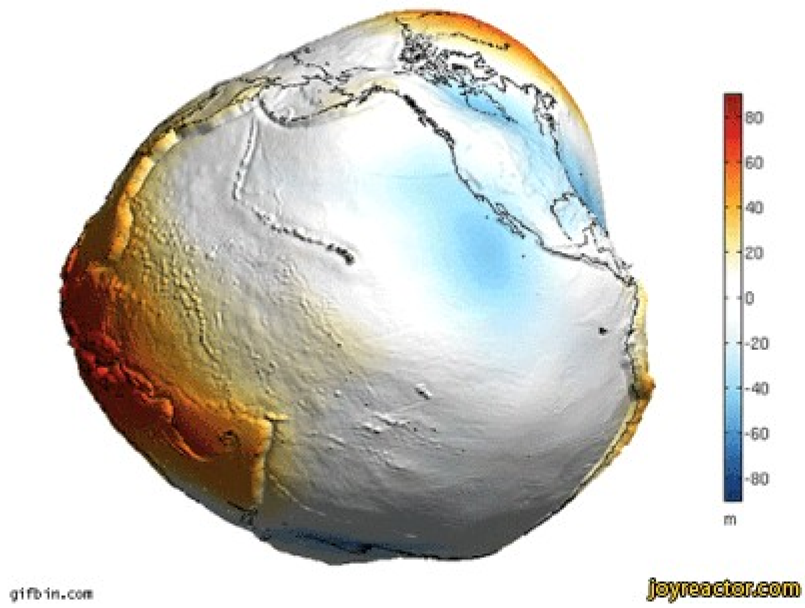
Gravity field of an irregular body: perturbations on satellite orbits.



Gravity field and Laplace equation:

$$\boldsymbol{G} = -\gamma \frac{m}{r^3}\boldsymbol{r} \qquad \boldsymbol{G} = -\nabla V$$

$$\Phi_G = \int_S \boldsymbol{G}\cdot\boldsymbol{n}\,ds = -\gamma 4\pi m \qquad \textit{Gauss theorem}$$

$$\int_S \boldsymbol{G}\cdot\boldsymbol{n}\,ds = \int_V \nabla\cdot\boldsymbol{G}\,dV \qquad \textit{Divergence theorem}$$

$$\int_V \nabla\cdot\boldsymbol{G}\,dV = -\gamma 4\pi \int_V \rho\,dV \quad \Rightarrow$$

$$\nabla\cdot\boldsymbol{G} = -4\pi\gamma\rho \quad \Rightarrow$$

$$\nabla^2 V = 4\pi\gamma\rho$$

If ρ = 0 (vacuum) then

$$\nabla^2 V = 0$$

Laplace equation

Outside a planet, the gravity field satisfies the Laplace equation which must be connected to the solution inside the planet where the density is not 0.

Solutions of the Laplace equation in spherical coordinates: $\nabla^2 V = 0$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial u}\left[(1-u^2)\frac{\partial V}{\partial u}\right] + \frac{1}{(1-u^2)}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$u = \cos\theta$$

$$V(r,u,\phi) = r^n S_n(u,\phi)$$

Separation of variables where S_n is a superficial spherical harmonic, depending only on the angles, solution of the Legendre equation

$$\frac{\partial}{\partial u}\left[(1-u^2)\frac{\partial S_n}{\partial u}\right] + \frac{1}{(1-u^2)}\frac{\partial^2 S_n}{\partial \phi^2} + n(n+1)S_n = 0$$

The general solution includes both terms in r^n and $r^{-(n+1)}$

$$V(r,u,\phi) = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right) S_n(u,\phi)$$

All A_n are =0 because they lead to terms which grow getting farther out. Only the B_n contributes to the physical solution. In case of **cylindrical symmetry** (independent from φ) the solution to the Legendre equation are the Legendre polynomials $P_n(\cos\theta)$.

$$V(r,\theta) = -\frac{Gm}{r}\left(1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\cos\theta)\right)$$

Where R is the equatorial radius.

