = \frac{\textit{gas pressure}}{\textit{magnetic pressure}} = \frac{P}{B^2/2\mu_0}$$

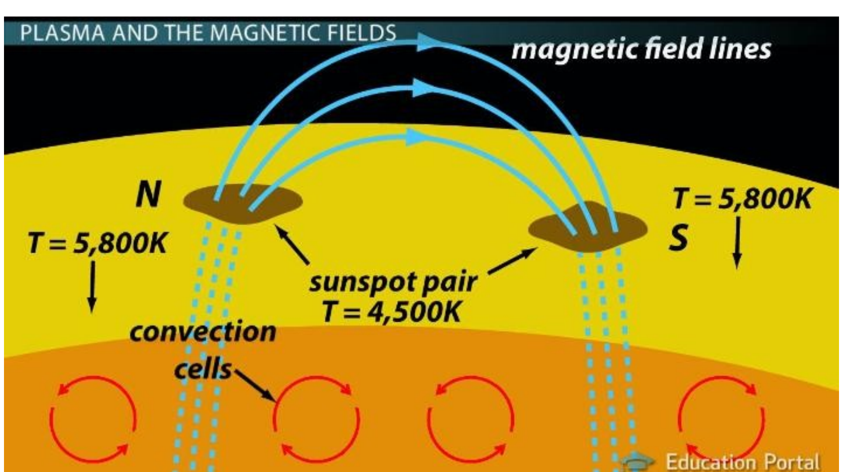
Solar corona: β ~ 3.5 x 10-3

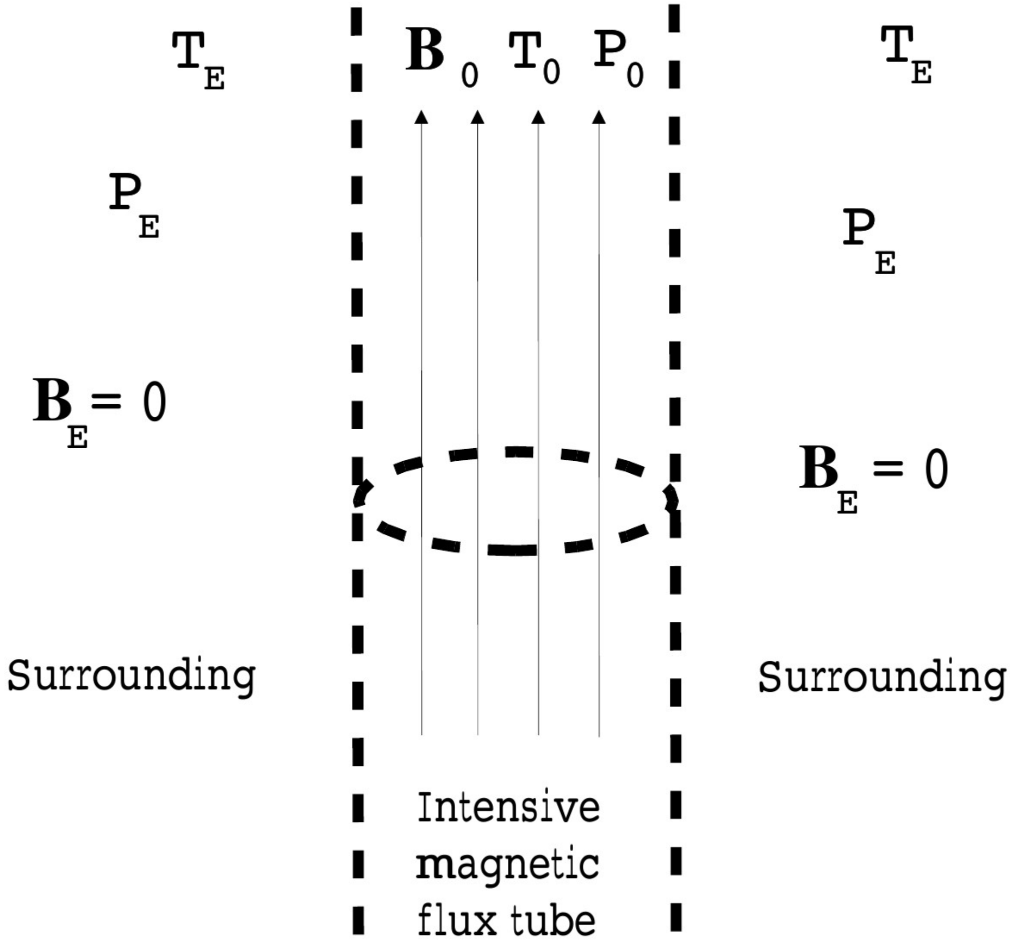
Solar wind (Earth orbit): β ~ 2

Why sunspots have a lower temperature?



