# CAPITOLO 8

- Secular theory for 2 planets
- Secular theory for the solar system
- Minor bodies secular evolution
- Secular resonances
- Chaos

**Solar System Dynamics, Murray & Dermott**

# Equations for the N-body problem:

$$\frac{d^2\mathbf{r_i}}{dt^2} = -G(M_0+m_i)\frac{\mathbf{r_i}}{r_i^3} - G\sum_{j=1,\, j\neq 1}^{N} m_j\left(\frac{\mathbf{r_j}-\mathbf{r_i}}{r_{i,j}^3} - \frac{\mathbf{r_j}}{r_j^3}\right)$$

**For only two planets they become:**

$$\frac{d^2\mathbf{r_1}}{dt^2} = -G(M_0+m_1)\frac{\mathbf{r_1}}{r_1^3} - G\,m_2\left(\frac{\mathbf{r_2}-\mathbf{r_1}}{r_{1,2}^3} - \frac{\mathbf{r_2}}{r_2^3}\right)$$

..and the same for planet 2. It can be written in a more compact form

$$\ddot{\mathbf{r}}_{\mathbf{i}} = \nabla_i(U_i+R_i)$$

With i=1,2

$$U_i = G\frac{(M_0+m_i)}{r_i}$$

$$R_i = G\frac{m_j}{|\mathbf{r_j}-\mathbf{r_i}|} - G\,m_j\frac{\mathbf{r_i}\cdot\mathbf{r_j}}{r_j^3}$$

**The first is the Keplerian term while R is the disturbing function made of a direct (the first) and indirect (the second) terms. The second is due to the use of a reference frame centered on the Sun giving origin to non-inertial forces.**

AIM: write equations for the evolution with time of the orbital elements (Lagrange equations) due to R. The first step is to write R as a function of orbital elements using a Fourier expansion since it is periodic in the angles.

$$R_{1,2} = G\, m_2 \sum_{j_1, j_2, l_1, l_2, m_1, m_2} S^{j_1, j_2, l_1, l_2, m_1, m_2}(a_1, a_2, e_1, e_2, i_1, i_2)$$
$$\cos(J_1 \lambda_1 + j_2 \lambda_2 + l_1 \varpi_1 + l_2 \varpi_2 + m_1 \Omega_1 + m_2 \Omega_2)$$

Now the Lagrange equations can be used to compute the variations of the orbital elements:

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial \lambda}$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}(1-\sqrt{1-e^2}) \frac{\partial R}{\partial \lambda} - . \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial \varpi}$$

$$\frac{di}{dt} = -\frac{tg(i/2)}{na^2\sqrt{1-e^2}} \left( \frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \varpi} \right) - \frac{1}{na^2\sqrt{1-e^2}\sin i} \frac{\partial R}{\partial \Omega}$$

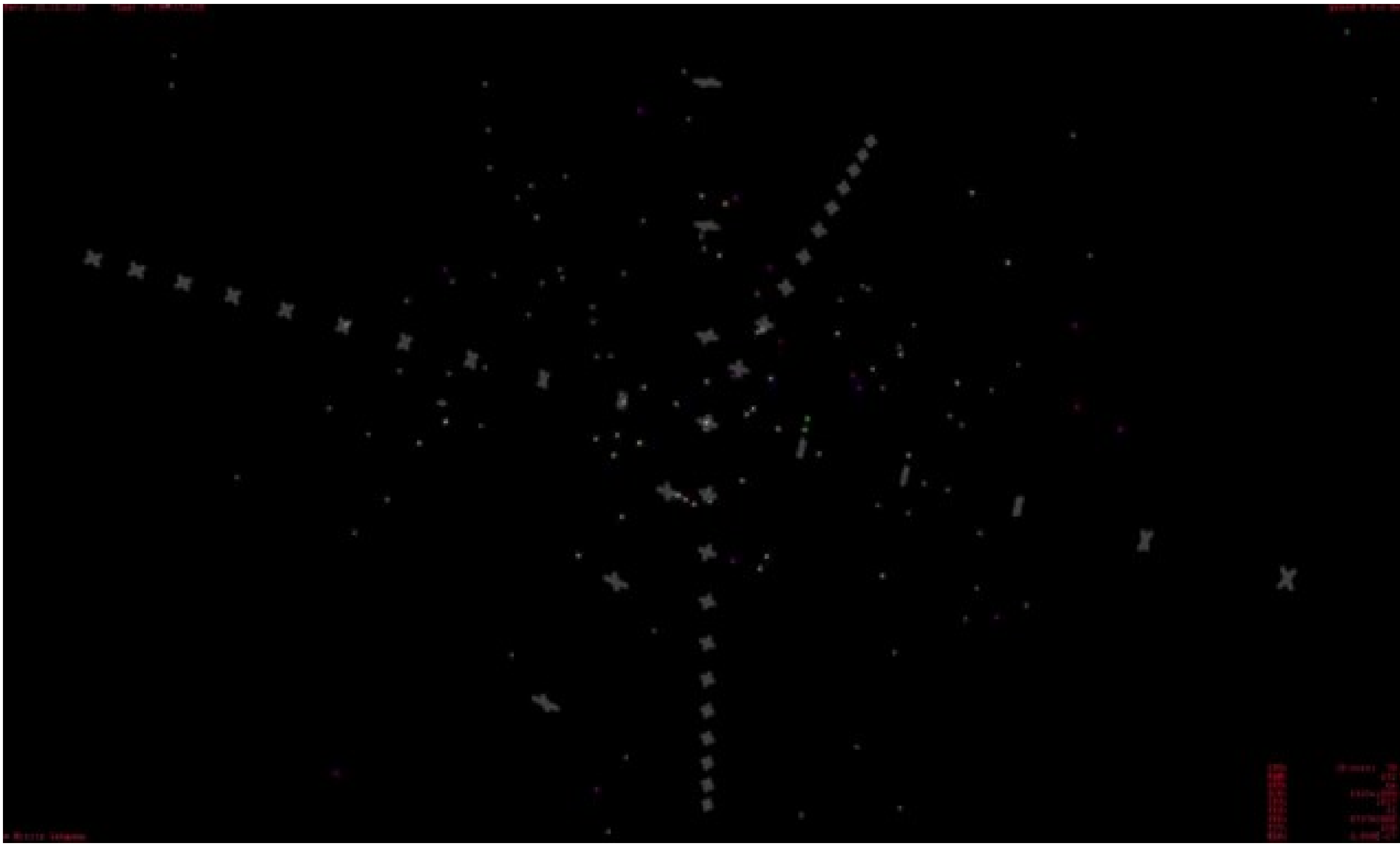
$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i} \frac{\partial R}{\partial i}$$

