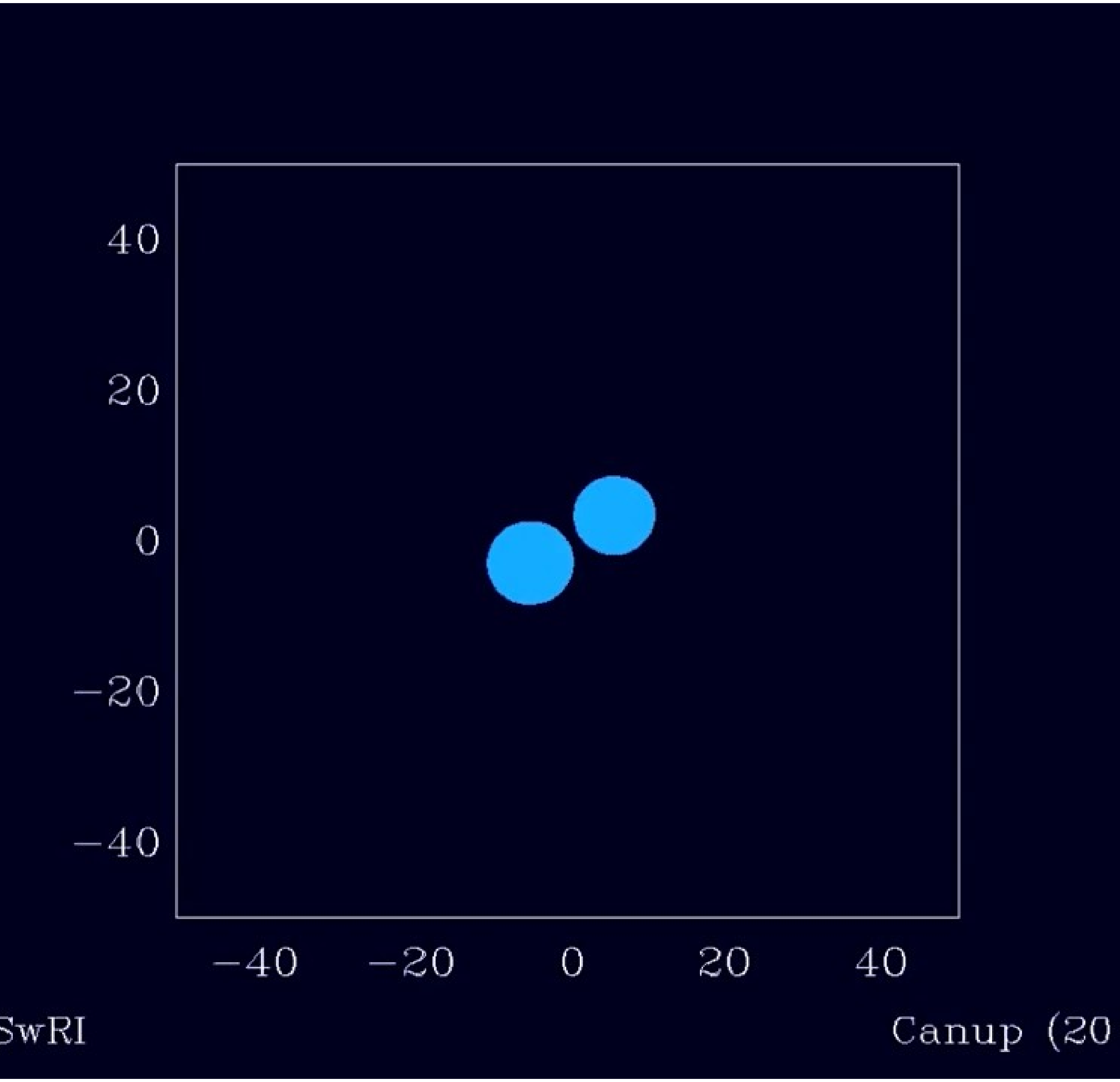
Chapter 5

- **The Moon**
- **Formation of the tidal bulge**
- **Tidal delay**
- **Dynamica evolution of the Earth-Moon system**
- **Tidal evolution of the giant planet satellites.**

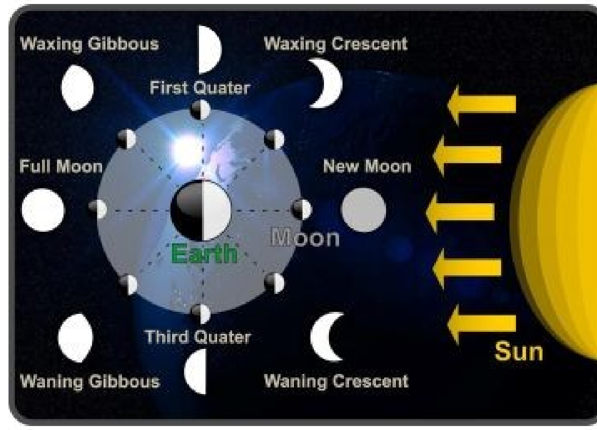
Murray & Dermot, Solar System Dynamics

The giant impact that formed the Moon from the primordial Earth.

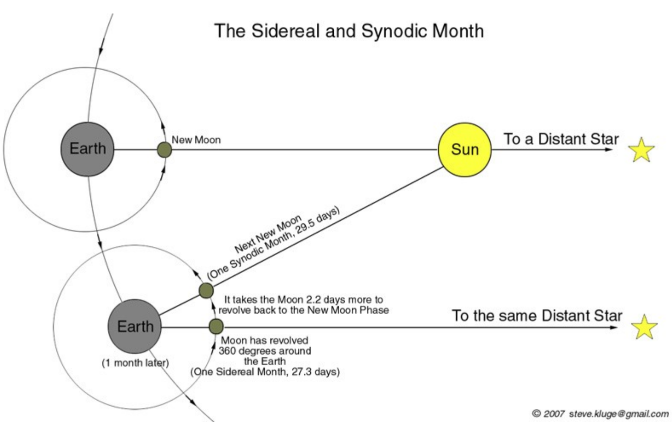


The Moon fact sheet:

