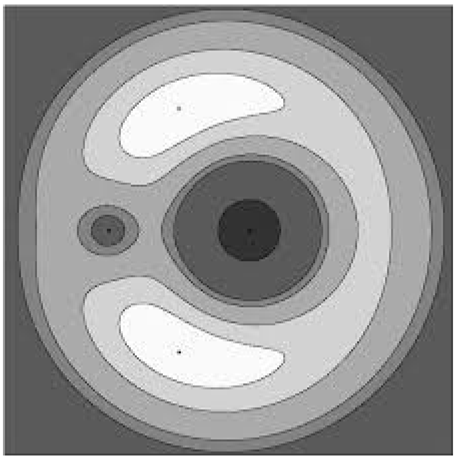
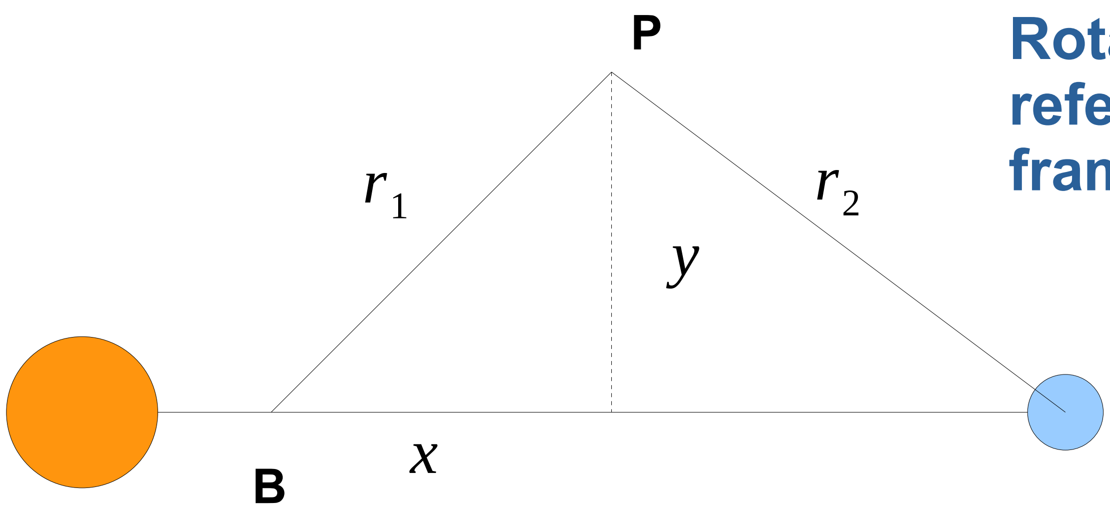
3body problem, restricted & planar & circular





Rotating reference frame ω=n

$$\ddot{\boldsymbol{r}}+2(\boldsymbol{\omega}\times\boldsymbol{v})+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})=-\nabla V$$

$$\boldsymbol{\omega}=\begin{pmatrix}0\\0\\n\end{pmatrix}\qquad \boldsymbol{v}=\begin{pmatrix}\dot{x}\\\dot{y}\\\dot{z}\end{pmatrix}$$

-Mass of test body=0
-Second body on circular orbit
-Planar (no inclination)

$$\ddot{x}-2n\dot{y}=\frac{\partial U}{\partial x}$$
$$\ddot{y}+2n\dot{x}=\frac{\partial U}{\partial y}$$
$$\ddot{z}=\frac{\partial U}{\partial z}$$

Hill's equations.

$$U=\frac{1}{2}n^2(x^2+y^2)+\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2} \qquad \textit{Pseudo-potential}$$

$$\frac{1}{2}(\dot{x}^2+\dot{y}^2)=U+\frac{C}{2} \qquad \textit{Jacoby constant}$$

$$\dot{x}\ddot{x} - \dot{x}\,2n\,\dot{y} = \dot{x}\frac{\partial U}{\partial x}$$

$$\dot{y}\ddot{y} + \dot{y}\,2n\,\dot{x} = \dot{y}\frac{\partial U}{\partial y}$$

$$\Rightarrow \quad \dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\frac{\partial U}{\partial x} + \dot{y}\frac{\partial U}{\partial y}$$

$$\int_{t_0}^{t_1} (\dot{x}\ddot{x} + \dot{y}\ddot{y})\,dt = \int_{t_0}^{t_1} \dot{x}\frac{\partial U}{\partial x}\,dt + \int_{t_0}^{t_1} \dot{y}\frac{\partial U}{\partial y}\,dt$$

$$\Rightarrow \frac{1}{2}(\dot{x}^2 + \dot{y}^2)\Big|_{x_0, y_0}^{x_1, y_1} = \int_{x_0}^{x_1} \frac{\partial U}{\partial x}\,dx + \int_{y_0}^{y_1} \frac{\partial U}{\partial y}\,dy$$

$$\frac{1}{2}\dot{x}^2 + \dot{y}^2 = U + \frac{C}{2}$$

Normalized units:

$$G = 1 \quad G(m_1 + m_2) = 1 \quad \bar{\mu} = \frac{m_2}{m_1 + m_2} G$$

$$\mu_1 = G m_1 = 1 - \bar{\mu} = x_2$$

$$\mu_2 = G m_2 = \bar{\mu} = x_1 \qquad x_1 = \bar{\mu} = \frac{m_2}{m_1 + m_2} \qquad x_2 = 1 - x_1$$

$$a = 1 \quad n = \sqrt{G\frac{(m_1 + m_2)}{a^3}} = 1$$

Adimensional numbers.

Hamiltonian approach

The Lagrangian of the system in the rotating reference frame is:

$$L = T + V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U(x,y) + U_c(x,y)$$

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \left(\frac{1-\mu}{r_1}\right) + \frac{\mu}{r_2} + \frac{1}{2}(x^2 + y^2) + x\dot{y} - y\dot{x}$$

The equations of motions are: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

The Hamiltonian is:

$$H = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \qquad where \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x$$

$$H = p_x \dot{x} + p_y \dot{y} - L =$$

$$\dot{x}^2 - \dot{x}y + x\dot{y} + \dot{y}^2 - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - U(x,y) - \frac{1}{2}(x^2 + y^2) - x\dot{y} + y\dot{x} =$$

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - U(x,y) - \frac{1}{2}(x^2 + y^2) = \frac{1}{2}v^2 - \bar{U}(x,y)$$

The Hamiltonian is the Jacoby integral

To go back to Hill's equations from the Hamiltonian, Hamilton equations must be used:

$$\dot{p}_x = -\frac{\partial H}{\partial x}$$

$$\dot{x} = \frac{\partial H}{\partial p_x}$$

$$\dot{p}_x = \frac{\partial L}{\partial x} = \dot{y} + \frac{\partial \overline{U}}{\partial x}$$

$$\dot{p}_y = \frac{\partial L}{\partial y} = -\dot{x} + \frac{\partial \overline{U}}{\partial y}$$

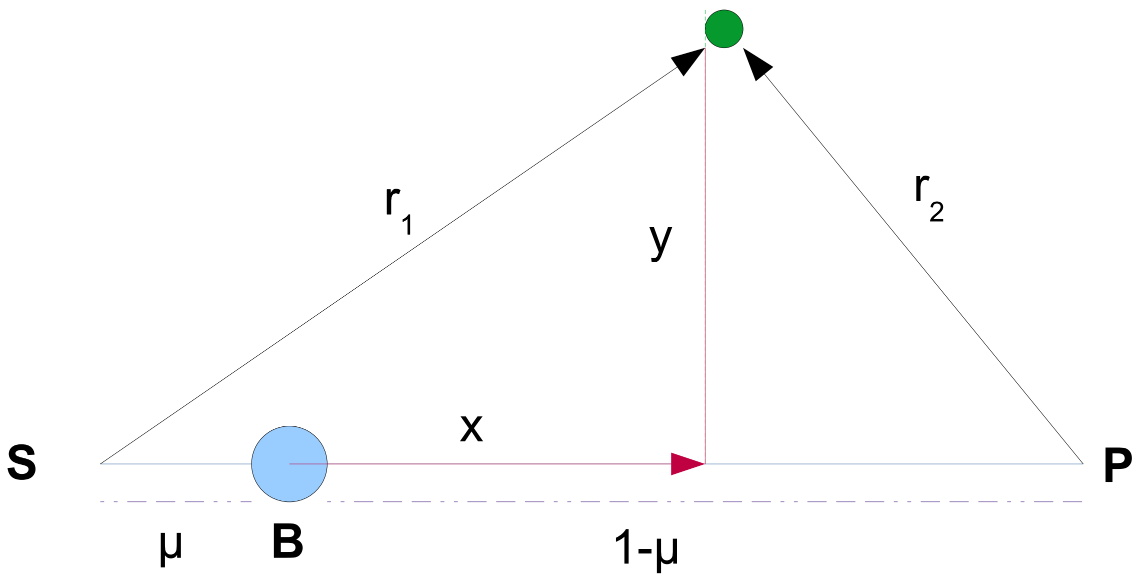
$$\ddot{x} - \dot{y} = \dot{y} + \frac{\partial \overline{U}}{\partial x}$$

$$\ddot{y} + \dot{x} = \dot{x} + \frac{\partial \overline{U}}{\partial y}$$

$$\ddot{x} - 2\dot{y} = \frac{\partial \overline{U}}{\partial x}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial \overline{U}}{\partial y}$$