Imagine a sunspot as a vertical magnetic flux tube.





Within the flux tube the magnetic field B_0 is vertical. In equilibrium conditions, i.e. the velocity **u** = 0 and also its time derivative:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\nabla\,\boldsymbol{u}\right)=0$$

So that:

$$\frac{1}{\mu_0}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}-\boldsymbol{\nabla}\left(P+\frac{B^2}{2\mu_0}\right)=0$$

Since the magnetic field is constant and vertical the *tension* term is = 0

$$\frac{1}{\mu_0}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}=0$$

$$\nabla\left(P+\frac{B^2}{2\mu_0}\right)=0$$

Which implies that this term is constant and has the same value inside and outside the flux tube.

$$P_E+\frac{B_E^2}{2\mu_0}=P_0+\frac{B_0^2}{2\mu_0} \quad \Rightarrow \quad P_E=P_0+\frac{B_0^2}{2\mu_0}$$

If also the density is equal inside and outside the tube and we recall the state equations:

$$P_E=\frac{\rho_E K_B T_E}{m_E} \qquad P_0=\frac{\rho_0 K_B T_0}{m_E}$$

$$\frac{T_0}{T_E}=1-\frac{B_0^2}{2\mu_0 P_E} \quad \Rightarrow \quad T_E>T_0$$

In the sun, ad example, T_0 ~ 3700 K while T_E ~ 5700 K

Non-ideal MHD, the magnetic diffusion:

$$\frac{1}{\sigma}\boldsymbol{J}=\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B} \qquad \nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \qquad \boldsymbol{J}=\frac{1}{\mu_0}\nabla\times\boldsymbol{B}$$

$$\nabla\times\boldsymbol{E}=-\frac{\partial\boldsymbol{B}}{\partial t} \quad\Rightarrow\quad \frac{\partial\boldsymbol{B}}{\partial t}=-\rho\nabla\times\boldsymbol{J}+\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$

$$\frac{\partial\boldsymbol{B}}{\partial t}=-\rho\nabla\times(\frac{1}{\mu_0}\nabla\times\boldsymbol{B})+\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$

$$\frac{\partial\boldsymbol{B}}{\partial t}=\frac{\rho}{\mu_0}\nabla^2 B-\frac{\rho}{\mu_0}\nabla(\nabla\cdot\boldsymbol{B})+\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$

This is =0 because of Maxwell’s equation $\nabla\cdot\boldsymbol{B}=0$

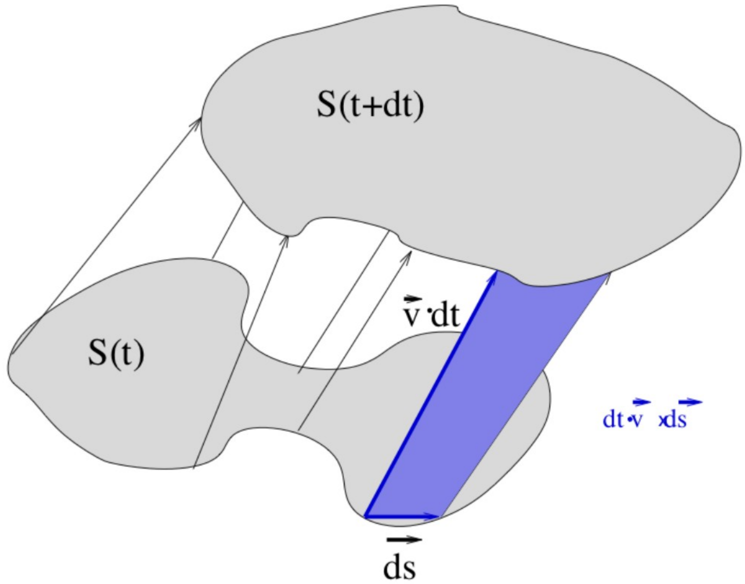
$$\frac{\partial\boldsymbol{B}}{\partial t}=\eta\nabla^2 B+\nabla\times(\boldsymbol{v}\times\boldsymbol{B})$$

where η is the diffusion coefficient.