The constant coefficients are the gravitational moments determined by the values of V at the planet surface and they are defined as:

$$J_n = -\frac{1}{m R^n} \int_0^R \int_{-1}^{1} r^n P_n(\cos\theta) \rho(r, \cos\theta) \cdot 2\pi r^2 d\cos\theta dr$$

This is a volume integral (even if bidemensional). The volume element is given by

$$dV_{ring} = 2\pi r \sin(\theta) r\, d\theta dr = -2\pi r^2 d\cos(\theta) d\theta dr$$

For n=1 → J_n = 0 since the expression for J_n calculates the CM position in the CM reference frame (i.e. 0). In case of North-South simmetry, all odd G-moments are =0.

Legendre polynomials:

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n (u^2 - 1)^n}{d u^n}$$ Rodrigues formula.

$$P_0(u) = 1$$

$$P_1(u) = u = \cos\theta$$

$$P_2(u) = \frac{1}{2}(3u^2 - 1) = \frac{1}{2}(3\cos^2\theta - 1) = \frac{1}{4}(2\cos 2\theta + 1)$$

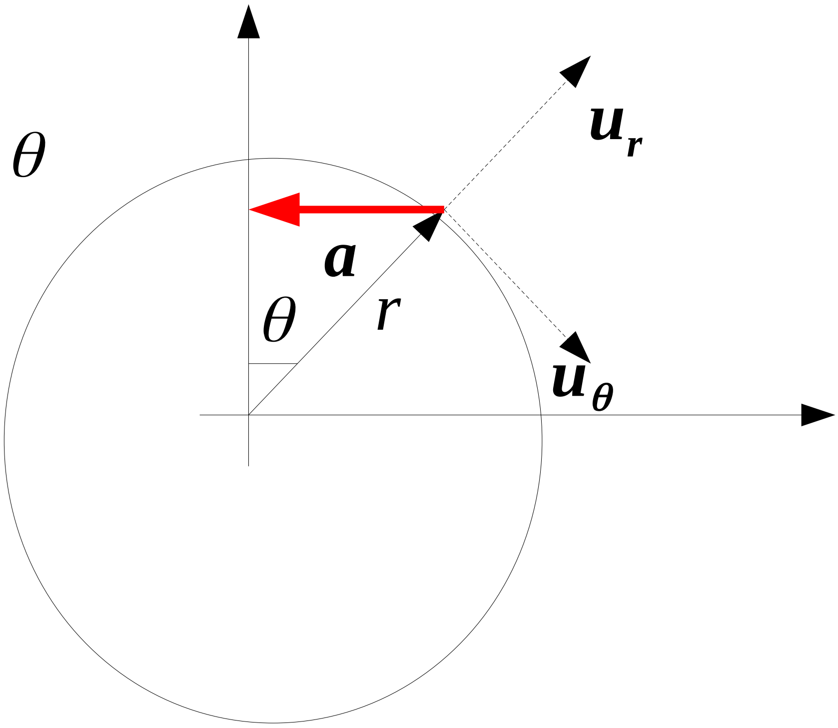
$$P_3(u) = \frac{1}{2}(5u^3 - 3u) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

Centrifugal potential

$$\boldsymbol{a} = -\Omega^2 r \sin\theta\, \boldsymbol{u} \quad \textit{centripetal acceleration}$$

$$a_r = -\Omega^2 r \sin^2\theta$$

$$a_\theta = -\Omega^2 r \sin\theta \cos\theta$$



$$\boldsymbol{F} = -\nabla V$$

$$\boldsymbol{F} = -m\,\boldsymbol{a} = -\boldsymbol{a}$$

for unit of mass

$$\nabla = \begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \phi} \end{pmatrix}$$

Centrifugal potential for mass unit:

$$V = -\frac{1}{2}\Omega^2 r^2 \sin^2\theta$$

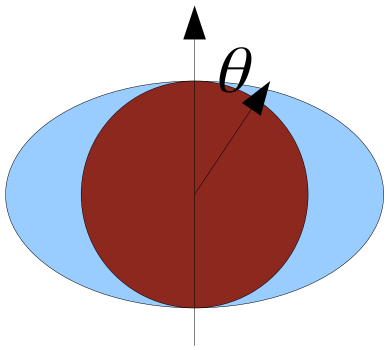
For a rotating body, the potential for mass unit is:

$$V = -\frac{GM}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta$$

Rotational deformation of a planet (pg. 150, M-D)