Hill's equations in normalized units



$$x_1 = \frac{m}{M} x_2 = \frac{m}{M}(1 - x_1) \Rightarrow x_1(M+m) = m \Rightarrow x_1 = \frac{m}{M+m} = \bar{\mu}$$

$$\ddot{x} - 2\dot{y} = x - \frac{(1-\bar{\mu})(x+\bar{\mu})}{r_1^3} - \frac{\bar{\mu}(1-\bar{\mu}-x)(-1)}{r_2^3}$$

$$\ddot{y} + 2\dot{x} = y - \frac{(1-\bar{\mu})\,y}{r_1^3} - \frac{\bar{\mu}\,y}{r_2^3}$$

$$r_1 = \sqrt{(x+\bar{\mu})^2 + y^2}$$

$$r_2 = \sqrt{(x-1+\bar{\mu})^2 + y^2}$$

$$U(x,y) = \frac{1}{2}(x^2+y^2) + \frac{(1-\bar{\mu})}{r_1} + \frac{\bar{\mu}}{r_2}$$

$$W(x,y) = -U(x,y)$$

Lagrangian equilibrium points.

$$\ddot{x}=0 \quad \dot{x}=0$$
$$\ddot{y}=0 \quad \dot{y}=0$$

$$\Rightarrow \quad \nabla U = -\nabla W = 0$$

$$\nabla U = \boldsymbol{r} - \frac{(1-\bar{\mu})\boldsymbol{r}_1}{r_1^3} - \frac{\bar{\mu}\boldsymbol{r}_2}{r_2^3}$$

$$\boldsymbol{r} = (1-\bar{\mu})\boldsymbol{r}_1 + \bar{\mu}\boldsymbol{r}_2$$

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \boldsymbol{r}_1 = \begin{pmatrix} x+\bar{\mu} \\ y \end{pmatrix} \quad \boldsymbol{r}_2 = \begin{pmatrix} x-1+\bar{\mu} \\ y \end{pmatrix}$$

$$\Downarrow$$

$$(1-\bar{\mu})\boldsymbol{r}_1\left(1-\frac{1}{r_1^3}\right) + \bar{\mu}\boldsymbol{r}_2\left(1-\frac{1}{r_2^3}\right) = 0$$

Triangular Lagrangian points L_4 and L_5: $\mathbf{r}_1 = \mathbf{r}_2 = 1$

Solving equation along x-axis L_1, L_2, L_3:

$$x = -1 - \bar{\mu} + \frac{7}{12}\bar{\mu} + O(\bar{\mu}^3) \qquad L_3$$

$$x = 1 - \bar{\mu} \pm \left(\frac{\bar{\mu}}{3}\right)^{\frac{1}{3}} + O(\bar{\mu}^{\frac{2}{3}}) \qquad L_1,\ L_2$$

Around these points move Trojan asteroids.

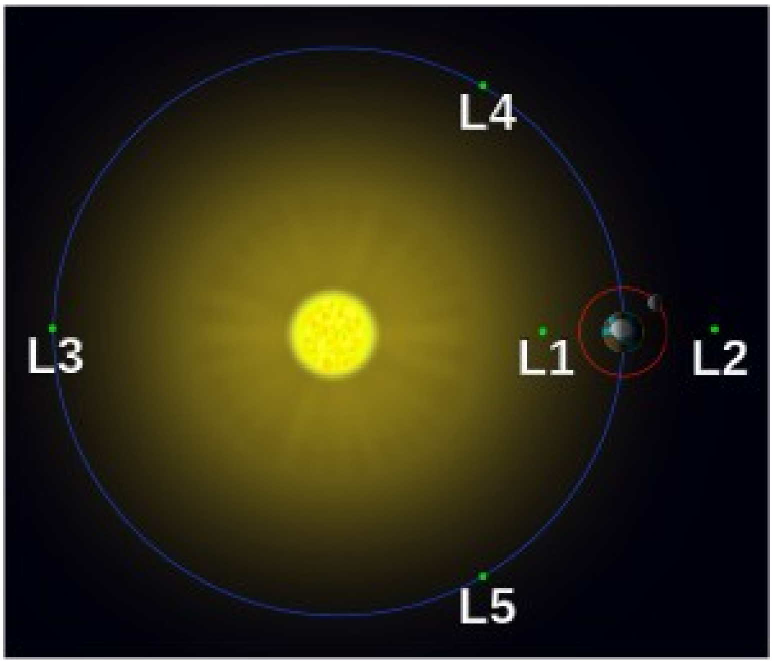
$$L_4 \Rightarrow \begin{pmatrix} \frac{1}{2} - \bar{\mu} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \cdot a$$

$$L_5 \Rightarrow \begin{pmatrix} \frac{1}{2} - \bar{\mu} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \cdot a$$

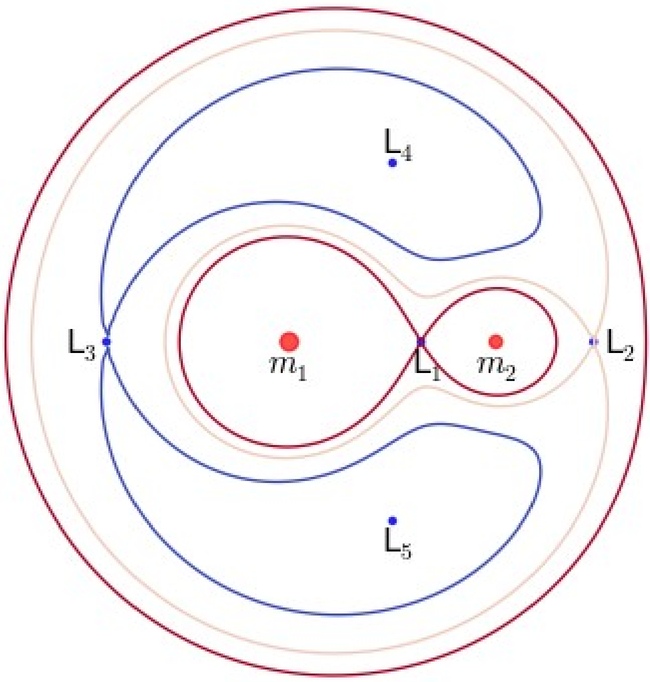
These lagrangian points define Hill's sphere