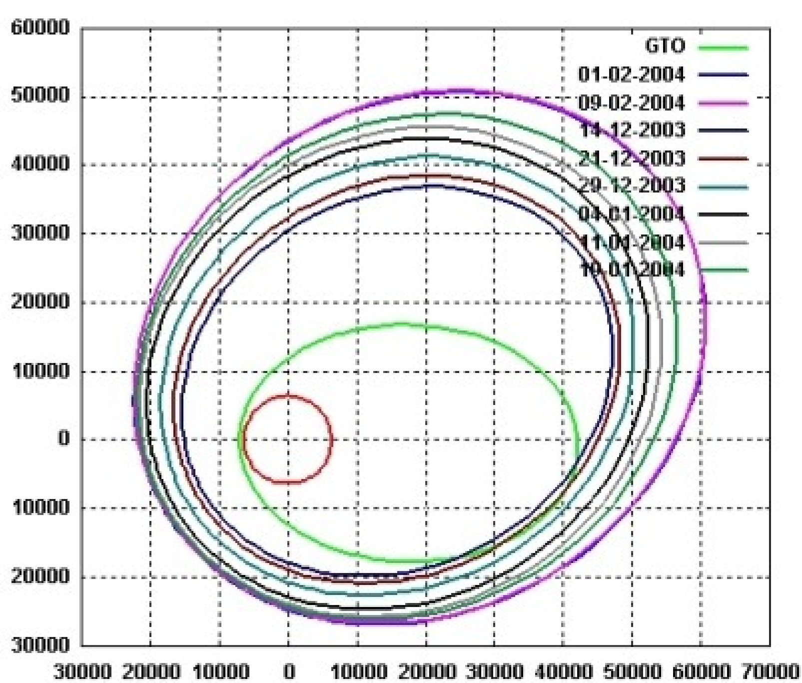
$$\frac{d\varpi}{dt} = \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial e} + \frac{tg(i/2)}{na^2\sqrt{1-e^2}} \frac{\partial R}{\partial i}$$

$$\frac{d\epsilon}{dt} = -\frac{2}{na} \frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2 e} \frac{\partial R}{\partial e} + \frac{tg(i/2)}{na^2\sqrt{1-e^2}} \frac{\partial R}{\partial i}$$

ε denotes the mean longitude at epoch i.e. the longitude of the body at the moment in which time is started.

# Osculating orbits





## ..on how to derive Lagrange's equations.

$$x = x(a, e, i, \tilde{\omega}, \Omega, \epsilon; t)$$
$$y = y(a, ....; t)$$
$$z = z(a, ....; t)$$

$$\ddot{\boldsymbol{r}} = \nabla(U_K + R)$$

**$U_K$ is the Keplerian 2-body potential.**

$$\frac{dx}{dt} = \sum_{i=1}^{6} \frac{\partial x}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} + \frac{\partial x}{\partial t}$$

**The same for y and z, 3 equations.**

**For an osculating orbit, which locally approximates the evolution of r for a limited amount of time**

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} \quad \rightarrow \quad \sum_{i=1}^{6} \frac{\partial x}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} = 0 \quad 3 \;\; equations$$

$$\frac{d^2 x}{dt^2} = \sum_{i=1}^{6} \frac{\partial \dot{x}}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} + \frac{\partial \dot{x}}{\partial t}$$

$$\frac{\partial \dot{x}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{dx}{dt}\right) = \frac{\partial^2 x}{\partial t^2}$$

**For the osculating orbital elements.**

$$\sum_{i=1}^{6} \frac{\partial \dot{x}}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t} = \frac{\partial R}{\partial x}$$

**...and similar equations along y and z.**

**In all, 6 equations for the 6 orbital parameters!**

**To obtain simple equations from the Fourier development, 1) the disturbing function is averaged over the fast angles (the anomalies of the two planets)**

$$\bar{R}_i = \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} R_{i,j} \; d\lambda_1 d\lambda_2$$

After the averaging the semimajor axis of the planets are constant since

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial \bar{R}}{\partial \lambda} = 0$$

**And 2) only 2nd order terms in eccentricity and inclination are retained (2nd order theory). R becomes then:**

$$\bar{R}_j = n_j a_j^2 \left[ \frac{1}{2} A_{jj} e_j^2 + A_{jk} e_j e_k \cos(\varpi_1 - \varpi_2) \right.$$
$$\left. \frac{1}{2} B_{jj} i_j^2 + B_{jk} i_j i_k \cos(\Omega_1 - \Omega_2) \right]$$