Initial assumption: planet covered by ocean, *a* is its equatorial radius.

$$r = a + \delta r(\theta)$$



The ocean relaxes to the equipotential surface where $V=V_0$

$$V_0 = -\frac{GM}{a} + \frac{GM}{a^2}\delta r - \frac{1}{2}\Omega^2 a^2 \sin^2\theta + -\Omega^2 a \sin^2\theta \cdot \delta r + \ldots (\delta r^2 \ \text{is small})$$

For most planets the rotation is slow so:

$$\Omega^2 a \ll \frac{GM}{a^2} \quad \rightarrow \quad q = \frac{\Omega^2 a^3}{G\,M} \ll 1$$

$$\Omega^2 a \sin^2\theta \cdot \delta r \ll \frac{GM}{a^2} \quad \text{this term is neglected}$$

$$\delta r = \left(V_0 + \frac{GM}{a}\right)\frac{a^2}{GM} + \frac{1}{2}\frac{\Omega^2 a^4}{GM}\sin^2\theta = r_0 + \frac{1}{2} q\, a \sin^2\theta$$

Flattening f

$$f = \frac{r_{eq} - r_{pol}}{r_{eq}} = \frac{1}{2}\frac{\Omega^2 a^3}{GM} = \frac{q}{2}$$

$$\left(V_0 + \frac{GM}{a}\right) \ll 1$$

Because V_0 is the superficial potential ~ -GM/a

Previous calculations do not take into account that when the planet is rotationally deformed, its gravity field is not anymore GM/r. Better approximation is to include the quadrupole term J_2.

$$V(r,\theta)=-\frac{GM}{r}+\left(\frac{a}{r}\right)^2 P_2(\cos\theta)\,J_2\frac{GM}{r}$$

$$P_2(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1) \;\Rightarrow\; P_2(\cos\theta)-1=-\frac{3}{2}\sin^2\theta$$

$$V_c(r,\theta)=-\frac{1}{2}\Omega^2 r^2\sin^2\theta=\frac{1}{3}\Omega^2 r^2\left[P_2(\cos\theta)-1\right]$$

$$V(r,\theta)=-\frac{GM}{r}+\left(J_2\frac{GMa^2}{r^3}+\frac{1}{3}\Omega^2 r^2\right)P_2(\cos\theta)\,J_2-\frac{1}{3}\Omega^2 r^2$$

Full potential in quadrupole approximation. Assuming that

$$r=a+\delta r(\theta) \;\Rightarrow\; r(\theta)=a+K-\left[J_2+\frac{1}{3}q\right]a\,P_2(\cos\theta)$$

$$f=\frac{r_{eq}-r_{pol}}{r_{eq}}=\frac{3}{2}J_2+\frac{1}{2}q$$

Earth

$$f_{computed}=0.003349$$
$$f_{observed}=0.003353$$

Jupiter

$$f_{computed}=0.06670$$
$$f_{observed}=0.06487$$

TABLE 6.2 Gravitational Moments and the Moment of Inertia Ratio.

Body	J_2 $(\times10^{-6})$	J_3 $(\times10^{-6})$	J_4 $(\times10^{-6})$	J_5 $(\times10^{-6})$	J_6 $(\times10^{-6})$	q_r	Λ_2	$I/(MR^2)$	Refs
Mercury	60 ± 20					1.0×10^{-6}	60	0.33	1
Venus	4.46 ± 0.03	-1.93 ± 0.02	-2.38 ± 0.02			6.1×10^{-8}	73	0.33	1
Earth	1 082.627	-2.532 ± 0.002	-1.620 ± 0.003	-0.21	0.65	3.45×10^{-3}	0.314	0.33	1
Moon	203.43 ± 0.09					7.6×10^{-6}	26.8	0.393	1, 2
Mars	1960.5 ± 0.2	31.5 ± 0.5	-15.5 ± 0.7			4.57×10^{-3}	0.429	0.366	1
Jupiter	14736 ± 1	0	-587 ± 5	0	31 ± 20	0.089	0.165	0.254	1
Saturn	16298 ± 10	0	-915 ± 40	0	103 ± 50	0.155	0.105	0.210	1
Uranus	3343.4 ± 0.3	0	-28.9 ± 0.5			0.029	0.113	0.23	1
Neptune	3411 ± 10	0	-35 ± 10			0.026	0.131	0.23	1
Io[a]	1863 ± 90							0.375 ± 0.005	3
Europa	438 ± 9							0.348 ± 0.002	4
Ganymede	127 ± 3							0.311 ± 0.003	5
Callisto	34 ± 5							0.358 ± 0.004	6

1: Yoder (1995). 2: Konopliv *et al.* (1998). 3: Anderson *et al.* (1996a). 4: Anderson *et al.* (1998a). 5: Anderson *et al.* (1996b). 6: Anderson *et al.* (1998b).