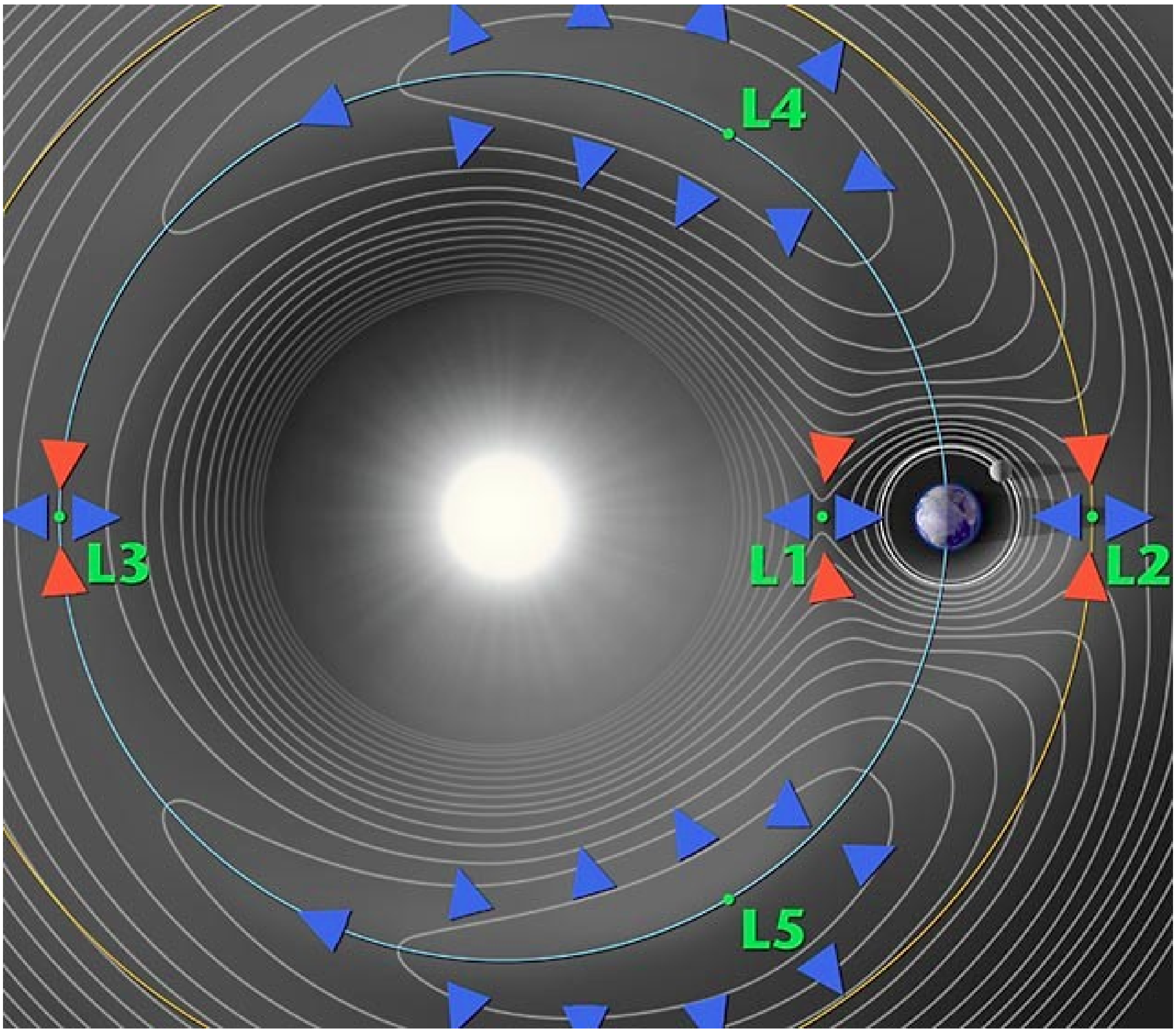
$$L_1 \Rightarrow \begin{pmatrix} 1 - \bar{\mu} + \left(\frac{\bar{\mu}}{3}\right)^{\frac{1}{3}} \\ 0 \end{pmatrix} \cdot a \qquad L_2 \Rightarrow \begin{pmatrix} 1 - \bar{\mu} - \left(\frac{\bar{\mu}}{3}\right)^{\frac{1}{3}} \\ 0 \end{pmatrix} \cdot a$$

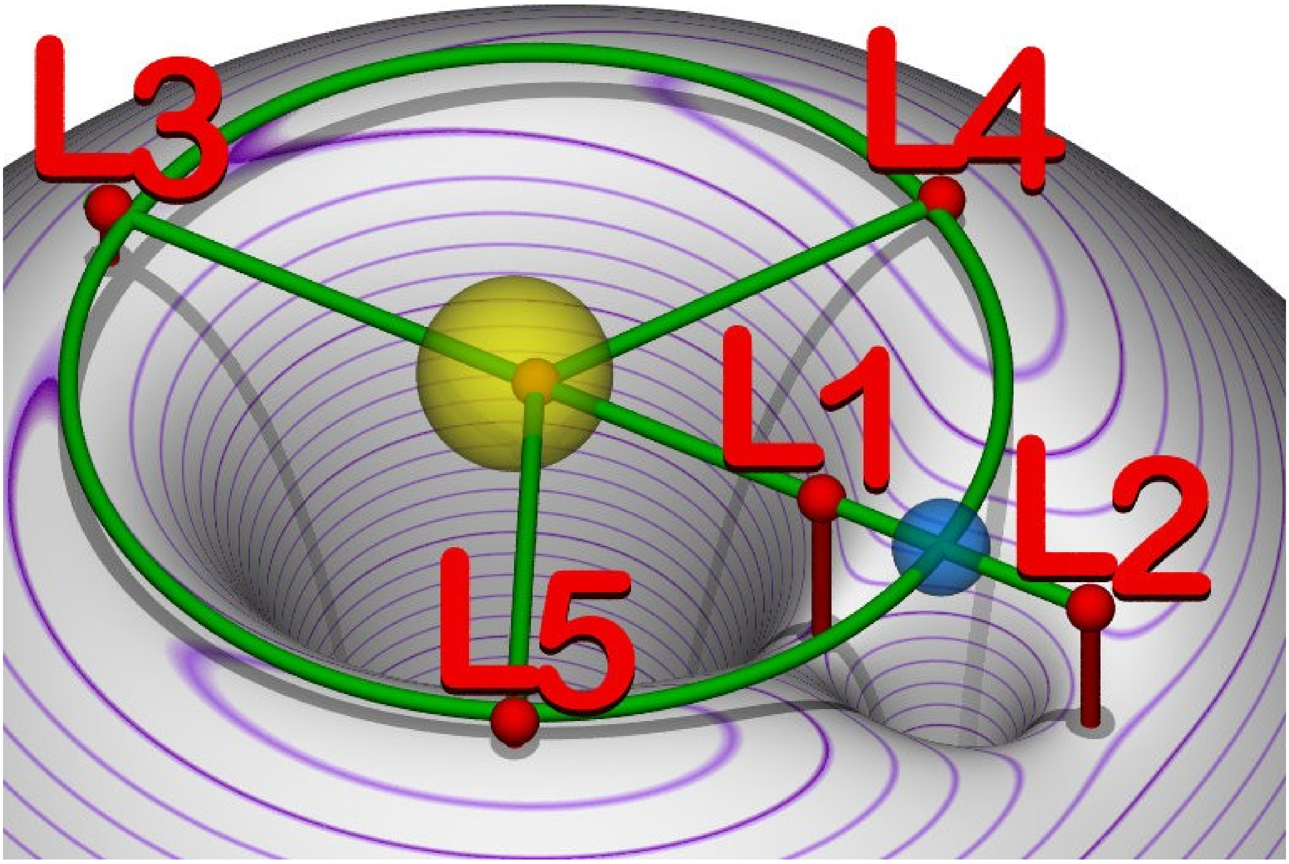
$$L_3 \Rightarrow \begin{pmatrix} -1 - \frac{5}{12}\bar{\mu} \\ 0 \end{pmatrix} \cdot a$$

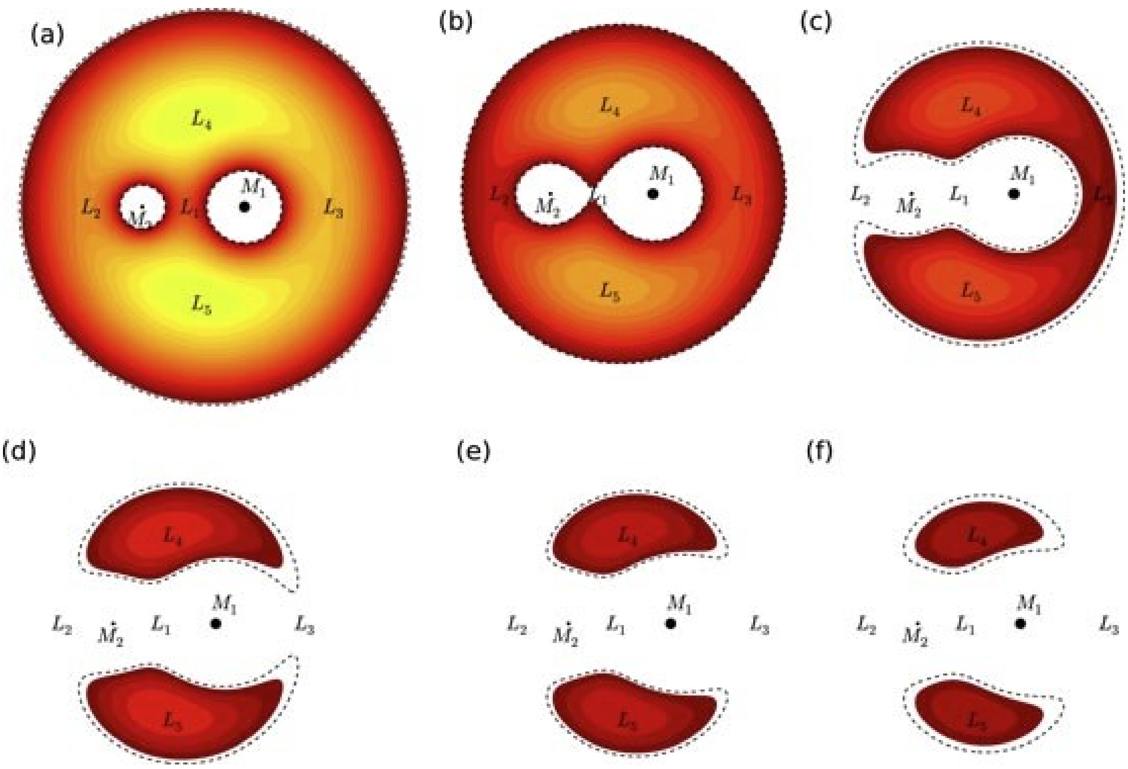


0 velocity curves: They cannot be crossed by a body!! They define the regions were stable motion is possible