Alfven's theorem and freezing of magnetic field lines

In a perfectly conducting plasma (MHD) field lines move with the plasma flow.

The curve c encloses the surface S which moves with the plasma. In the time interval *dt* an element of c which is determined by the vector ***ds*** sweeps an area (blue in the fig) equal to



$$Area = (\mathbf{v}\,dt)\times \mathbf{ds}$$

The flux through S is given by $\iint_S \mathbf{B}\cdot\mathbf{dS}$ where **dS** is the surface element and **ds** the line element.

and its change with time is given by $\frac{d}{dt}\left(\iint_S \mathbf{B}\cdot\mathbf{ds}\right)$

The flux element exiting (or entering) through the blue area is

$$\mathbf{B}\cdot((\mathbf{v}\,dt)\times\mathbf{ds}) = -((\mathbf{v}\,dt)\times\mathbf{B})\cdot\mathbf{ds}$$

The total flux is then $-\int_c ((\mathbf{v}\,dt)\times\mathbf{B})\cdot\mathbf{ds}$

The change in the flux is due to 1) a change in time of B or 2) to a motion of the boundary of the surface S, and then curve c. As a consequence, we may write:

$$\frac{d}{dt}\left(\iint_S \boldsymbol{B}\cdot\boldsymbol{dS}\right)=\iint_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot\boldsymbol{dS}-\int_c \boldsymbol{v}\times\boldsymbol{B}\cdot\boldsymbol{ds}$$

Using the Stokes' theorem, the second term can be transformed in a surface integral

$$\frac{d}{dt}\left(\iint_S \boldsymbol{B}\cdot\boldsymbol{dS}\right)=\iint_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot\boldsymbol{dS}-\iint_S \nabla\times(\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{dS}$$

Combining the two integrals and reminding the induction equation we get:

$$\frac{d}{dt}\left(\iint_S \boldsymbol{B}\cdot\boldsymbol{dS}\right)=\iint_S \left(\frac{\partial \boldsymbol{B}}{\partial t}-\nabla\times(\boldsymbol{v}\times\boldsymbol{B})\right)\cdot\boldsymbol{dS}=0$$

The magnetic flux through a closed circuit does not change if the circuit moves with the plasma, so B is frozen on the plasma.

Consideriamo ora $u_{\parallel} = \bar{u} \cdot \hat{b}$

$u_{\perp} = \| \bar{u} - u_{\parallel} \hat{b} \|$

$$m \dot{\bar{u}} \cdot \hat{b} = m \dot{u}_{\parallel} = q \left(\hat{b} \cdot \hat{b} \, (\bar{E} \cdot \hat{b}) + \underbrace{(\bar{u} \times \bar{B}) \cdot \hat{b}}_{=0} \right) =$$

$$= q E_{\parallel}$$

ma $u_{\parallel} = v_{\parallel}$ perché

$$\bar{u} \cdot \hat{b} = \bar{v} \cdot \hat{b} - \overbrace{\left(\frac{\bar{E} \times \bar{B}}{B^2} \right) \cdot \hat{b}}^{=0}$$

quindi $\boxed{v_{\parallel} = \frac{q}{m} E_{\parallel} t + v_{\parallel 0}}$

Ora: per ottenere $u_{\perp}$ si sottrae a $\dot{\bar{u}}$ $\dot{u}_{\parallel}$ lungo $\hat{b}$ =>

$$m \dot{\bar{u}} - m \dot{u}_{\parallel} \hat{b} = q \left(\hat{b} \, (\bar{E} \cdot \hat{b}) + \bar{u} \times \bar{B} - q E_{\parallel} \hat{b} \right) =$$

$$= q (\bar{u} \times \bar{B}) = q \, \bar{u}_{\perp} \times \bar{B}$$

$$m \dot{\bar{u}}_{\perp} = q \, \bar{u}_{\perp} \times \bar{B}$$

Nel Sd.R in moto con $\bar{v}_{d} = \frac{\bar{E} \times \bar{B}}{...}$ la

3) il guiding center si muove lungo le linee di campo con $v_{\parallel}$ ($\parallel$ a $\bar{B}$)