[a] J_2 was determined from C_{22} assuming hydrostatic equilibrium, in which case $J_2 = (10/3)C_{22}$.

(q_r: $\frac{\Omega R^3}{GM}$)

• Se simmetria NORD-SUD → i termini J_n con n dispari ≡ 0

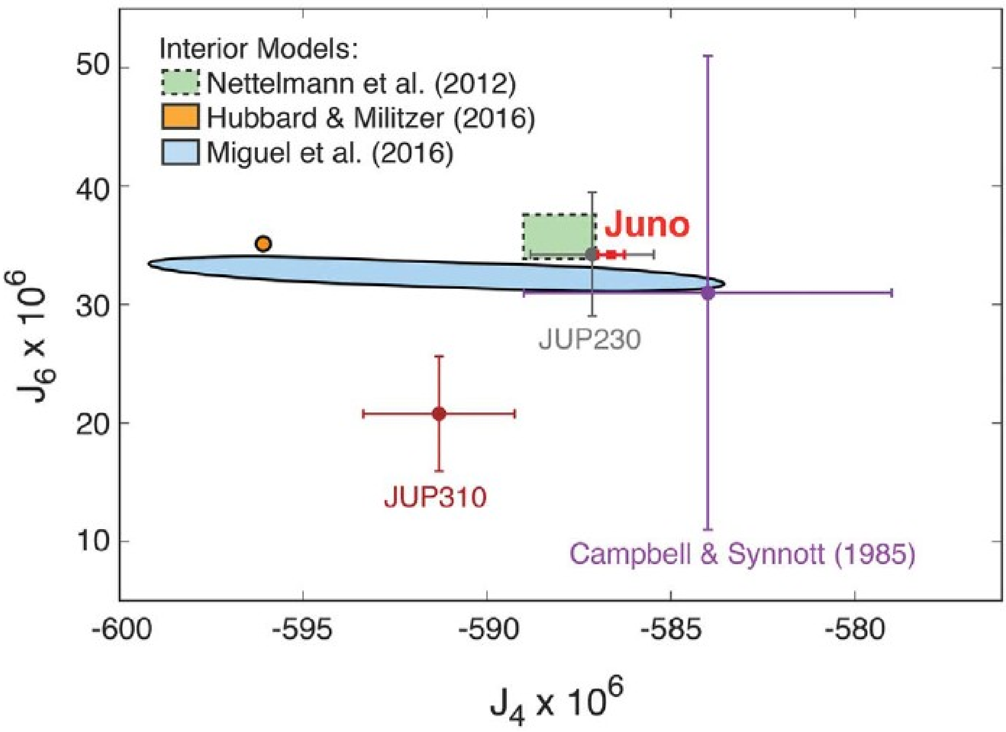


Fig. 5. Jupiter gravitational field coefficients J_4 and J_6. Pre-Juno observations from Campbell and Synnott (1985) (*26*) (purple), Jacobson (2003) (*27*) (JUP230, gray), and Jacobson (2013) (*28*) (JUP310, brown) are compared to Juno's preliminary measurement (red); values are given in table S3. Overlain are model predictions by Nettelmann *et al.* (2012) (*29*), Hubbard and Militzer (2016) (*39*), and Miguel *et al.* (2016) (*40*).

JUNO data (Bolton et al. 2017, Science 356, 821)

Disturbing function due to irregular shape of a planet.