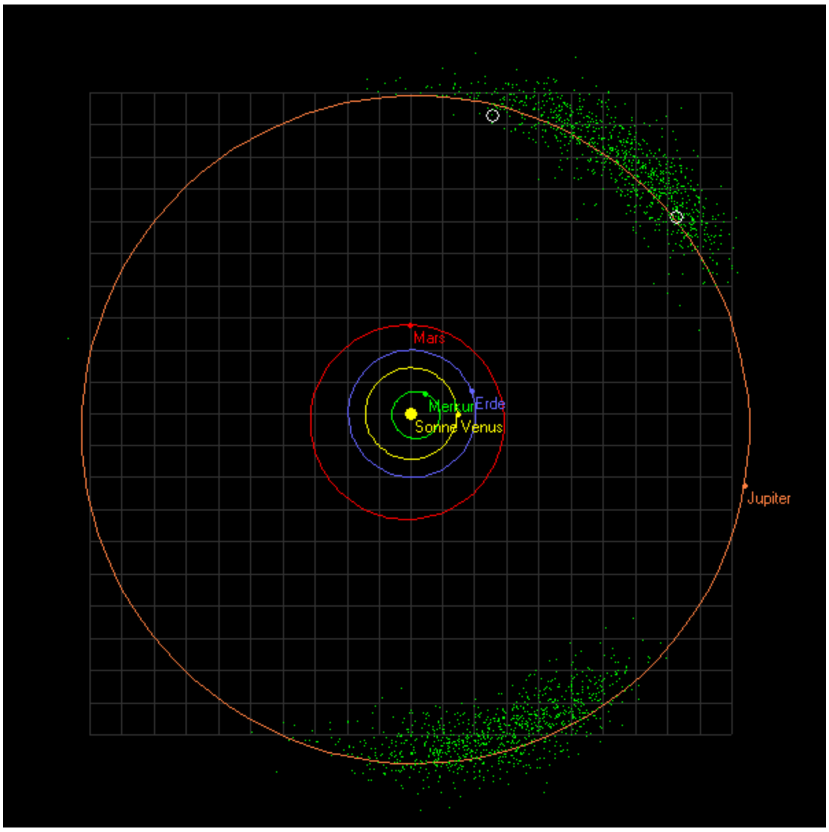


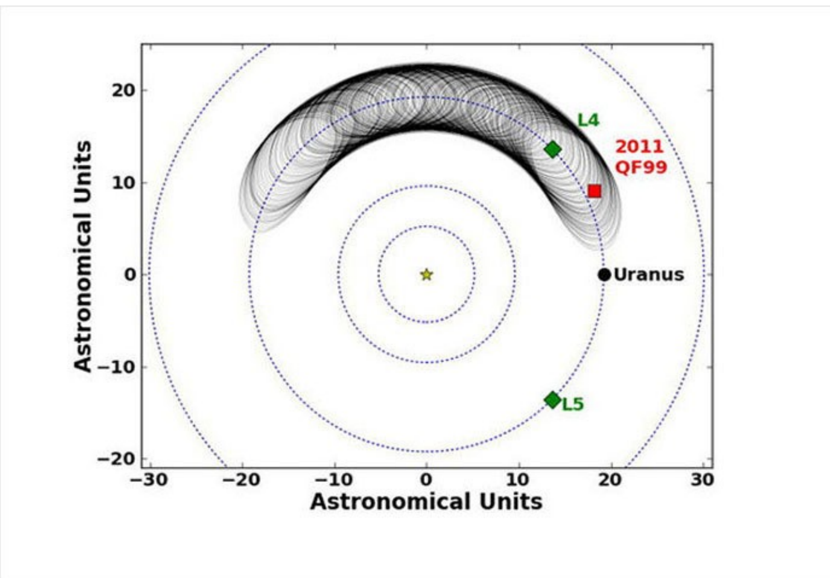




Different curves for different value of the Jacoby constant C = 2U

Mars, Jupiter, Uranus and Neptune have Trojan asteroids.





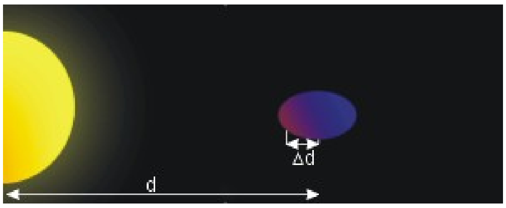
Hill's lobe: where the gravity of the body dominates respect to the central (more massive) body.

$$r_{Hill} = 1 - \mu_2 + \left(\frac{\mu_2}{3}\right)^{1/3}$$

$$r_{Hill} = \left(\frac{m_2}{m_1 + m_2} \cdot \frac{1}{3}\right)^{1/3} \cdot a$$

$$r_{Hill}^{Jupiter} = 0.355 \ \ au$$

$$r_{Hill}^{Earth} = 0.01 \ \ au$$



$$r_{Roche} = R_M \left(2 \frac{\rho_M}{\rho_m}\right)^{(1/3)} \qquad R_M \ \ primary \ \ body$$

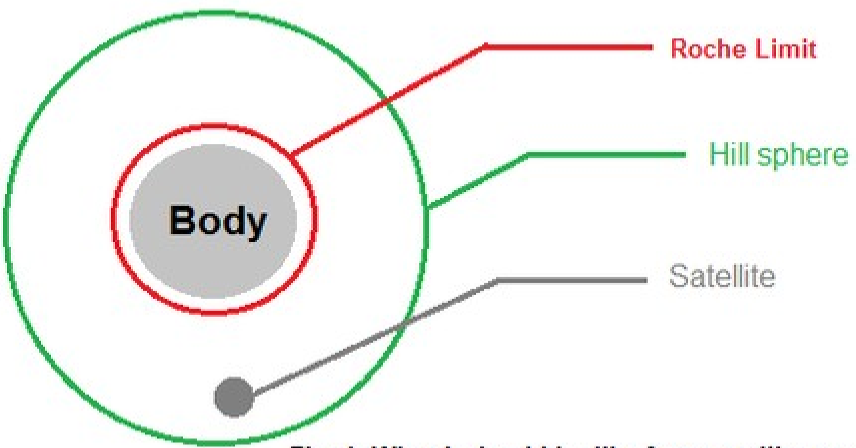
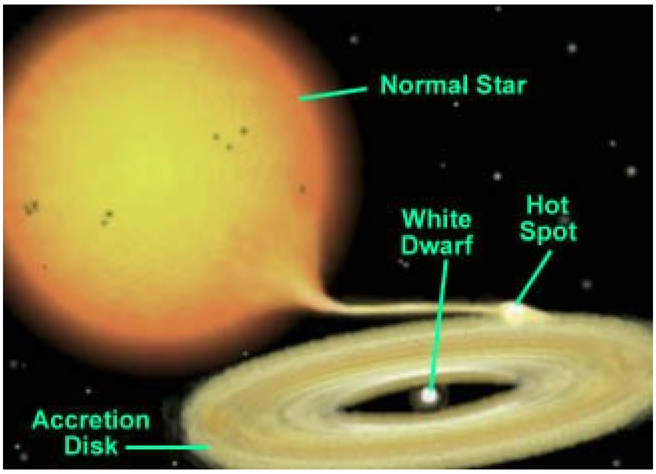
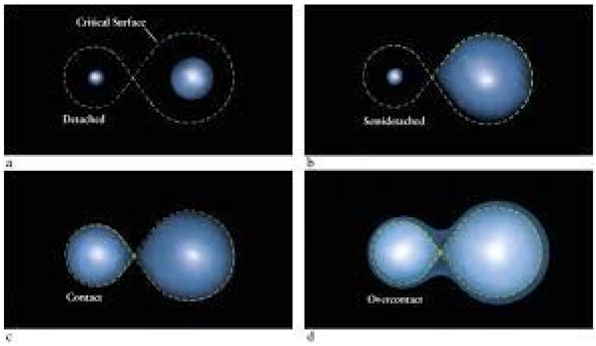
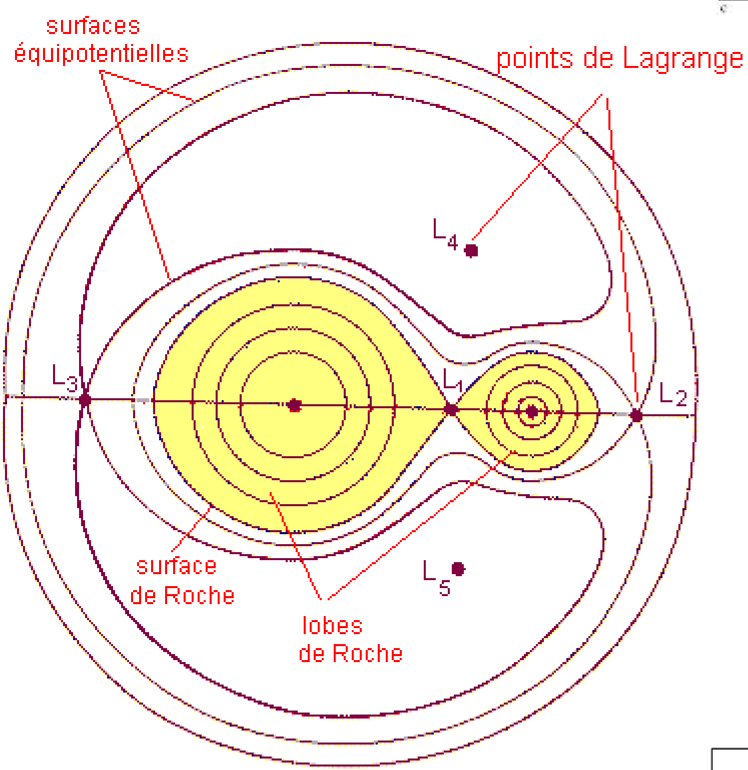