- $M = 7.349 \; 10^{22}$ kg ($0.012 \; M_E$)
- $R = 1738$ km
- $\rho = 3400$ kg/m^3
- $a = 384400$ km
- $e \approx 0.055$
- $i \approx 5^o$
- $T_{rev} = 27.32$ days

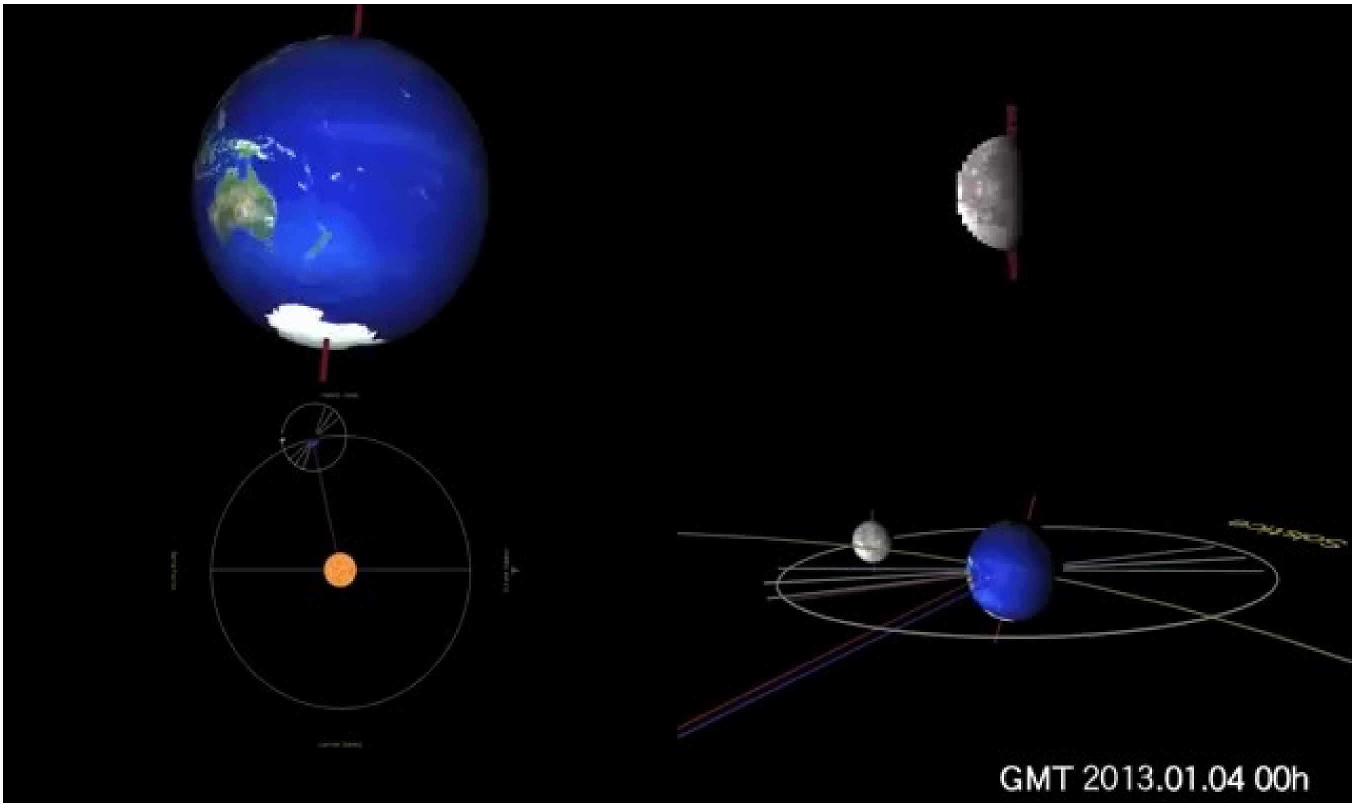


The Moon is in a synchronous rotation state, i.e. the rotation period is equal to the orbital period and it always show the same face to the Earth.



Sidereal month: 27.3 days
Synodic month: 29.5 days.