Allora, nel SdR originale il guiding center si muove con velocità

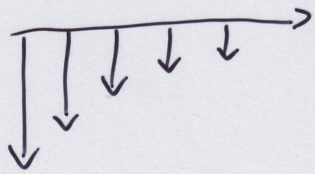
$$\bar{v}_{gc} = v_{\parallel} \hat{b} + \frac{\bar{E} \times \bar{B}}{B^2}$$

$$\boxed{V_E = \frac{\bar{E} \times \bar{B}}{B^2}}$$

VELOCITÀ di DRIFT

4)

•) Drift dovuto a $\bar{\nabla} B$



Campo NON omogeneo

(Si assume che $R_L \ll$ scala di variazione di $\bar{B}$)

$$\frac{r_L}{B}|\bar{\nabla} B| \ll 1$$

Si sviluppa la $\bar{v}$ della particella come una serie

$$\bar{v} = \bar{v}_0 + \bar{v}_1 + \bar{v}_2 \dots$$

Termini via via più piccoli.

dove il termine 0 corrisponde al moto giromagnetico. Si assume anche che:

$$\bar{B} = B_{gc}\,\hat{z} + (y - y_{gc})\frac{dB}{dy}\cdot\hat{z}$$

dove y_{gc} è il guiding center della particella all'inizio $(t=0)$

5) Le eq. del moto sono: $\left(m\dot{\bar v} = q\bar v \times \bar B\right)$

$$m\dot v_x = q v_y \left[B_{gc} + (y - y_{gc})\frac{dB}{dy}\right]$$

$$m\dot v_y = -q v_x \left[B_{gc} + (y - y_{gc})\frac{dB}{dy}\right]$$

Introduciamo nelle equazioni lo sviluppo in serie fino al I° ORDINE (si trascura il II°)

$$m\dot v_{x0} + m\dot v_{x1} = q\left(v_{y0} + v_{y1}\right)\left[B_{gc} + (y_0 + y_1 - y_{gc})\frac{dB}{dy}\right]$$

$$m\dot v_{y0} + m\dot v_{y1} = -q\left(v_{x0} + v_{x1}\right)\left[B_{gc} + (y_0 + y_1 - y_{gc})\frac{dB}{dy}\right]$$

- Assumiamo che v_{x0} e v_{y0} e y_0 corrispondano al moto giromagnetico
- si trascura $y_1 \cdot \frac{dB}{dy}$ perché di II° ordine

  (piccoli rispetto al moto giromagnetico)

SOLUZIONE ORD. 0 => moto giromagnetico

SOLUZIONE 1 => moto di drift

6)

$$m\dot{v}_{x1} = q\, v_{y1} B_{gc} + q\, v_{y0} (y_0 - y_{gc}) \frac{dB}{dy}$$

$$m\dot{v}_{y1} = -q\, v_{x1} B_{gc} - q\, v_{x0} (y_0 - y_{gc}) \frac{dB}{dy}$$

(si trascura $q v_{y1} (y_0 - y_{gc}) \frac{dB}{dy}$ perché II ordine, lo stesso per $q v_{x1} (y_0 - y_{gc}) \frac{dB}{dy}$).

→ Si esegue <u>media</u> delle quantità nelle equazioni su molti periodi di moto giromagnetico.

Ora $m\langle v_{x1}\rangle$ e $m\langle v_{y1}\rangle$ sono quantità piccole (I° ord) e la media $m\langle \dot{v}_{x1}\rangle$ ad esempio rappresenta la piccola variazione di una quantità piccola (II° ord).

In altre parole, la variazione di $\langle v_x\rangle$ rispetto al periodo giromagnetico è piccola.

7) Allora:

$$q\langle v_{y1}\rangle B_{gc} + q\langle v_{y0}(y_0 - y_{gc})\rangle \frac{dB}{dy} = 0$$

$$-q\langle v_{x1}\rangle B_{gc} - q\langle v_{x0}(y_0 - y_{gc})\rangle \frac{dB}{dy} = 0$$