**The coefficients $A_{jk}$ depend only on the masses and semimajor axes of the planets and are then constant in this approximation.**

$$A_{jj}=\frac{n_j}{4}\frac{m_j}{M_0+m_j}\alpha_{12}\bar{\alpha}_{12}b_{3/2}^{(1)}(\alpha_{12})$$

$$A_{jk}=-\frac{n_j}{4}\frac{m_k}{M_0+m_j}\alpha_{12}\bar{\alpha}_{12}b_{3/2}^{(2)}(\alpha_{12})$$

$$B_{jj}=-\frac{n_j}{4}\frac{m_j}{M_0+m_j}\alpha_{12}\bar{\alpha}_{12}b_{3/2}^{(1)}(\alpha_{12})$$

$$B_{jk}=\frac{n_j}{4}\frac{m_k}{M_0+m_j}\alpha_{12}\bar{\alpha}_{12}b_{3/2}^{(1)}(\alpha_{12})$$

**The differences between the A and B coefficients are in the sign and in the Laplace coefficients *b*. In $B_{JK}$ *b* is the same as in $B_{JJ}$ while in $A_{JK}$ it is different from that in $A_{JJ}$. The Laplace coefficients are defined as:**

$$\frac{1}{2}b_s^{(j)}(\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(j\psi)}{(1-2\alpha\cos(\psi)+\alpha^2)^s}d\psi$$

**While the constants $\alpha_{ij}$ are defined as:**

$$if\ a_1<a_2 \quad \alpha_{12}=\frac{a_1}{a_2} \qquad\qquad \begin{array}{ll} if\ j=1 & \bar{\alpha}_{12}=\alpha_{12} \\ if\ j=2 & \bar{\alpha}_{12}=1 \end{array}$$

Chapter 6, Murray & Dermott, Solar System Dynamics.

$$\frac{de_j}{dt} = -\frac{1}{n_j a_j^2 e_j} \frac{\partial \bar{R}_{ji}}{\partial \varpi_j}$$

$$\frac{di_j}{dt} = -\frac{1}{n_j a_j^2 i_j} \frac{\partial \bar{R}_{ji}}{\partial \Omega_j}$$

$$\frac{d\Omega_j}{dt} = \frac{1}{n_j a_j^2 i_j} \frac{\partial \bar{R}_{ji}}{\partial i_j}$$

$$\frac{d\varpi_j}{dt} = \frac{1}{n_j a_j^2 e_j} \frac{\partial \bar{R}_{ji}}{\partial e_j}$$

**n.b. In deriving these equations, terms proportional to powers higher than the second in eccentricity and inclination ($e_j^3$, $i_j^3$,….) are dropped. For example, the following term is neglected because higher order terms in *i,e* come out.**

**As an example, in the derivative of i there is the following term which is neglected because of higher order.**

$$\frac{tg\left(\frac{i}{2}\right)}{n_j a_j^2 \sqrt{(1-e_j^2)}} \frac{\partial \bar{R}_{j,i}}{\partial \varpi_j} =$$

$$\pm \frac{tg\left(\frac{i}{2}\right)}{n_j a_j^2 \sqrt{(1-e_j^2)}} n_j a_j^2 A_{ji} e_j e_i \sin(\varpi_1 - \varpi_2) \propto e^2 \cdot i$$

Previous equations have a problem: when either e or i are small and tend to 0 they become singular (1/0). To prevent this from happening, new non-singular variables are introduced.

$$h_j = e_j \sin(\varpi_j) \qquad p_j = i_j \sin(\Omega_j)$$
$$k_j = e_j \cos(\varpi_j) \qquad q_j = i_j \cos(\Omega_j)$$

The averaged second-order distubing function in these variables becomes

$$\bar{R}_j = n_j a_j^2 \left[ \frac{1}{2} A_{jj} (h_j^2 + k_j^2) + A_{jk} (h_j h_i + k_j k_i) \right.$$
$$\left. \frac{1}{2} B_{jj} (p_j^2 + q_j^2) + B_{jk} (p_j p_i + q_j q_i) \right]$$

$$\textit{remember}: \quad \cos(x-y) = \cos x \cos y + \sin x \sin y$$

7)

$$\dot{h}_1 = \frac{\partial h_1}{\partial e_1}\frac{de_1}{dt} + \frac{\partial h_1}{\partial \tilde{\omega}_1}\cdot\frac{d\tilde{\omega}_1}{dt} = \frac{h_1}{e_1}\dot{e}_1 + K_1\dot{\tilde{\omega}}_1 =$$

$$= -\frac{h_1}{m_1 a_1^2 e_1^2}\frac{\partial \bar{R}_1}{\partial \tilde{\omega}} + \frac{K_1}{m_1 a_1^2 e_1}\frac{\partial \bar{R}}{\partial e_1} =$$

$$= \frac{h_1}{m_1 a_1^2 e_1^2}\left(m_1 a_1^2 A_{12} e_1 e_2 \sin(\tilde{\omega}_1 - \tilde{\omega}_2)\right) +$$

$$+ \frac{K_1}{m_1 a_1^2 e_1} m_1 a_1^2 \left(A_{11} e_1 + A_{12} e_2 \cos(\tilde{\omega}_1 - \tilde{\omega}_2)\right) =$$

$$= \frac{e_2}{e_1} h_1 A_{12} \sin(\tilde{\omega}_1 - \tilde{\omega}_2) + K_1 A_{11} +$$

$$+ K_1 A_{12} \frac{e_2}{e_1} \cos(\tilde{\omega}_1 - \tilde{\omega}_2) =$$

$$= K_1 A_{11} + \Big( A_{12} e_2 \sin(\tilde{\omega}_1) \sin(\tilde{\omega}_1 - \tilde{\omega}_2) +$$

$$+ e_2 A_{12} \cos(\tilde{\omega}_1) \cos(\tilde{\omega}_1 - \tilde{\omega}_2) =$$

$$= K_1 A_{11} + A_{12} e_2 \Big( \sin(\tilde{\omega}_1) \sin(\tilde{\omega}_1) \cos(\tilde{\omega}_2) +$$

$$- \sin(\tilde{\omega}_1) \cos(\tilde{\omega}_1) \sin(\tilde{\omega}_2) + \cos(\tilde{\omega}_1) \cos(\tilde{\omega}_1) \cos(\tilde{\omega}_2) +$$

$$+ \sin(\tilde{\omega}_1) \cos(\tilde{\omega}_1) \sin(\tilde{\omega}_2) \Big) = K_1 A_{11} + K_2 A_{12}$$