Solar eclipse: anytime the nodal line of the Moon orbit passes in front of the Earth.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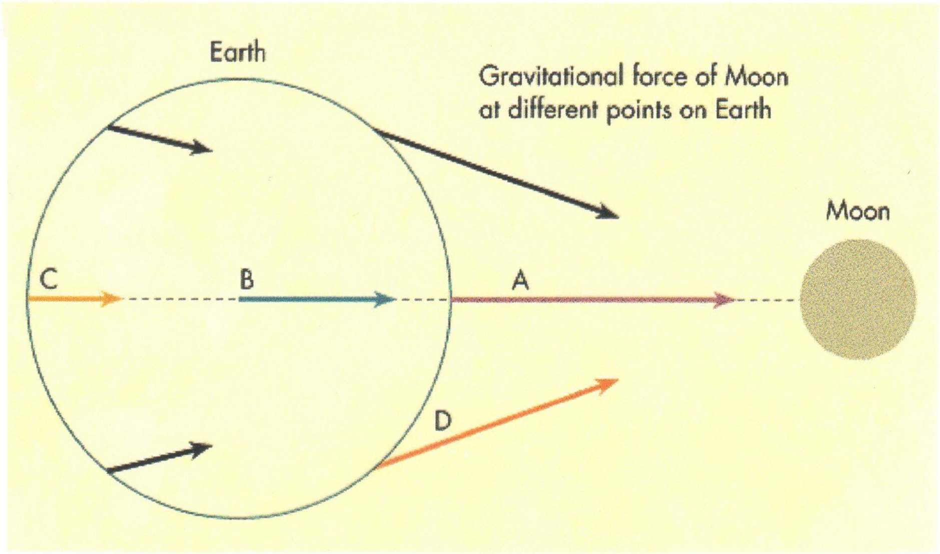
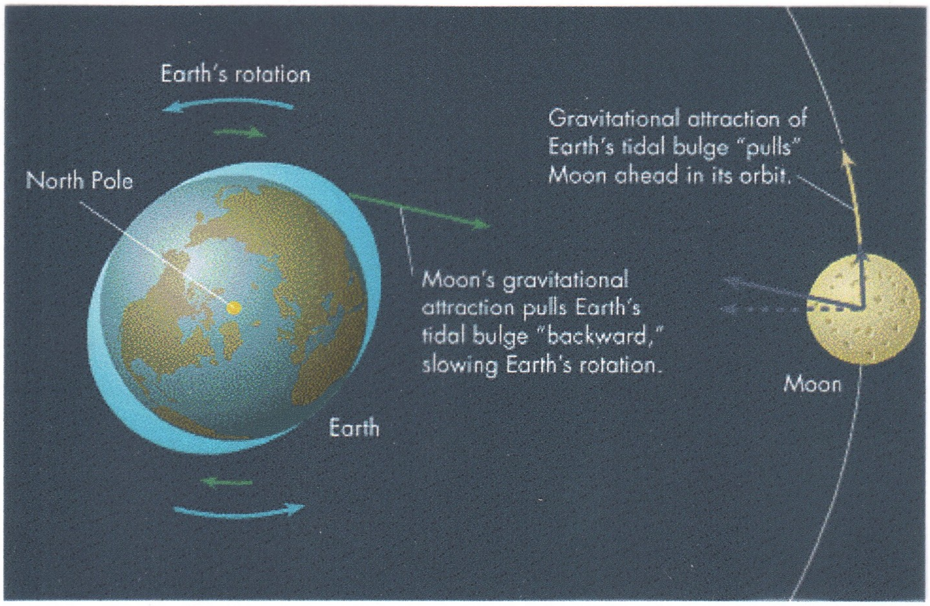
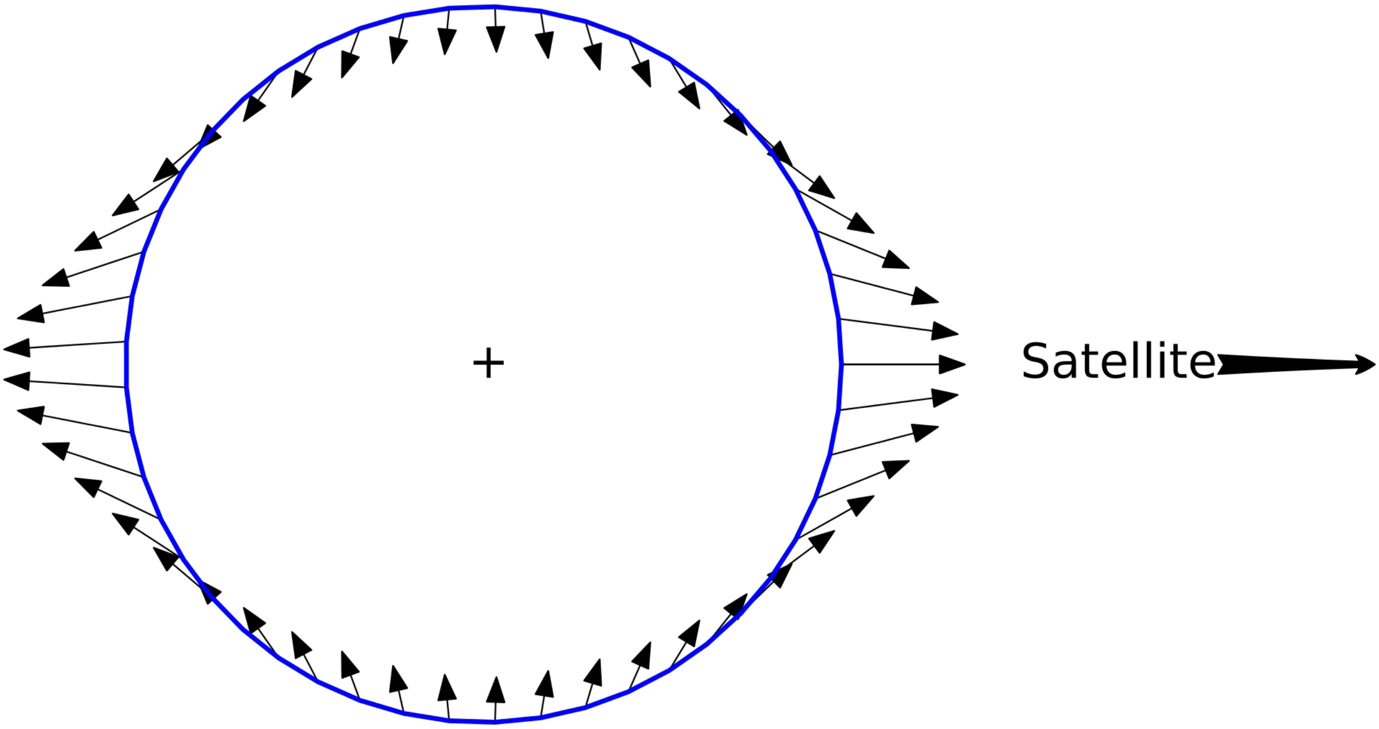
Tidal effects due to the gravity gradient. They occur if the size of the orbits is comparable to the size of the bodies.