If a planet has an irregular shape, which are the dynamical consequences on the orbit of a satellite?

$$\ddot{\mathbf{r}} = -\nabla U \qquad U = -G\frac{(M+m)}{r} \qquad 2 \;\; bodies$$

$$\ddot{\mathbf{r}}_i = -G(M+m_i)\frac{\mathbf{r}_i}{r_i^3} + G m_j \left(\frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ij}^3} - \frac{\mathbf{r}_j}{r_j^3} \right) \qquad 3 \;\; bodies$$

$$\ddot{\mathbf{r}} = -\nabla (U + R) \qquad R \quad disturbing \;\; function$$

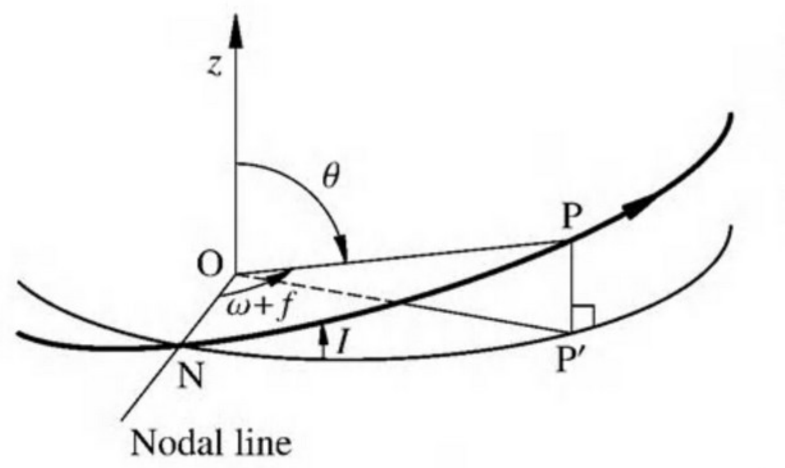
Disturbing function due to shape:

$$\ddot{\mathbf{r}}_i = -\nabla \left(U + \frac{G m}{r} J_2 \left(\frac{R}{r}\right)^2 P_2(\cos\theta) \right) = -\nabla (U + R)$$

The disturbing function R must be inserted in the Lagrange equations to compute the variations of the orbital elements with time. We have considered the J_2 term due to flattening of the planet, additional terms may be considered if the planet has a more complex shape. Next step: express R in function of the orbital elements to be used in the Lagrange equation.

$$V(r,\theta) = -\frac{GM}{r}\left(1 - J_2\left(\frac{R}{r}\right)^2 P_2(\cos\theta)\right)$$

$$R = \frac{GM}{r} J_2\left(\frac{R}{r}\right)^2 P_2(\cos\theta) = \frac{GMJ_2R^2}{2r^3}(3\cos^2\theta - 1)$$



From spherical trigonometry,

$$\cos(\theta) = \sin(i)\sin(\omega + f)$$

$$A = \frac{GMJ_2R^2}{2}$$

$$R = \frac{A}{r^3}(3\sin^2(i)\sin^2(\omega+f) - 1) =$$

$$\frac{A}{r^3}\left(\frac{3}{2}\sin^2(i)(1-\cos(2\omega+2f)) - 1\right)$$

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\sin^2(\omega+f) = \frac{1-\cos(2\omega+2f)}{2}$$

From Bertotti, Farinella & Vokroulicki, Physics of the Solar System, pg. 332

The disturbing function is averaged on the short period of the mean anomaly M of the satellite:

$$\overline{R}=\frac{1}{2\pi}\int_0^{2\pi}\frac{A}{r^3}\left(\frac{3}{2}\sin^2(i)(1-\cos(2\omega+2f))-1\right)dM$$

Since $\cos(2\omega+2f)=\cos(2\omega)\cos(2f)-\sin(2\omega)\sin(2f)$ *prostaferesi*

$$\overline{R}=\frac{A}{2\pi}\left[\left(\frac{3}{2}\sin^2(i)-1\right)\int_0^{2\pi}\frac{dM}{r^3}+\right.$$

$$+\frac{3}{2}\sin^2(i)\int_0^{2\pi}\cos(2\omega)\cos(2f)\frac{dM}{r^3}+$$

$$\left.-\frac{3}{2}\sin^2(i)\int_0^{2\pi}\sin(2\omega)\sin(2f)\frac{dM}{r^3}\right]$$