Fig. 1. What it should be like for a satellite to exist.

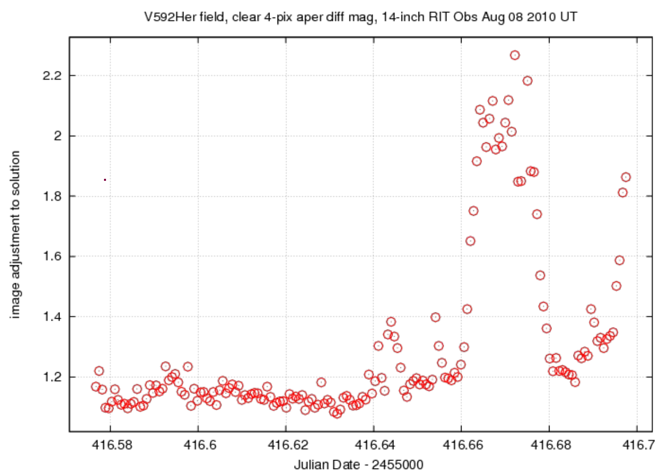
Cataclismic variables.







Potentiel de gravitation dans un système binaire



o) INVARIANTE di TISSERAND

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$$\bar{V}_{rot} = \bar{V}_{iner.} - \bar{V}_{s.rif.} \qquad \bar{V}_{srif} = \begin{pmatrix} -m\bar{y} \\ m\bar{x} \end{pmatrix}$$

$$\begin{cases} \dot{x} = \dot{\bar{x}} + m\bar{y} \\ \dot{y} = \dot{\bar{y}} - m\bar{x} \\ \dot{z} = \dot{\bar{z}} \end{cases} \Rightarrow (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = (\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2) + -2m(\bar{x}\dot{\bar{y}} - \dot{\bar{x}}\bar{y}) + m^2(\bar{x}^2 + \bar{y}^2)$$

$$\frac{1}{2}v^2 = \frac{1}{2}\bar{v}^2 - m h_z + \frac{m^2}{2}(\bar{x}^2 + \bar{y}^2)$$

$$\frac{m^2}{2}(\bar{x}^2 + \bar{y}^2) = \frac{m^2}{2}(x^2 + y^2)$$

r è uguale!

$$\frac{1}{2}v^2 - \frac{m^2}{2}(x^2 + y^2) = C + \frac{GM}{r_1} + \frac{Gm}{r_2} = \frac{1}{2}\bar{v}^2 - m h_z$$

$$\frac{1}{2}\bar{v}^2 = GM\left(\frac{1}{r_1} - \frac{1}{2a}\right) \qquad h_z = \sqrt{GMa(1-e^2)}\cos i$$

$$C + \frac{Gm}{r_2} = -GM\left(\frac{1}{2a} + \sqrt{\frac{a(1-e^2)}{a_J^3}}\cos i\right)$$

Se $r_2 \simeq a_J$

$$\frac{a_J}{a} + 2\sqrt{\frac{a(1-e^2)}{a_J}}\cos i = T \quad \text{costante!}$$

$$T = -\frac{m}{M} - \frac{2a_J C}{GM}$$

↑ Per riconoscere comete dopo close encounter.