Tidal bulge:

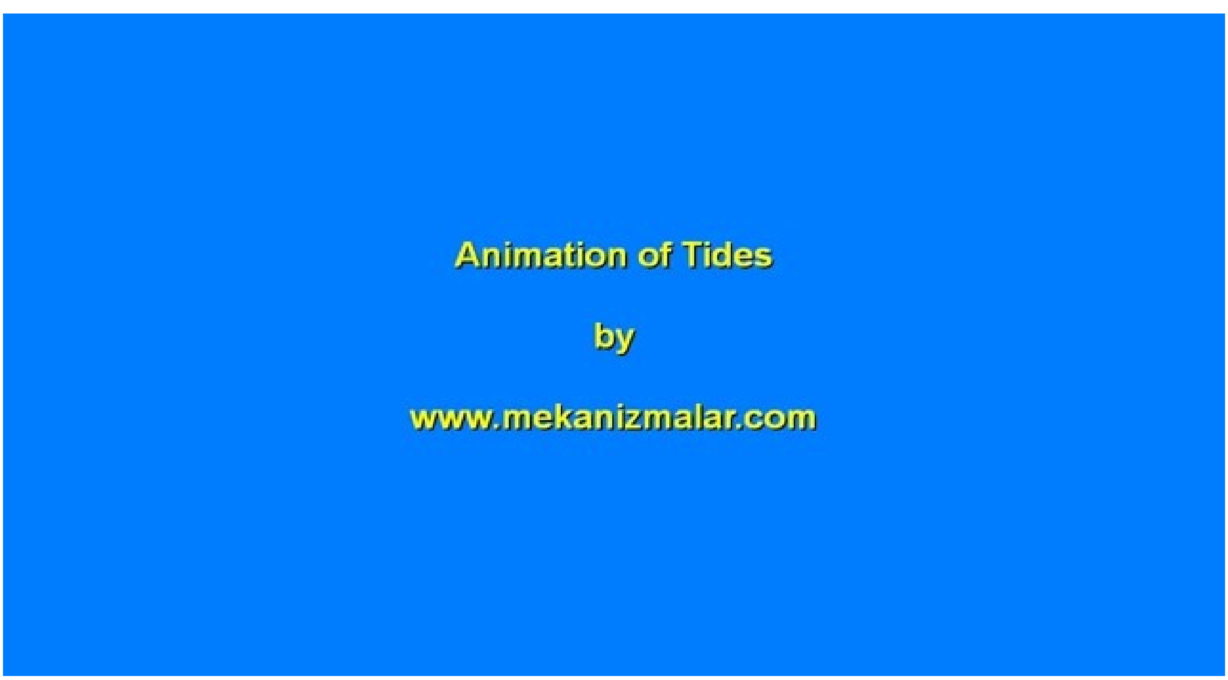




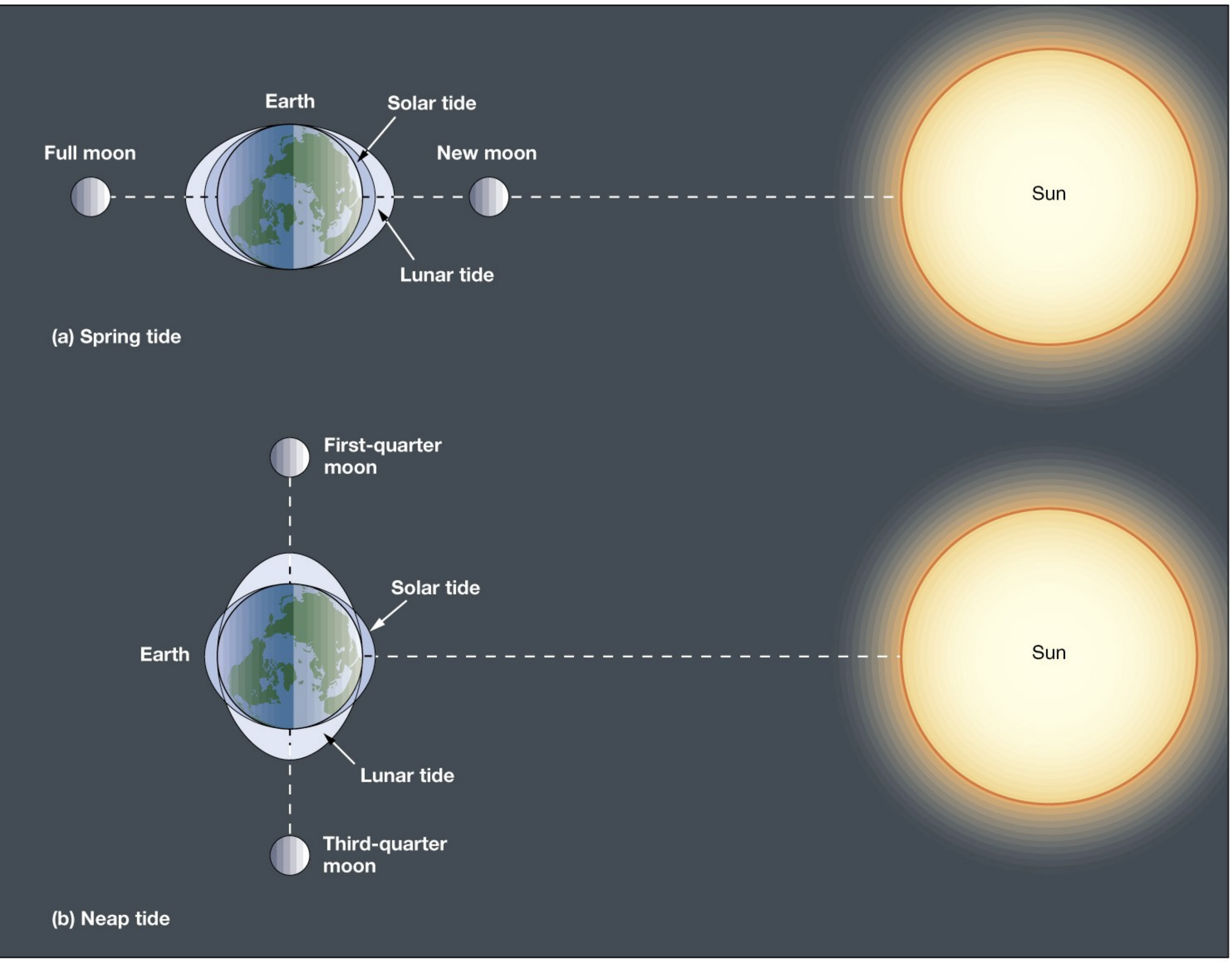
The equipotential surface is not a sphere but it becomes an ellipsoid.



Additional forces respect to the pure radial term of point mass.



On the Earth surface the tides are a combination of the Moon and Solar tides and they depend on the mutual phases.