The final equations for h,k,p and q are a system of first order differential equations with constant coefficients:

$$\dot{h}_1 = A_{11} k_1 + A_{12} k_2 \qquad = \frac{1}{n_1 a_1^2} \frac{\partial \bar{R}_1}{\partial k_1}$$

$$\dot{h}_2 = A_{21} k_1 + A_{22} k_2 \qquad = \frac{1}{n_2 a_2^2} \frac{\partial \bar{R}_2}{\partial k_2}$$

$$\dot{k}_1 = -A_{11} h_1 - A_{12} h_2 \qquad = -\frac{1}{n_1 a_1^2} \frac{\partial \bar{R}_1}{\partial h_1}$$

$$\dot{k}_2 = -A_{21} h_1 - A_{22} k_2 \qquad = -\frac{1}{n_2 a_2^2} \frac{\partial \bar{R}_2}{\partial h_2}$$

A more compact form is:

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = A \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \qquad \begin{pmatrix} \dot{k}_1 \\ \dot{k}_2 \end{pmatrix} = -A \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

It is called linear theory because the elements of the matrix A are constants. The solution is a combination of sin & cos

$$h_j = \sum_{i=1}^{2} e_{ji} \sin(g_i t + \beta_i)$$

$$k_j = \sum_{i=1}^{2} e_{ji} \cos(g_i t + \beta_i)$$

$$g_i = eigenvalues \ \ of \ \ A$$

$$e_{ji} = eigenvectors \ \ of \ \ A$$

For the variables p and q,

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = B \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = -B \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$p_j = \sum_{i=1}^{2} i_{ji} \sin(f_i t + \gamma_i)$$

$$q_j = \sum_{i=1}^{2} i_{ji} \cos(f_i t + \gamma_i)$$

$$f_i = eigenvalues\ \ of\ \ B$$
$$i_{ji} = eigenvectors\ \ of\ \ B$$

However, one eigenvalue of B is =0!

$$\begin{pmatrix} B_{11}-f & B_{12} \\ B_{21} & B_{22}-f \end{pmatrix} = 0 \ \Rightarrow f^2 - f(B_{11}+B_{22}) + B_{11}B_{22} - B_{12}B_{21} = 0$$

But since in the matrix B there is only one Laplace coefficient, the product:

$$B_{11}B_{22} - B_{12}B_{21} = 0$$

$$f_1 = 0 \qquad f_2 = B_{11} + B_{22}$$

**This is a consequence of the angular momentum conservation of the 2-planet system. If the inclination of one planet changes, the other must act to maintain constant the initial angular momentum. Only 1 degree of freedom.**

# Solution for Jupiter and Saturn

$$m_1/m_c = 9.54786 \times 10^{-4} \qquad m_2/m_c = 2.85837 \times 10^{-4}$$

$$a_1 = 5.202545 \text{ AU} \qquad a_2 = 9.554841 \text{ AU}$$

$$n_1 = 30.3374\ ^\circ\text{y}^{-1} \qquad n_2 = 12.1890\ ^\circ\text{y}^{-1}$$

$$e_1 = 0.0474622 \qquad e_2 = 0.0575481$$

$$\varpi_1 = 13.983865\ ^\circ \qquad \varpi_2 = 88.719425\ ^\circ$$

$$I_1 = 1.30667\ ^\circ \qquad I_2 = 2.48795\ ^\circ$$

$$\Omega_1 = 100.0381\ ^\circ \qquad \Omega_2 = 113.1334\ ^\circ$$

$$\alpha = a_1/a_2 = 0.544493$$

$$b_{3/2}^{(1)} = 3.17296 \qquad b_{3/2}^{(2)} = 2.07110$$

$$\mathbf{A} = \begin{pmatrix} +0.00203738 & -0.00132987 \\ -0.00328007 & +0.00502513 \end{pmatrix} \ ^\circ\text{y}^{-1}$$

$$\mathbf{B} = \begin{pmatrix} -0.00203738 & +0.00203738 \\ +0.00502513 & -0.00502513 \end{pmatrix} \ ^\circ\text{y}^{-1}$$

## Fundamental frequencies:

$g_1 = 9.63435 \times 10^{-4}\ ^\circ\mathrm{y}^{-1}$ $\qquad g_2 = 6.09908 \times 10^{-3}\ ^\circ\mathrm{y}^{-1}$