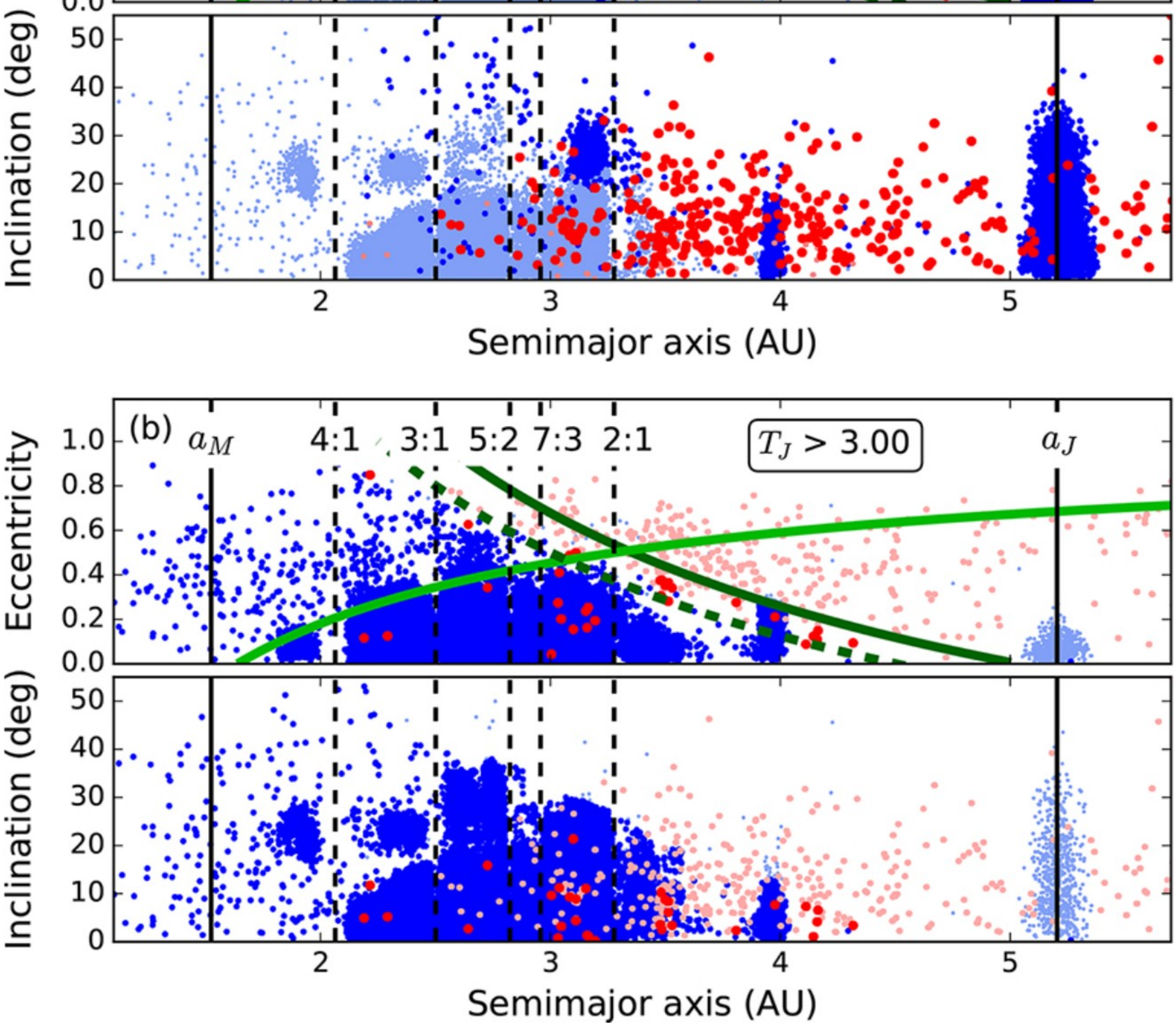


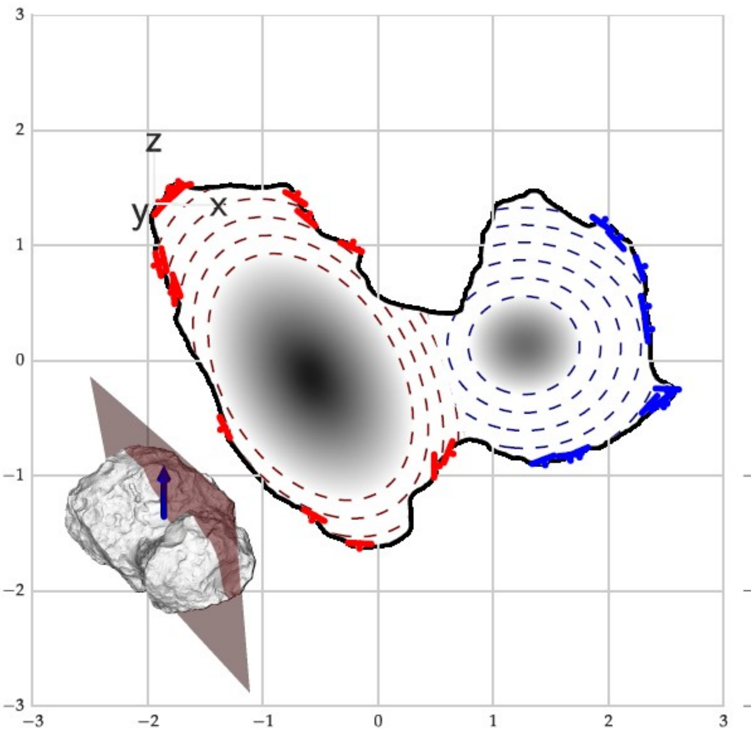
Figure 1: Plots of a vs. e (top half of each panel) and i (bottom half of each panel) for the first 50 000 numbered asteroids (pale blue dots) and all comets catalogued by the Minor Planet Center as of 2014 April 1 (pale red dots), where asteroids and comets with T_J values of (a) $T_J < 3.00$, and (b) $T_J > 3.00$ are highlighted with dark blue and dark red dots, respectively. Solid vertical lines mark a for Mars and Jupiter (a_M and a_J), while the 4:1, 3:1, 5:2, 7:3, and 2:1 MMRs with Jupiter are marked with dashed vertical lines. The loci of Mars-crossing orbits (where $q = Q_M$) and Jupiter-crossing orbits (where $Q = q_J$) are marked with light green and dark green curved solid lines, respectively, on each a-e plot, while the loci of orbits for which objects can potentially come within 1.5 Hill radii of Jupiter ($Q = q_J - 1.5R_H$) are marked with dark green dashed lines.

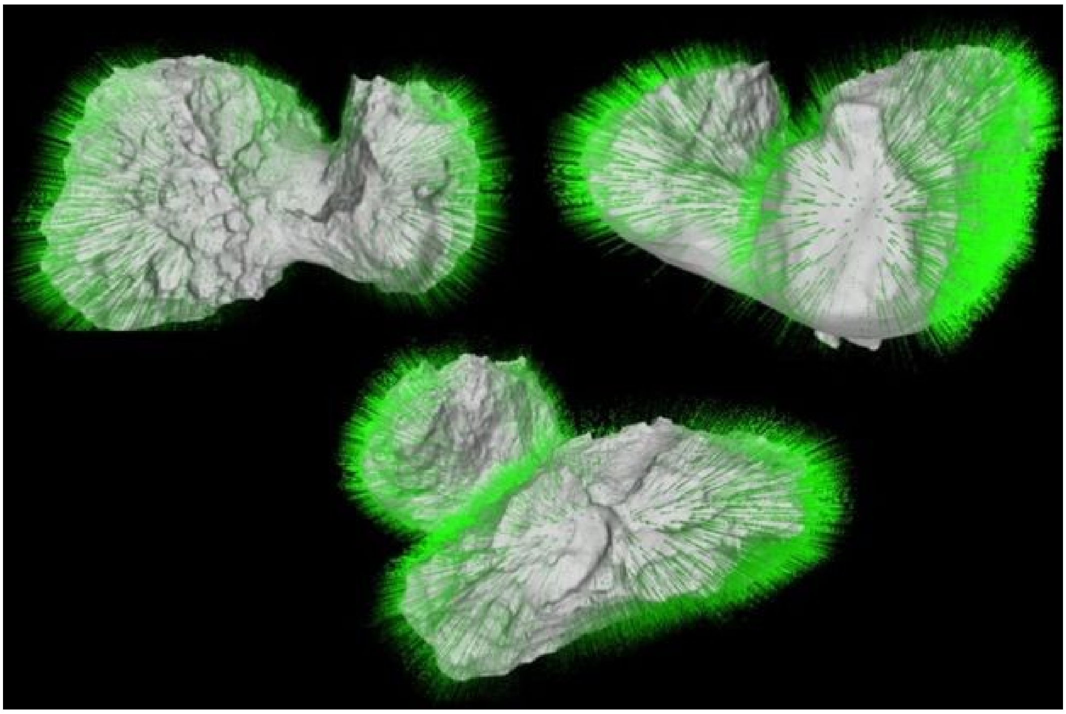
From Hsieh & Haghighipour, 2016, Icarus, Volume 277, p. 19-38

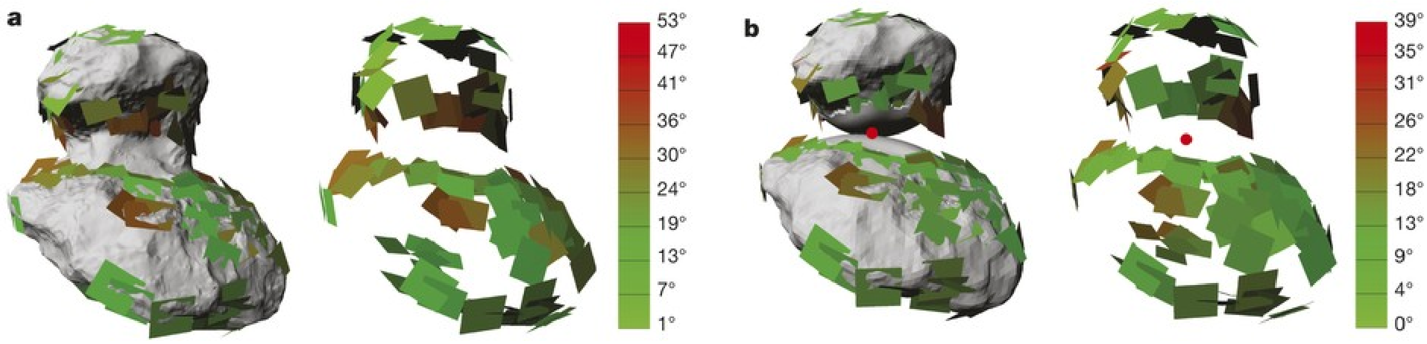
Most comets have T_J < 3 while most asteroids have T_J > 3 but there are exceptions. Upper plot, comets in red with T_J < 3, on the bottom plot asteroids with T_J > 3 (blue dots)

P67: the comet explored by ROSETTA mission

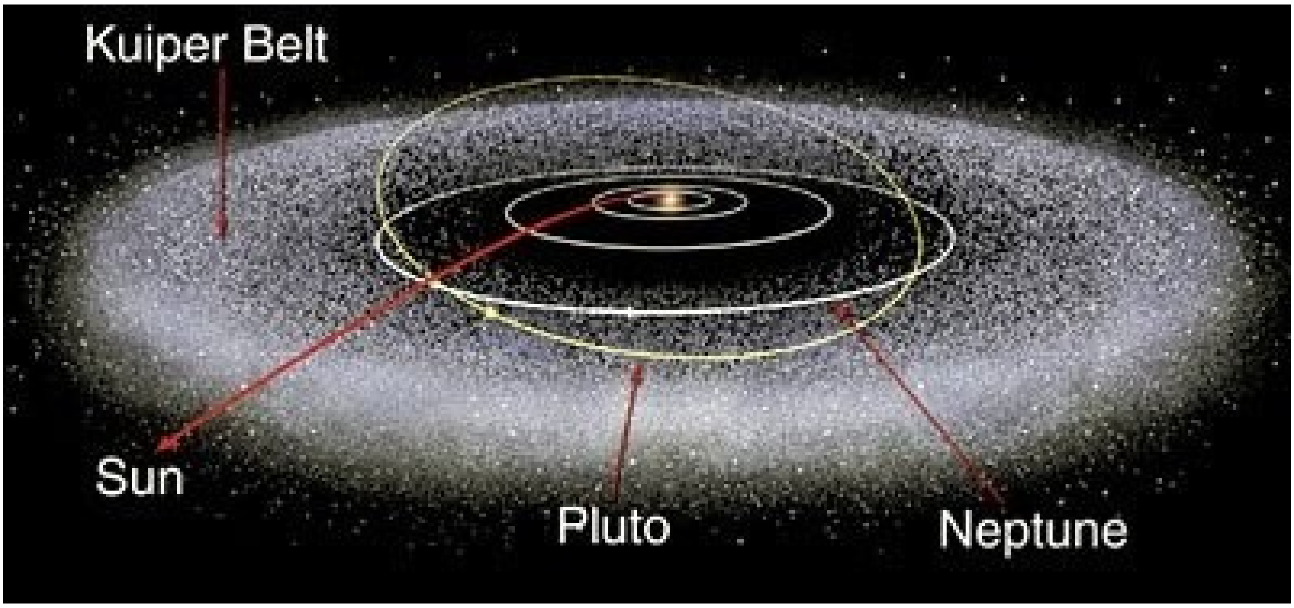
The comet is made of two pieces (lobes) which independently formed in the outer Kuiper Belt.