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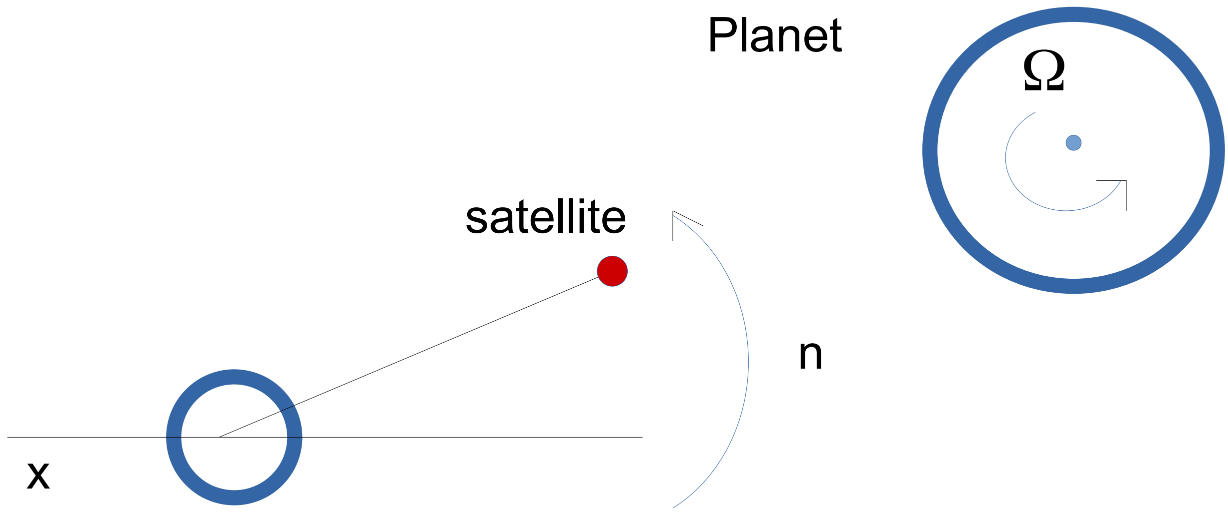
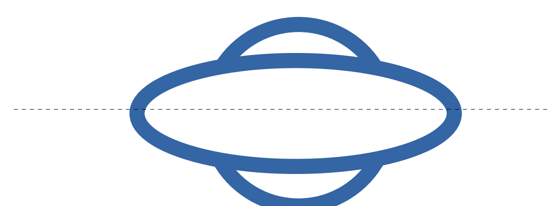
Simple 1-D model to explain the tidal advance or delay



The planet behaves as an harmonic oscillator:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + F_0\cos(\omega t)$$

- **x = shift from equilibrium**
- **m = planet inertia against deformation**
- **k = restoring force**
- **ω = $n - \Omega$**
- **β = dissipative term**
- **ω_0 = k/m proper Earth frequency**



$$\frac{d^2x}{dt^2} + \omega_0^2 x + \frac{1}{\tau}\frac{dx}{dt} = \frac{F_0}{m}\cos(\omega t)$$

- **ω_0 = natural frequency of the planet: $\omega_0^2 = k/m$**
- **τ = damping timescale $\tau = m/\beta$**

Solution:

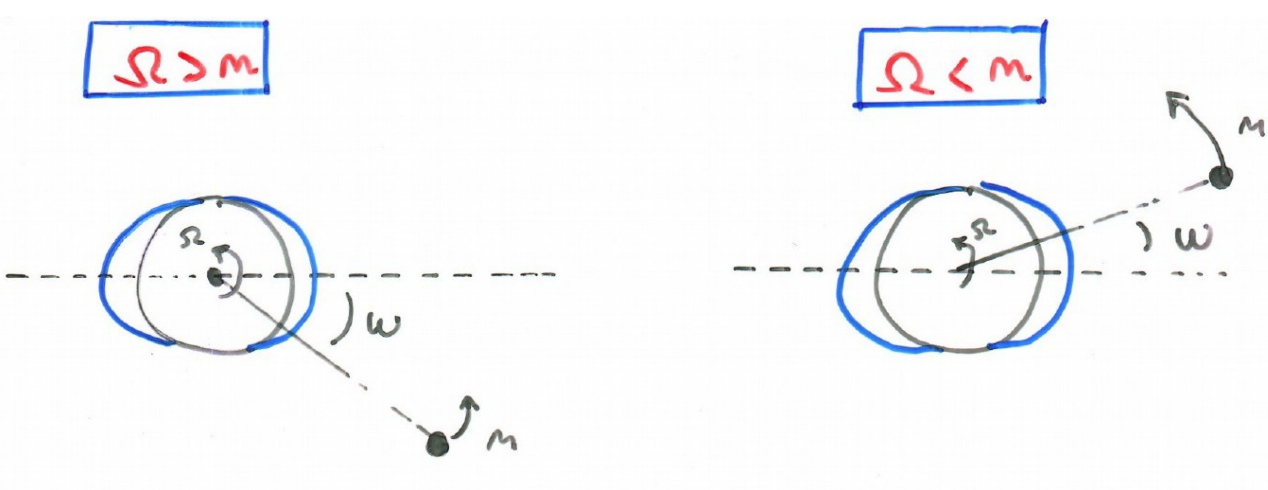
$$x(t) = A(\omega)\cos(\omega t + \delta(\omega))$$

$$A(\omega) = \frac{F_0}{m}\left[(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right]^{-1/2}$$

$$\sin(\delta(\omega)) = -\frac{\omega}{\tau}\left[(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega}{\tau}\right)^2\right]^{-1/2}$$

Dissipation into heat gives a positive τ so: $-\pi < \delta \leq 0$

The tide reaches its peak AFTER the Moon has passed: tidal delay (or advance). It depends on the difference between n and Ω (either positive or negative).