$f_1 = 0$ $\qquad f_2 = -7.06251 \times 10^{-3}\ ^\circ\mathrm{y}^{-1}$

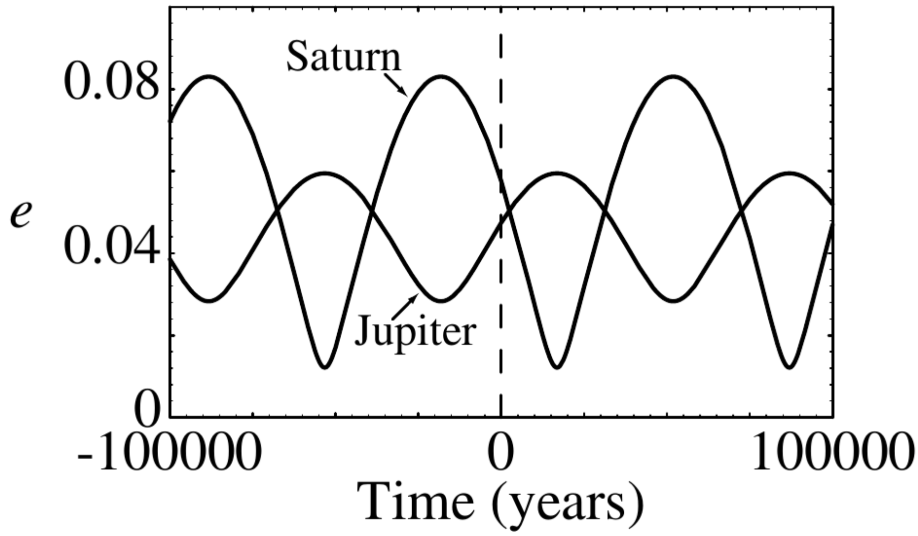
## Eigenvectors:

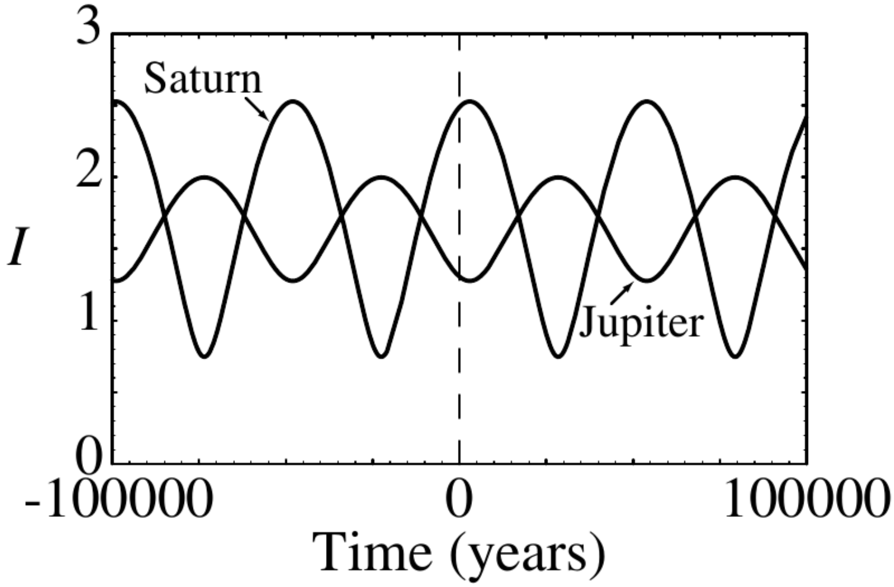
$$\begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} = \begin{pmatrix} -0.0438821 \\ -0.0354375 \end{pmatrix} \qquad \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix} = \begin{pmatrix} 0.0155788 \\ -0.047581 \end{pmatrix}$$

$$\begin{pmatrix} I_{11} \\ I_{21} \end{pmatrix} = \begin{pmatrix} 0.0285689 \\ 0.0285689 \end{pmatrix} \qquad \begin{pmatrix} I_{12} \\ I_{22} \end{pmatrix} = \begin{pmatrix} -0.00629766 \\ 0.015533 \end{pmatrix}$$

## Constant coefficients:

$\beta_1 = -146.892^\circ \quad \beta_2 = -53.3565^\circ \quad \gamma_1 = 105.74^\circ \quad \gamma_2 = 126.825^\circ$




$$e_1(t)^2 = (e_{11}^2 + e_{12}^2) + 2\, e_{11}\, e_{12} \cos((g_1 - g_2)\, t + \beta_1 - \beta_2)$$
$$e_2(t)^2 = (e_{21}^2 + e_{22}^2) + 2\, e_{21}\, e_{22} \cos((g_1 - g_2)\, t + \beta_1 - \beta_2)$$

**The eccentricities of the two planets oscillate in antiphase with the same frequency $g_1$ - $g_2$**

## The secular solution for all 8 planets of the solar system is:

$$h_j = e_j \sin(\varpi_j) = \sum_{k=1}^{N} M_{jK} \sin(g_k t + \beta_k)$$

$$k_j = e_j \cos(\varpi_j) = \sum_{i=1}^{N} M_{jk} \cos(g_k t + \beta_k)$$

$$p_j = \sin(i_j/2) \sin(\Omega_j) = \sum_{k=1}^{N} N_{jK} \sin(f_k t + \gamma_k)$$

$$q_j = \sin(i_j/2) \cos(\Omega_j) = \sum_{i=1}^{N} N_{jk} \cos(f_k t + \gamma_k)$$

| $k$ | $g_k$ | $\beta_k(^\circ)$ | $s_k$ | $\delta_k(^\circ)$ |
|---|---|---|---|---|
| 1 | 5.5964 | 112.08 | −5.6174 | 348.60 |
| 2 | 7.4559 | 200.51 | −7.0795 | 273.25 |
| 3 | 17.3646 | 305.12 | −18.8512 | 240.20 |
| 4 | 17.9156 | 335.38 | −17.7482 | 303.75 |
| 5 | 4.2575 | 30.65 | 0.0000 | 107.58 |
| 6 | 28.2455 | 128.09 | −26.3450 | 307.29 |
| 7 | 3.0868 | 121.36 | −2.9927 | 320.62 |
| 8 | 0.6726 | 74.06 | −0.6925 | 203.90 |

**List of fundamental frequencies of the SS given in arcsec/year and computed respect to the ecliptic reference frame. S5 is 0 for the conservation of angular momentum reducing the number of independent frequencies to 7 for i,Ω.**