$$\frac{dM}{r^3}=\frac{1}{r^3}\frac{dM}{df}\cdot df$$

$$\frac{dM}{df}=n\frac{dt}{df}=\frac{n}{\dot{f}}$$

$$\frac{dM}{r^3}=\frac{n}{r^3\dot{f}}df=\frac{n}{(r^2\dot{f})r}df=\frac{n}{rh}df$$

$$r=\frac{a(1-e^2)}{(1+e\cos f)}$$

$$h=\sqrt{(\mu a(1-e^2))}$$

$$\frac{n}{h}=\frac{1}{a^2\sqrt{(1-e^2)}}$$

$$\frac{dM}{r^3}=\frac{n(1+e\cos f)}{ha(1-e^2)}df=\frac{(1+e\cos f)}{a^3(1-e^2)^{3/2}}df$$

$$\int_0^{2\pi}\frac{dM}{r^3}=\int_0^{2\pi}\frac{(1+e\cos f)}{a^3(1-e^2)^{3/2}}df=\frac{2\pi}{a^3(1-e^2)^{3/2}}$$

$$\int_0^{2\pi} \frac{\cos(2\omega)\cos(2f)(1+e\cos(f))}{a^3(1-e^2)^{3/2}} df \quad =$$

$$\frac{\cos(2\omega)}{a^3(1-e^2)^{3/2}} \int_0^{2\pi} \cos(2f)\, df + e \int_0^{2\pi} \cos(2f)\cos(f)\, df = 0$$

$$\sin(2\omega) \int_0^{2\pi} \sin(2f) \frac{dM}{r^3} \quad =$$

$$\sin(2\omega) \int_0^{2\pi} \frac{\sin(2f)(1+e\cos(f))}{a(1-e^2)^{3/2}} df = 0$$

$$\overline{R} = \frac{GMJ_2 R^2}{2\cdot 2\pi} \left(\frac{3}{2}\sin^2(i) - 1 \right) \frac{2\pi}{a^3(1-e^2)^{3/2}}$$

$$\overline{R} = \frac{GMJ_2}{2a^3} R^2 \left(\frac{3}{2}\sin^2(i) - 1 \right) \frac{1}{(1-e^2)^{3/2}}$$

The disturbing function is inserted in the Lagrange equations to get the variations of the orbital elements.

$$\frac{da}{dt}=\frac{2}{na}\frac{\partial R}{\partial \lambda}$$

$$\frac{de}{dt}=-\frac{\sqrt{1-e^2}}{na^2 e}(1-\sqrt{1-e^2})\frac{\partial R}{\partial \lambda}-.\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \varpi}$$

$$\frac{di}{dt}=-\frac{tg(i/2)}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \lambda}+\frac{\partial R}{\partial \varpi}\right)-\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega}$$

$$\frac{d\Omega}{dt}=-\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}$$

$$\frac{d\varpi}{dt}=\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e}+\frac{tg(i/2)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}$$

$$\frac{d\epsilon}{dt}=-\frac{2}{na}\frac{\partial R}{\partial a}+\frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2 e}\frac{\partial R}{\partial e}+\frac{tg(i/2)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}$$

Changes in the osculating orbital elements due to the J_2 term

$$\frac{da}{dt}=0 \quad \frac{de}{dt}=0 \quad \frac{di}{dt}=0$$

The shape of the orbit does not change, but the orientation does….

$$\frac{d\tilde{\varpi}}{dt}=3nJ_2\left(\frac{R}{a}\right)^2\frac{\left(1-\frac{5}{4}\sin(i)^2\right)}{(1-e^2)^2}$$

It is = 0 when

$$\sin(i)=\frac{2}{\sqrt{5}}\Rightarrow i\sim 63.4^o$$

$$\frac{d\Omega}{dt}=-\frac{3}{2}nJ_2\left(\frac{R}{a}\right)^2\frac{\cos(i)}{(1-e^2)^2}$$

It is =0 when i=90°
maximum when i=0°

Very good predictions for artificial satellites and the satellited of the outer planets. For the Moon, the solar perturbations are strong and change the values of both