It is usually introduced a dissipation function Q given by

$$Q=\frac{2\pi E_0}{\Delta E}=\frac{2\pi E_0}{-\int_0^T \dot{E}\,dt}$$

Where ΔE is the energy dissipated in 1 period of the satellite and E_0 is the maximum energy stored in the bulge. The value of E_0 can be computed assuming that in 1 dimension the Earth acts like a spring with given k (related to its proper oscillation frequency):

$$E_0=\int_0^A kx\,dx=\int_0^A \omega_0^2 m\,x\,dx=\frac{1}{2}m\omega_0^2 A^2$$

This is the potential energy against the recoil gravity force of the planet. The energy dissipated is given by the work done by the viscous force :

$$dW=-\dot{x}\frac{m}{\tau}dx \;\Rightarrow\; \dot{E}=\frac{dW}{dt}=-\frac{m}{\tau}\dot{x}^2=-\frac{m}{\tau}(A\omega)^2\sin(\omega t+\delta)^2$$

$$\langle\dot{E}\rangle=-\frac{m}{\tau}(A\omega)^2\frac{1}{T}\int_0^T \sin(\omega t)^2\,dt=-\frac{1}{2}\frac{m}{\tau}(\omega A)^2$$

$$\int_0^T \sin(\omega t)^2 dt=\frac{1}{2}t|_0^T-\frac{\sin(2\omega t)}{4a}|_0^T=\frac{1}{2}T$$

The energy dissipated in a cycle is then:

$$\Delta E=-\left(-\dot{E}\frac{2\pi}{\omega}\right)=\frac{1}{2}\frac{m}{\tau}(\omega A)^2\frac{2\pi}{\omega}=\frac{\pi m}{\omega\tau}A^2\omega^2$$

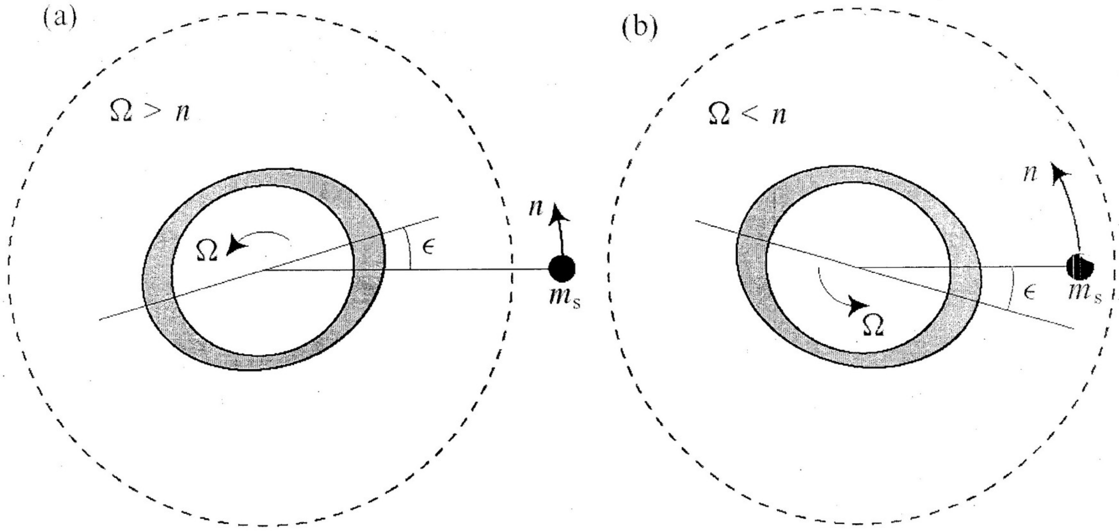
$$\boxed{Q=\frac{\tau}{\omega}\omega_0^2}$$

There is a link between Q and δ:

$$\sin\delta = -\frac{1}{Q}$$

Assuming that ω_0 >> ω i.e. the recoil force of the planet has a frequency which is larger than the frequency of the orbital motion of the satellite, we get

$$\sin\delta = -\frac{\omega}{\tau}\left[\left(\omega_0^2 - \omega^2\right)^2 + \left(\frac{\omega}{\tau}\right)^2\right]^{-\frac{1}{2}} \simeq -\frac{\omega}{\tau}\frac{1}{\omega_0^2} = -\frac{1}{Q}$$



Advanced tide: the tidal bulge acts on the satellite accelerating it while it reduces the rotation rate of the planet.

Delayed tide: the tidal bulge acts on the satellite decelerating it while it speeds up the rotation rate of the planet.

$$\dot{E} = \mathrm{T}\,n - \mathrm{T}\,\Omega = -\mathrm{T}(\Omega - n) < 0$$

$$\dot{E} = -\mathrm{T}\,n + \mathrm{T}\,\Omega = \mathrm{T}(\Omega - n) < 0$$

Interaction between orbital and rotational motion: mechanical energy is dissipated as heat by the tide on Earth. The total energy of the system decreases while the total angula momentum is constant because the system is isolated.

$$E=\frac{1}{2}I_P\Omega^2-G\frac{m_p m_s}{2a}$$

$$G(m_p+m_s)=n^2a^3$$

$$\dot{E}=I_P\Omega\dot{\Omega}+G\frac{m_p m_s}{2a^2}\dot{a}=I_P\Omega\dot{\Omega}+\frac{1}{2}\frac{m_p m_s}{(m_p+m_s)}n^2a\dot{a}<0$$

$$L=I_p\Omega+\frac{m_p m_s}{(m_p+m_s)}h=I_p\Omega+\frac{m_p m_s}{(m_p+m_s)}na^2$$

$$h=na^2\sqrt{1-e^2}\quad for\quad e\approx 0\quad\Rightarrow h=na^2$$

$$\dot{L}=I_p\dot{\Omega}+\frac{1}{2}\frac{m_p m_s}{(m_p+m_s)}na\dot{a}=0$$

$$\frac{d}{dt}(a^2n)=2a\frac{da}{dt}n+a^2\frac{dn}{dt}$$

$$\frac{dn}{da}\frac{da}{dt}=\frac{dn}{dt}=\frac{-3}{2}\frac{n}{a}\frac{da}{dt}\quad\Rightarrow\quad\frac{d}{dt}(a^2n)=\frac{1}{2}an\dot{a}$$

From the conservation of momentum:

$$I_p \dot{\Omega} = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a}$$

Inserting this expression in the energy derivative:

$$\dot{E} = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a} \Omega + \frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n^2 a \dot{a}$$

Finally we get:

$$\boxed{\dot{E} = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a} (\Omega - n) < 0}$$

Prograde satellites:

- $n < \Omega \rightarrow$ **da/dt > 0** The satellite migrates outward.
- $n > \Omega \rightarrow$ **da/dt < 0** The satellite migrates inward. (Phobos, Mars satellite, will impact the planet)
- $N = \Omega \rightarrow$ **da/dt=0** dE/dt = 0 Synchronous orbit, no evolution (Pluto & Charon)

Retrograde satellites:

- $n < 0$, $\Omega - n > 0 \rightarrow$ **da/dt <0** The satellite **always** migrate inward and impact the planet (example of Triton, satellite of Neptune).