Ora la media di $v_{y0}(y_0 - y_{gc})$ è nulla perché

$$v_{y0} = \pm i v_\perp e^{i(\omega_c t + \delta)}$$

$$y_0 - y_{gc} = \pm \frac{v_\perp}{\omega_c} e^{i(\omega_c t + \delta)}$$

← PARTE REALE

I due termini sono sfasati di 90°, sono oscillanti e quindi la media è 0. $\Rightarrow \langle v_{y1}\rangle = 0$

$$\langle v_{x1}\rangle = -\langle v_{x0}(y_0 - y_{gc})\rangle \frac{dB}{dy}$$

dove $\langle v_{x0}(y_c - y_{gc})\rangle = \frac{v_\perp^2}{\omega} \langle e^{2i(\omega_c t + \delta)}\rangle =$

$$= \frac{v_\perp^2}{2\omega_c}$$

*

- $\langle \mathrm{Re}\left(i v_\perp e^{i(\omega_c t+\delta)}\right) \cdot \mathrm{Re}\left(\frac{v_\perp}{\omega_c} e^{i(\omega_c t+\delta)}\right) \rangle =$

$$= \frac{v_\perp^2}{\omega_c}\langle -\sin\varphi \cos\varphi\rangle = -\frac{v_\perp^2}{\omega_c}\frac{1}{2\pi}\int_0^{2\pi}\sin\varphi\cos\varphi\, d\varphi = 0$$

- $\langle v_{x0}\cdot(y_c - y_{gc})\rangle = \frac{v_\perp^2}{\omega_c}\langle \mathrm{Re}\left(e^{i(\omega_c t+\delta)}\right)\cdot \mathrm{Re}\left(e^{i(\omega_c t+\delta)}\right)\rangle =$

$$= \frac{v_\perp^2}{\omega_c}\frac{1}{2\pi}\int_0^{2\pi}\cos^2\varphi\, d\varphi = \frac{v_\perp^2}{\omega_c}\frac{\pi}{2\pi} = \frac{v_\perp^2}{2\omega_c}$$

8)

$$\langle v_{x_1} \rangle = \frac{v_\perp^2}{2\omega_c} \frac{1}{B_{gc,i}} \frac{dB}{dy}$$

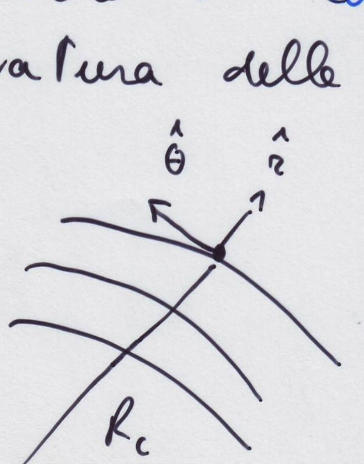
La particella quindi 'drifta' in una direzione ⊥ a y e z quindi dove B è costante e quindi $B_{gc,i} = B_{gc}$

La forma generale è:

$$\bar{v}_{grad} = \frac{v_\perp^2}{2\omega_c} \frac{\bar{B} \times \bar{\nabla} B}{B^2} = \frac{v_\perp^2 m}{2q} \frac{\bar{B} \times \bar{\nabla} B}{B^3}$$

•) DRIFT dovuto alla CURVATURA del CAMPO.

Introduciamo un sistema di coordinate cilindriche che localmente approssimano la curvatura delle linee di campo B



9)

Ad ordine 0, le particelle si muovono lungo le linee di campo (direzione $\hat{\Theta}$) con velocità $v_{\parallel}\hat{b}$ e $v_{\perp}$. Ci si pone in un sistema di Rif. che si muove solidalmente con le linee di campo.

$$\bar{F}_{cf} = \frac{m v_{\parallel}^2 \hat{r}}{R_c} = m v_{\parallel}^2 \frac{\bar{R}_c}{R_c^2}$$

con R_c raggio di curvatura. In presenza di una forza si utilizza:

$$\bar{v}_{curv} = \frac{(\bar{F} \times \bar{B})}{q B^2} = \frac{m v_{\parallel}^2}{q B^2} \frac{\bar{R}_c \times \bar{B}}{R_c^2} = m v_{\parallel}^2 \frac{\bar{B} \times \bar{n}}{R_c q B^2}$$

$$\bar{n} = \frac{\bar{R}_c}{R_c}$$

Ora $\frac{\bar{R}_c}{R_c^2} = (\hat{b} \cdot \bar{\nabla})\hat{b}$

$$\bar{v}_{curv} = \frac{m v_{\parallel}^2}{q B^2} \bar{B} \times [(\hat{b} \cdot \bar{\nabla})\hat{b}]$$