$P_{\tilde{\varpi}},\ P_{\Omega}$

$$P_{\tilde{\varpi}}=5.98\ \ yr$$
$$P_{\Omega}=18.60\ \ yr$$

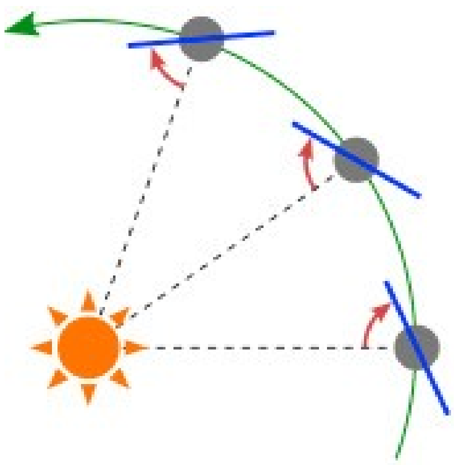
They are smaller than predicted by the above equations.

SSO: Sun Synchronous Orbits

$$\frac{d\Omega}{dt}=-\frac{3}{2}nJ_2\left(\frac{R}{a}\right)^2\frac{\cos(i)}{(1-e^2)^2}=\frac{2\pi}{365.25}$$

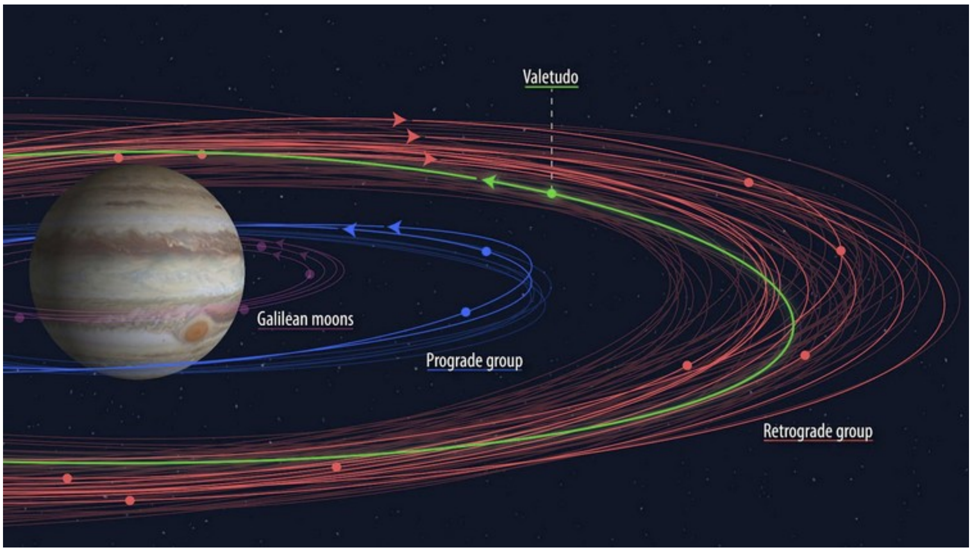
Orbits per day	Period (h)		Altitude (km)	Maximal latitude	Inclin-ation
16	$1\frac{1}{2}$	= 1:30	274	83.4°	96.6°
15	$1\frac{3}{5}$	= 1:36	567	82.3°	97.7°
14	$1\frac{5}{7}$	≈ 1:43	894	81.0°	99.0°
13	$1\frac{11}{13}$	≈ 1:51	1262	79.3°	100.7°
12	2		1681	77.0°	103.0°
11	$2\frac{2}{11}$	≈ 2:11	2162	74.0°	106.0°
10	$2\frac{2}{5}$	= 2:24	2722	69.9°	110.1°
9	$2\frac{2}{3}$	= 2:40	3385	64.0°	116.0°
8	3		4182	54.7°	125.3°
7	$3\frac{3}{7}$	≈ 3:26	5165	37.9°	142.1°

i must be larger than 90° to grant positive precession.