How to compute the derivative of the semimajor axis as a function of time? 3 are the basic equations

$$\dot{E}=-\frac{1}{2}\frac{m_p m_s}{m_p+m_s}n\,a\,\dot{a}(\Omega-n)$$

$$\dot{E}=-\mathrm{T}(\Omega-n)$$

$$\mathrm{T}=\frac{3}{2}k_2\frac{Gm_s^2}{a^6}R^5|\sin(\delta)|$$

δ is the tidal delay

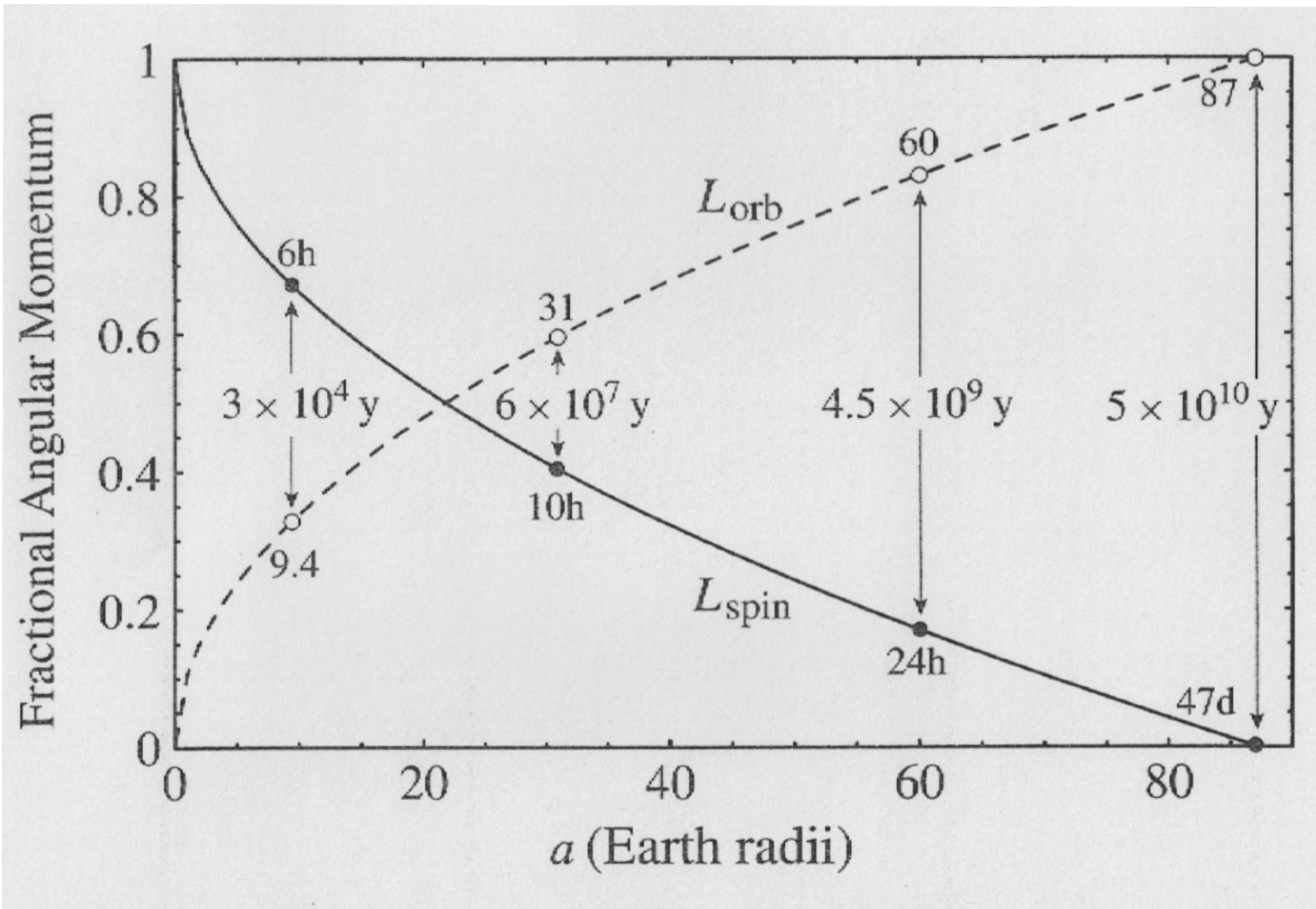
The first equation gives the effect of the energy dissipation on the orbit, the second links the torque to the energy dissipation, the third is the torque due to the deformed shape of the body (ellipsoid) which can be calculated using the Love number k_2.

$$\frac{da}{dt}=\frac{3k_2}{Q}\frac{m_s}{m_p}\left(\frac{R}{a}\right)^5 a\,n$$

Taking into account also satellites WITHIN the synchronous orbit:

$$\frac{da}{dt}=sign(\Omega-n)\frac{3k_2}{Q}\frac{m_s}{m_p}\left(\frac{R}{a}\right)^5 a\,n$$

Evoluzione mareale della luna



$$L_{orb} \propto h \simeq n a^2 = \sqrt{\mu a}$$

After 10^{10} years, the sun 1) enflates and will include the Earth 2) will lose half of its mass 3) will contract and become and white dwarf