Table 7.2: Coefficients $M_{j,k}$ of the Lagrange–Laplace type solution (7.10). The values with $i \geq 5$ and $k \geq 5$ are taken from Nobili *et al.* (1989); the others are taken from Laskar (1990), when reported, or Bretagnon (1974) otherwise. All values have been multiplied by $10^6$.

| $j \backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 185444 | −27700 | 1458 | −1428 | 36353 | 113 | 623 | 7 |
| 2 | 6668 | 20733 | −11671 | 13464 | 19636 | −551 | 614 | 11 |
| 3 | 4248 | 16047 | 9406 | −13159 | 18913 | 1506 | 650 | 12 |
| 4 | 650 | 2917 | 40133 | 49032 | 20300 | 7030 | 862 | 20 |
| 5 | −7 | −12 | −1 | 0 | 44187 | −15700 | 1814 | 58 |
| 6 | −6 | −12 | −7 | −7 | 32958 | 48209 | 1511 | 57 |
| 7 | 2 | 3 | 0 | 0 | −37587 | −1547 | 29033 | 1666 |
| 8 | 0 | 0 | 0 | 0 | 1881 | −103 | −3697 | 9118 |

**From "Modern Celestial Mechanics", Morbidelli 2002, Taylor&Francis**

Table 7.3: The same as Table 7.2, but for the coefficients $N_{j,k}$. The ecliptic reference frame for the year 2000 has been used for the computation of $N_{j,5}$.

| $j \backslash k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 39957 | 30169 | 1678 | 72261 | 13772 | −139 | −1665 | −724 |
| 2 | 6716 | −4045 | −9544 | −5759 | 13772 | −60 | −959 | −663 |
| 3 | 4960 | −3431 | 8760 | 4024 | 13772 | −1404 | −866 | −650 |
| 4 | 860 | −566 | −15421 | 34689 | 13772 | −4579 | −628 | −615 |
| 5 | −11 | 4 | 0 | −1 | 13772 | 3153 | −485 | −584 |
| 6 | −14 | 6 | −2 | −13 | 13772 | −7858 | −394 | −564 |
| 7 | 11 | −3 | 0 | 1 | 13772 | 353 | 8887 | 543 |
| 8 | 0 | 0 | 0 | 0 | 13772 | 38 | −1062 | 5790 |

**Ecliptic plane: orbital plane of Earth (at a given year, i.e. 2000)**

**Invariant plane: plane perpendicular to the total angula momentum of the SS**

# Equzioni secolari per tutti i pianeti del Sistema Solare

$$h_j = e_j \sin \tilde{\omega}_j = \sum_{k=1}^{N} M_{j,k} \sin(g_k t + \beta_k)$$

$$k_j = e_j \cos(\tilde{\omega}_j) = \sum_{k=1}^{N} M_{j,k} \cos(g_k t + \beta_k)$$

$$p_j = \sin\left(\frac{i_j}{2}\right) \sin \Omega_j = \sum_{k=1}^{N} N_{j,k} \sin(s_k t + \delta_k)$$

$$q_j = \sin\left(\frac{i_j}{2}\right) \cos \Omega_j = \sum_{k=1}^{N} N_{j,k} \cos(s_k t + \delta_k)$$

SOLUZIONE di LAGRANGE - LAPLACE

- Per la CONSERVAZIONE del MOMENTO ANGOLARE non tutti gli $s_k$ sono INDIPENDENTI:

$$L_x = \sum_{j=1}^{N} \mu_j \sqrt{G(M_0 + m_j) a_j (1 - e_j^2)} \sin i_j \cos \Omega_j$$

$$L_y = \sum_{j=1}^{N} \mu_j \cdot h_j \cdot \sin i_j \sin \Omega_j$$

$$\mu_j = \frac{M_0 m_j}{M_0 + m_j} \qquad h_j = \sqrt{G(M_0 + m_j) a_j (1 - e_j^2)}$$

↑ Da non confondere con le h, k....

$L_x$ e $L_y$ SONO COSTANTI

Per un dato $\bar{j}$

$$\tan \Omega_{\bar{j}} = \frac{L_y - \sum_{j \neq \bar{j}} \mu_j h_j \sin i_j \sin \Omega_j}{L_x - \sum_{j \neq \bar{j}} \mu_j h_j \sin i_j \cos \Omega_j}$$

Allora $\Omega_{\bar{j}}$ può essere calcolato sapendo gli altri $\Omega_{j \neq \bar{j}}$. Questo riduce a $N-1$ le frequenze del sistema. Per convenzione si pone $s_5 = 0$

- LA TEORIA SECOLARE viene di solito calcolata nel PIANO INVARIANTE $\Rightarrow$ piano $\perp$ al vettore momento angolare $\vec{L}$ totale del Sistema Solare.

- NB: Nel piano invariante tutti i $N_{j,5} = 0$ altrimenti

$$q_j = \sum_{k=1}^{N} N_{j,k} \cos(s_k t + \delta_k) = N_{j,5} \cdot \cos(\delta_k) + \cdots$$

Il termine costante $N_{j,5} \cos(\delta_k)$ darebbe una inclinazione forzata per tutti i $j$ e il piano invariante NON sarebbe $\perp$ a $\vec{L}$

(3)

• FREQUENZE FONDAMENTALI del SISTEMA SOLARE

| k | $g_k$ | $\beta_k(°)$ | $s_k$ | $\delta_k(°)$ |
|---|---|---|---|---|
| 1 | 5.5964 | 112.08 | −5.6174 | 348.60 |
| 2 | 7.4559 | 200.51 | −7.0795 | 273.25 |
| 3 | 17.3646 | 305.12 | −18.8512 | 240.20 |
| 4 | 17.9156 | 335.38 | −17.7482 | 303.75 |
| 5 | 4.2575 | 30.65 | 0.0000 | 107.58 |
| 6 | 28.2455 | 128.09 | −26.3450 | 307.29 |
| 7 | 3.0868 | 121.36 | −2.9927 | 320.62 |
| 8 | 0.6726 | 74.06 | −0.6925 | 203.90 |