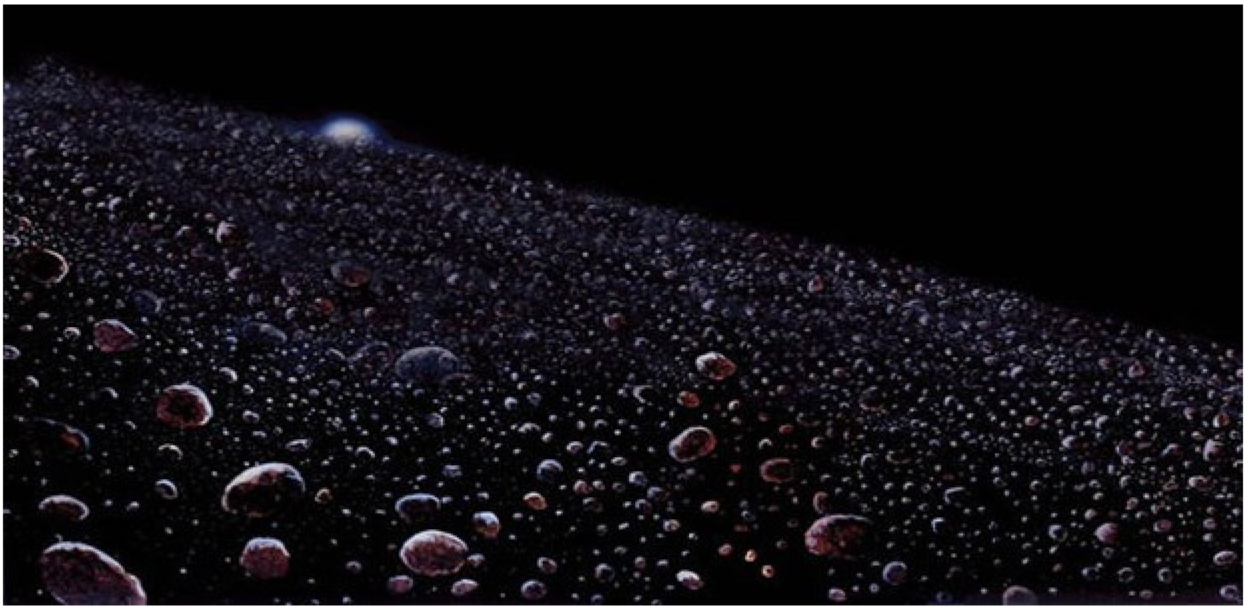




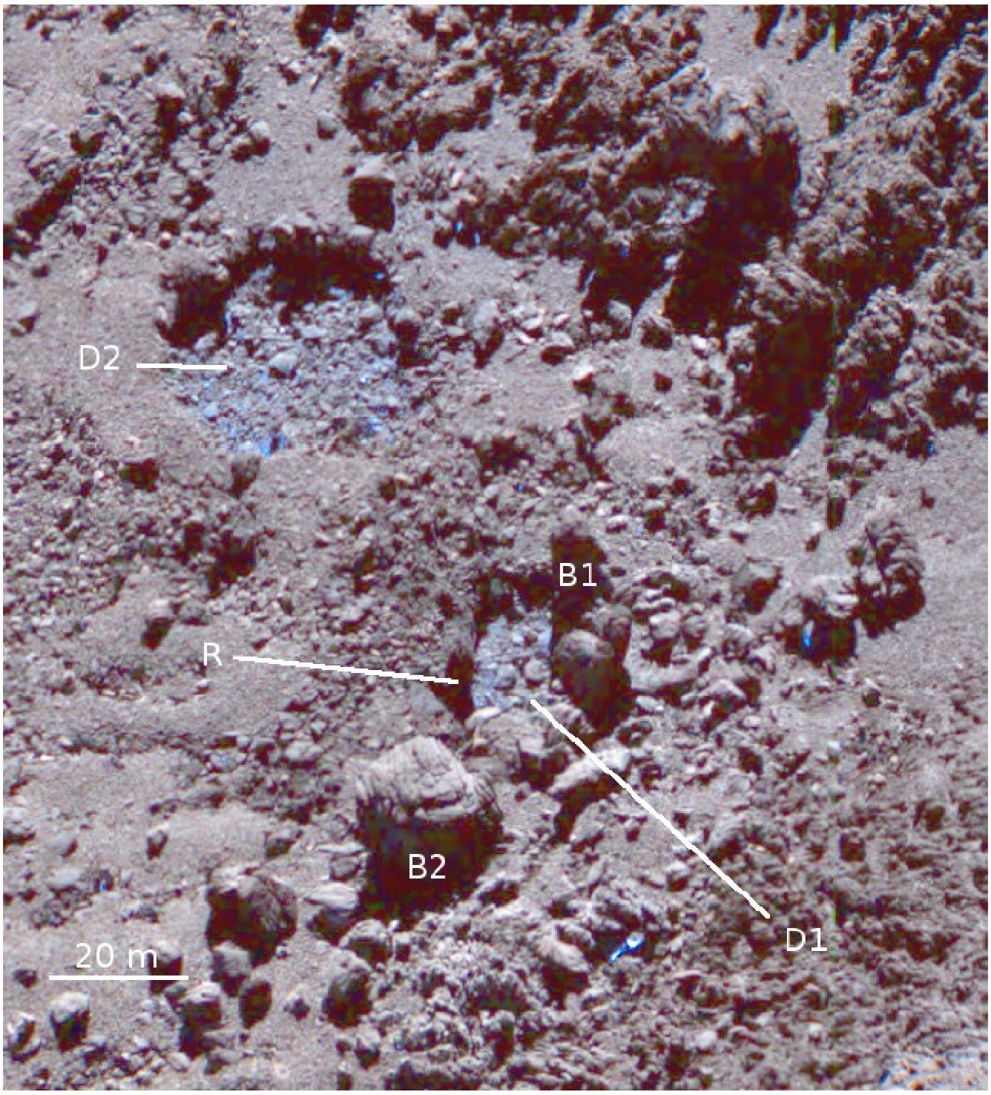


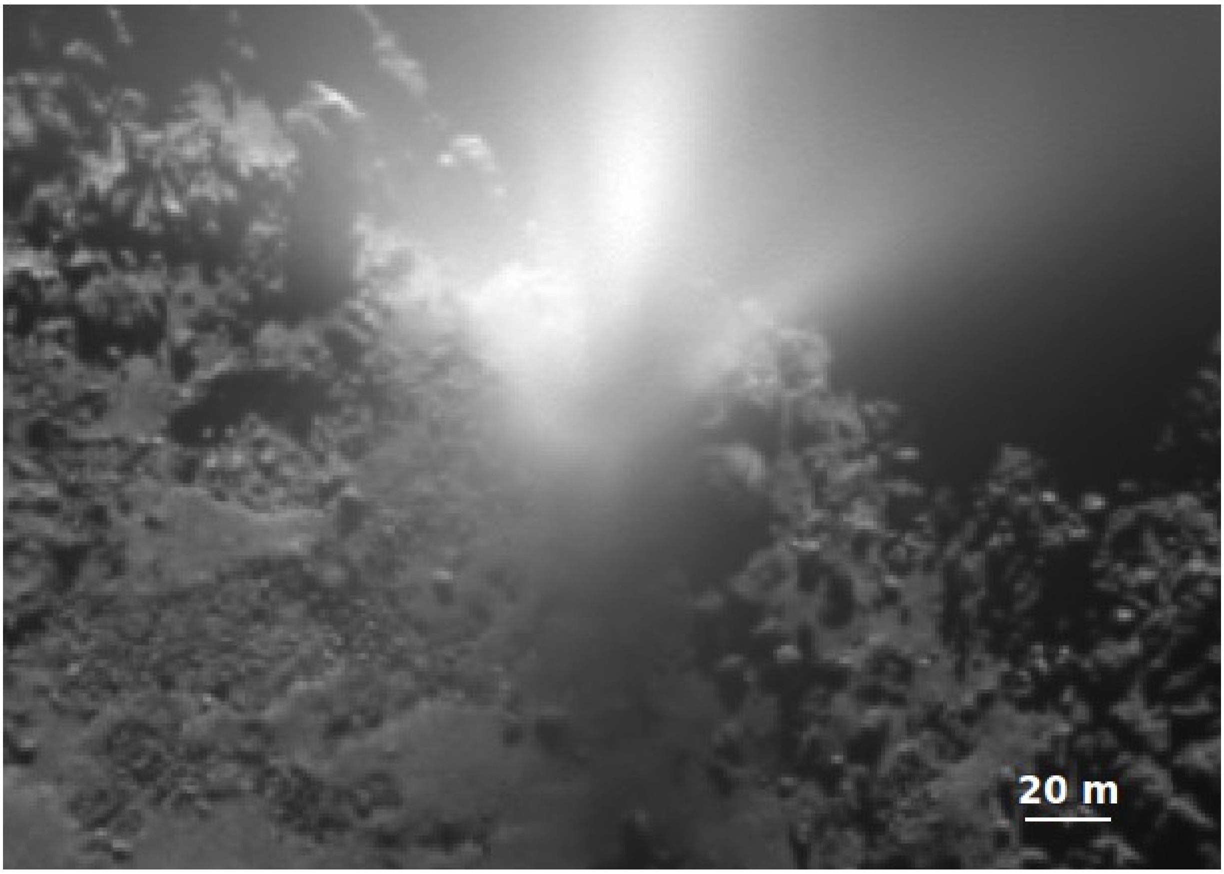
Formed in the Kuiper Belt, due to repeated close encounters with Neptune it was injected in a elliptic orbit crossing the inner regions of the SS.



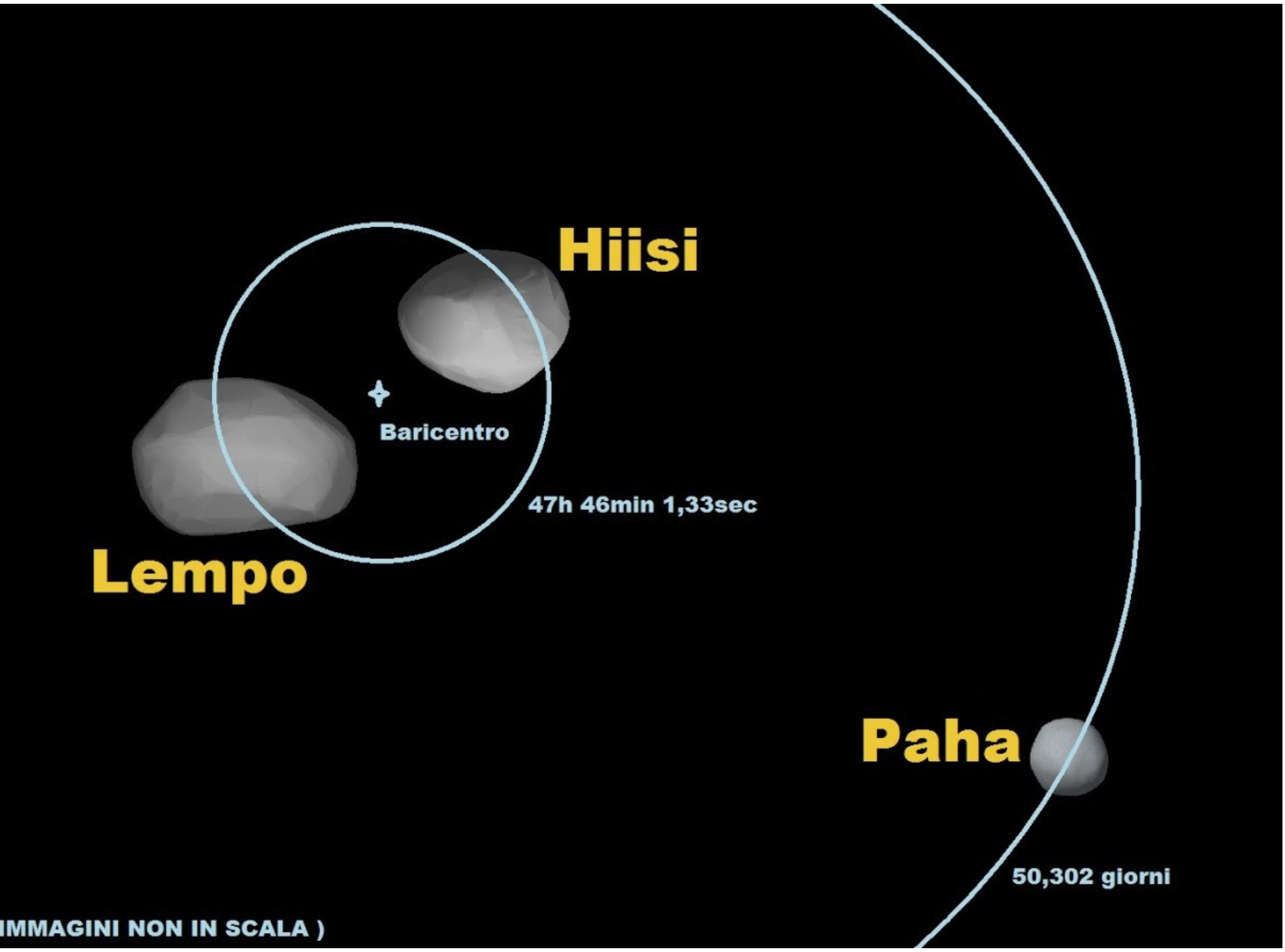


Close images of the comet surface.





A peculiar Plutino triple system: Lempo. Hiisi and Praha.



Trojan motion: effects of mass growth and migration of the planet. Trojan capture.

$$\ddot{\phi} + \frac{27}{4}\mu n_P^2 \phi = 0$$

With n_P mean motion of the planet and φ the difference between between the longitude of the planet and that of the body. This equation can be re-written in Hamiltonian form:

$$H = \frac{1}{2} a_P^2 \dot{\phi}^2 + \frac{1}{2}\omega a_P \phi^2$$

Where the libration frequency is given by

$$\omega^2 = \frac{27}{4}\mu n_P^2 \qquad \mu = \frac{m_P}{M_s + m_P}$$

It is the hamiltonian of a harmonic oscillator, the solution is

$$\phi(t) = \frac{A}{2}\cos(\omega t + \alpha)$$

Where A is the libration amplitude (the ½ is by definition of amplitude)

To test what it happens when we change either the semi-major axis or the mass of the planet we introduce an adiabatic invariant.