Rate ot outward migration of the Moon: from laser ranging it is estimated to be:

$$\boxed{\frac{da}{dt} \simeq 3.8\ cm/y}$$

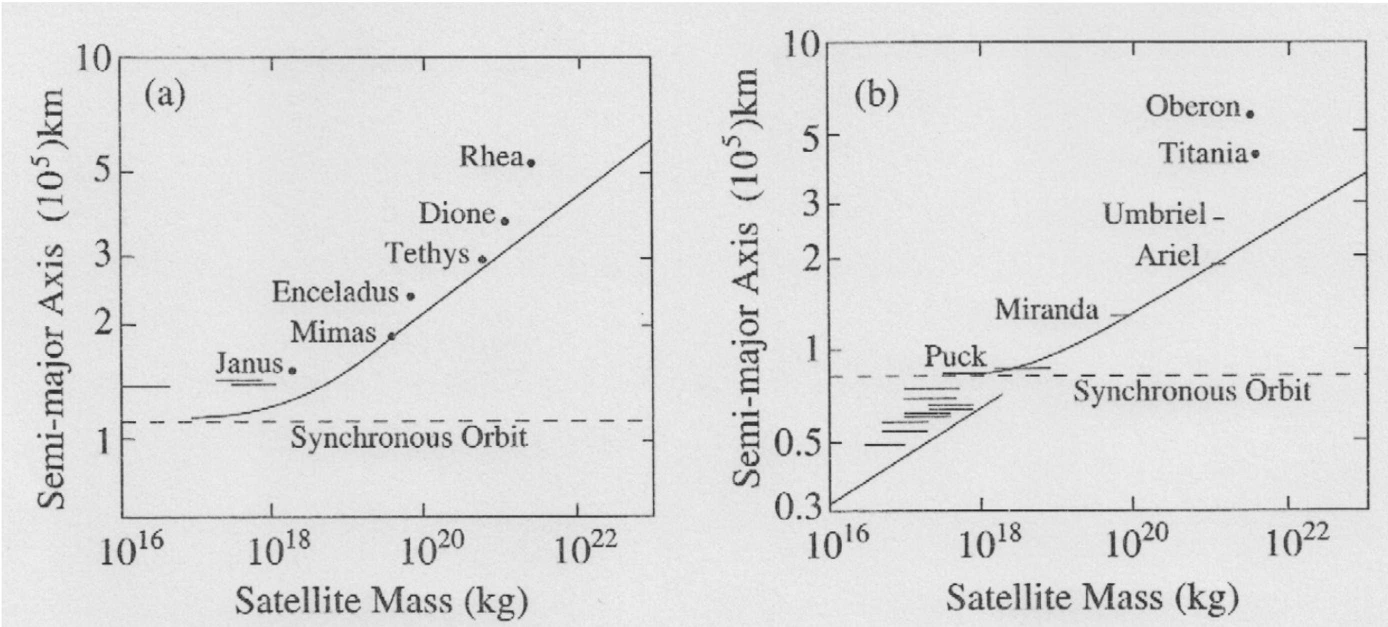
From the conservation of angular momentum:

$$I_p \dot{\Omega} = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a} = -\mathrm{T}$$

$$\boxed{\dot{\Omega} = -\frac{\mathrm{T}}{I_p} \simeq -6 \times 10^{-22} rad/s^2}$$

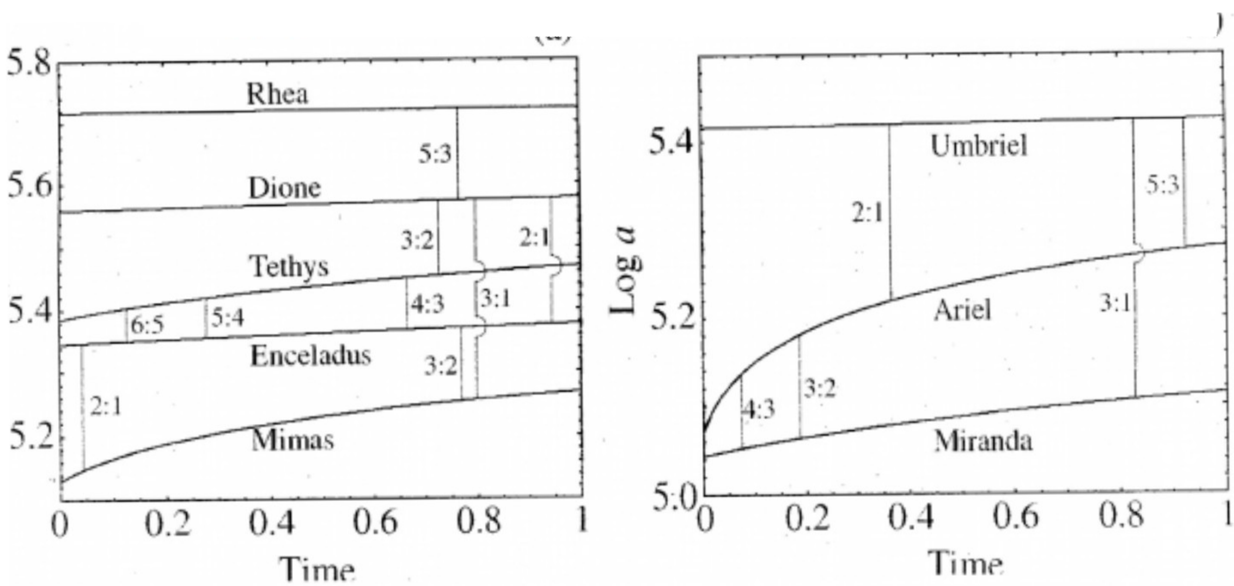
The day on Earth grows by about 1 hr every 150 Myr.

Uranus and Saturn satellites evolution.



Saturn

Uranus



- Any satellite ‘sees’ only its own tide. For ‘him’, the other tides average to 0.
- The more massive cause stronger tides and migrate outwards at a faster pace.
- On the way, possible trapping in mean motion resonances.