**Arcsec/year**

Arcsec = 1/3600

$N_{J,K} \times 10^5$ RISPETTO ECLITTICA

| | j\k | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| M | 1 | 39957 | 30169 | 1678 | 72261 | 13772 | −139 | −1665 | −724 |
| V | 2 | 6716 | −4045 | −9544 | −5759 | 13772 | −60 | −959 | −663 |
| T | 3 | 4960 | −3431 | 8760 | 4024 | 13772 | −1404 | −866 | −650 |
| Ma | 4 | 860 | −566 | −15421 | 34689 | 13772 | −4579 | −628 | −615 |
| G | 5 | −11 | 4 | 0 | −1 | 13772 | 3153 | −485 | −584 |
| S | 6 | −14 | 6 | −2 | −13 | 13772 | −7858 | −394 | −564 |
| U | 7 | 11 | −3 | 0 | 1 | 13772 | 353 | 8887 | 543 |
| N | 8 | 0 | 0 | 0 | 0 | 13772 | 38 | −1062 | 5790 |

$M_{J,K} \times 10^5$

| | j\k | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| M | 1 | 185444 | −27700 | 1458 | −1428 | 36353 | 113 | 623 | 7 |
| V | 2 | 6668 | 20733 | −11671 | 13464 | 19636 | −551 | 614 | 11 |
| T | 3 | 4248 | 16047 | 9406 | −13159 | 18913 | 1506 | 650 | 12 |
| Ma | 4 | 650 | 2917 | 40133 | 49032 | 20300 | 7030 | 862 | 20 |
| G | 5 | −7 | −12 | −1 | 0 | 44187 | −15700 | 1814 | 58 |
| S | 6 | −6 | −12 | −7 | −7 | 32958 | 48209 | 1511 | 57 |
| U | 7 | 2 | 3 | 0 | 0 | −37587 | −1547 | 29033 | 1666 |
| N | 8 | 0 | 0 | 0 | 0 | 1881 | −103 | −3697 | 9118 |

3-1)



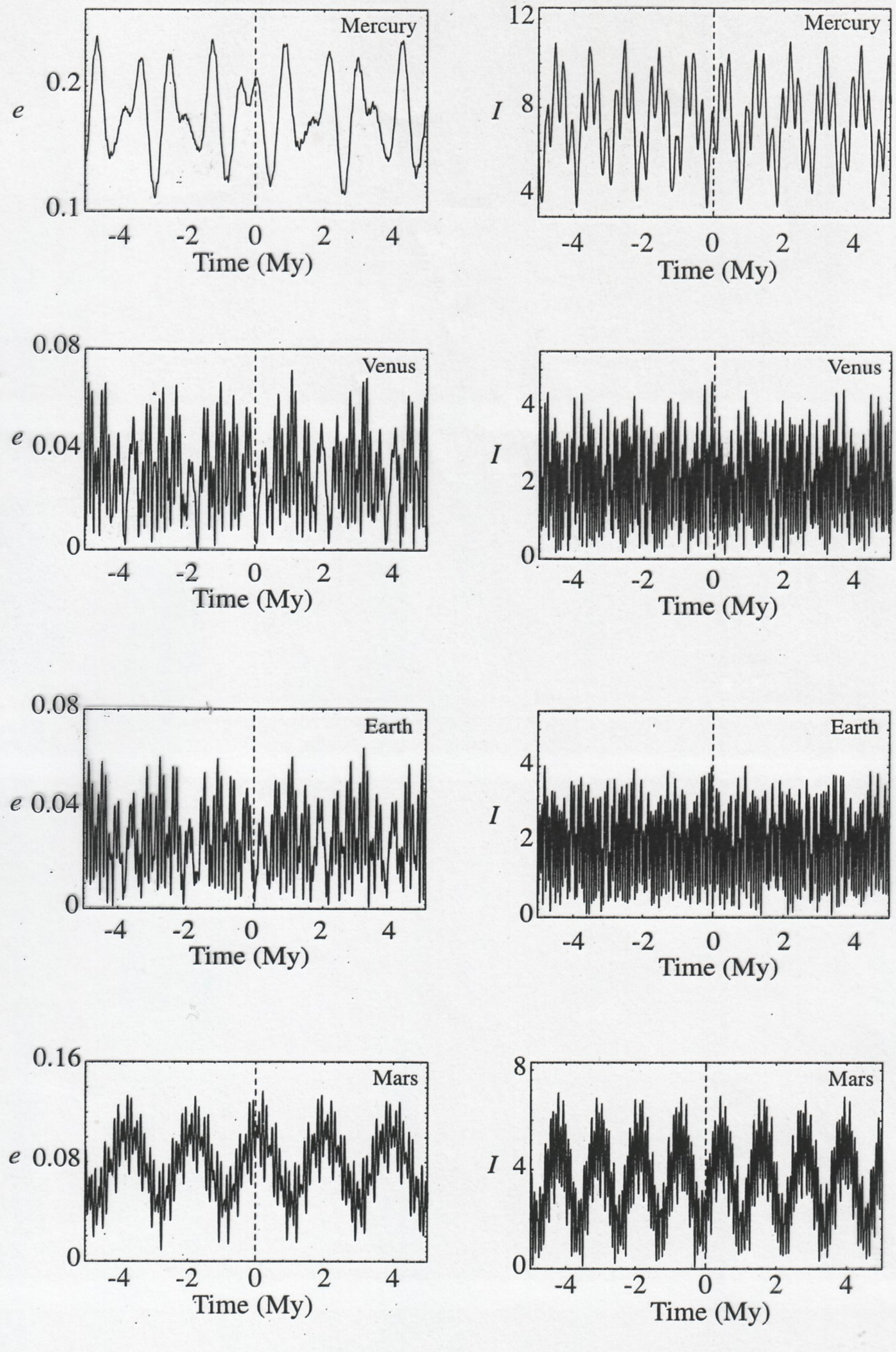



Fig. 7.9. Plots of the eccentricity and inclination (in degrees) of Mercury, Venus, Earth, and Mars over a period of 10 million years centred on AD 1900, according to the secular theory of Brouwer & van Woerkom (1950).