$$J = \int^{period} p\, dq$$

Assuming that in a period the relevant properties of the planet do not change i.e. a_p, A and ω are costant over a period T. Since:

$$p = a_P \dot{\phi} \qquad q = a_P \phi$$

the integral becomes

$$J=\int_0^{2\pi} a_P \dot{\phi}\, a_P\, d\phi = \int_0^{T} a_P^2 \frac{A^2}{4} \omega^2 \sin^2(\omega t+\alpha)\, dt$$

$$J=a_P^2 \frac{A^2}{4} \omega^2 \int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t+\alpha)\, dt$$

Recalling that $\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$

we get

$$J=a_P^2 \frac{A^2}{4} \omega^2 \frac{2\pi}{2\omega} = \frac{\pi}{4} A^2 a_P^2\, \omega =$$

$$\frac{\pi}{4} A^2 a_P^2 \sqrt{\left(\frac{27}{4}\frac{m_P}{m_P+M_s}\right)} \sqrt{\left(\frac{G(m_P+M_s)}{a_P^3}\right)} =$$

$$\frac{\pi}{4} A^2 \sqrt{\left(\frac{27}{4} G\right)}\, m_P^{1/2} a_P^{1/2}$$

The conservation of J in presence of adiabatic variations of both a_P and m_P leads to:

$$\frac{A_f}{A_i} = \left(\frac{a_{P,i}}{a_{P,f}}\right)^{(1/4)} \left(\frac{m_{P,i}}{m_{P,f}}\right)^{(1/4)}$$

This equation implies that either if the planet migrates inward or it grows in mass, the libration amplitude decreses (from Fleming & Hamilton, Icarus 148, 479, 2000) .