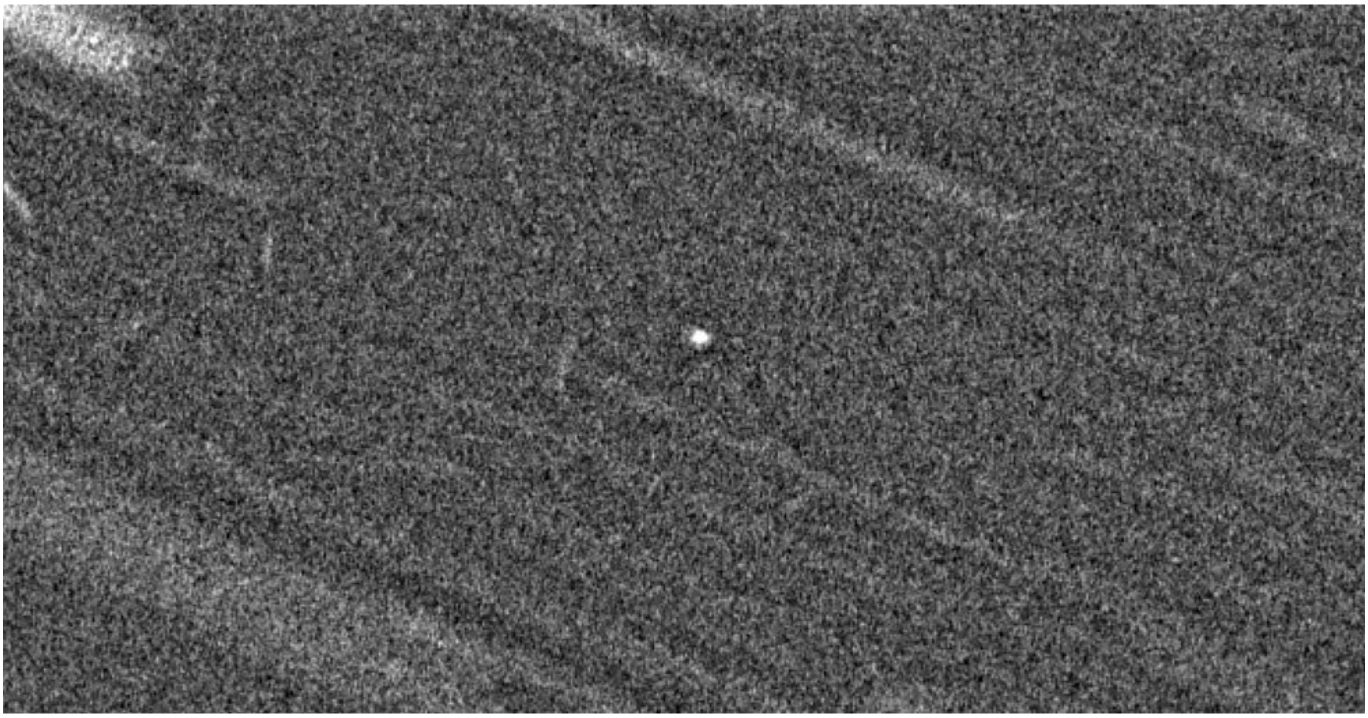
a) Sun-synchronous orbit

Jupiter rich satellite system.



Orbits of confirmed 79 moons of Jupiter….. BUT…...

Jupiter might have more than 500 irregular moons larger than 800 m in diameter



2) ESEMPIO 2: SATELLITI GPS

$a = 26400$ Km $\quad e = 0 \quad i = 55.5^\circ$

$n = 1.472 \times 10^{-4}\ s^{-1}$ T ~ 12 hr

$$\frac{d\Omega}{dt} = -\frac{3}{2} n J_2 \left(\frac{R}{a}\right)^2 \frac{\cos i}{(1-e^2)^2} = -7.903 \times 10^{-9}\ s^{-1}$$

$T_\Omega = 25$ yr

$$\frac{d\tilde{\omega}}{dt} = 3 n J_2 \left(\frac{R}{a}\right)^2 \frac{\left(1 - \frac{5}{4}\sin^2 i\right)}{(1-e^2)^2} = 4.205 \times 10^{-9}\ s^{-1}$$

$T_{\tilde{\omega}} = 47.3$ yr

Non ha senso [Se $e = 0$, $\tilde{\omega} = \Omega$]

Aumenta a → diminuisce rilevanza del J_2, più lenta la precessione.

	a (Km)	e	i	T_Ω
Metis	127.979	0	0.001°	42 day
Io	421 600	0.004	0.04°	138 day
Callisto	1.883.000	0.007	0.281°	24 yr

Satelliti di Giove.

Metis è destinato a impattare su Giove perche' interno all'orbita sincrona. Forse e sorgente degli anelli di Giove.