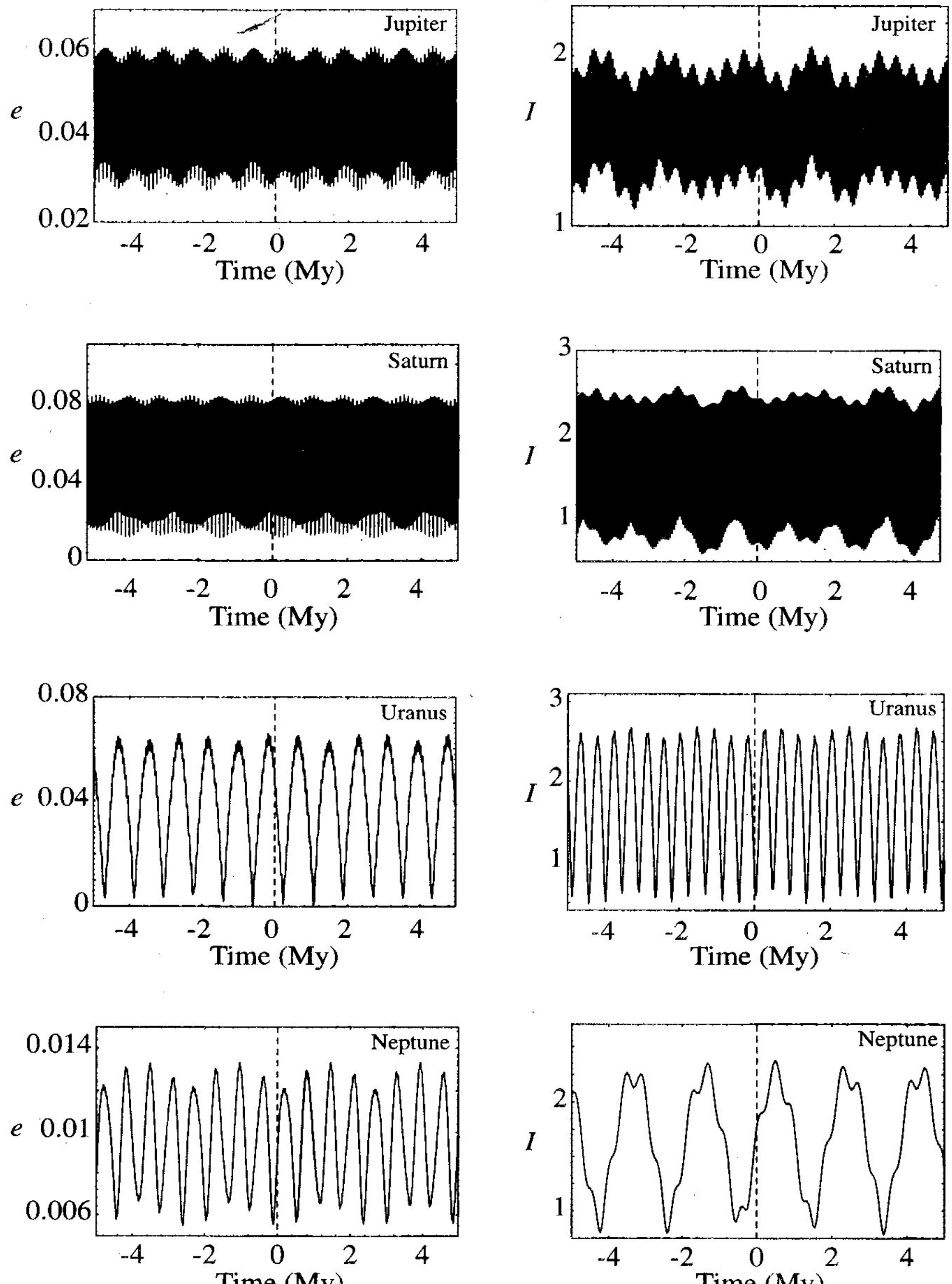
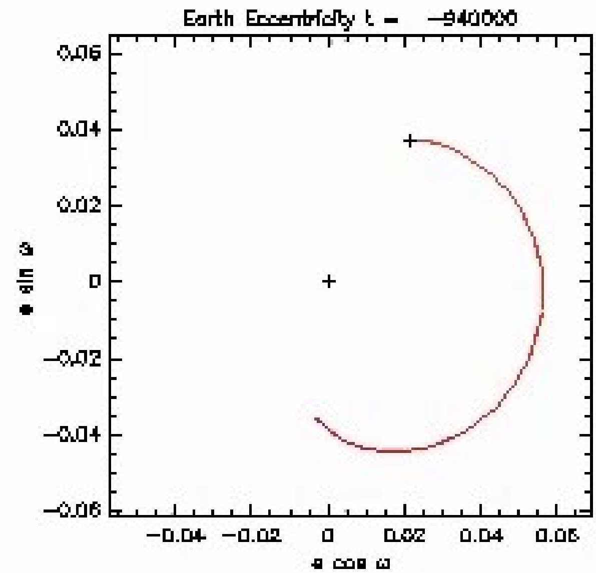
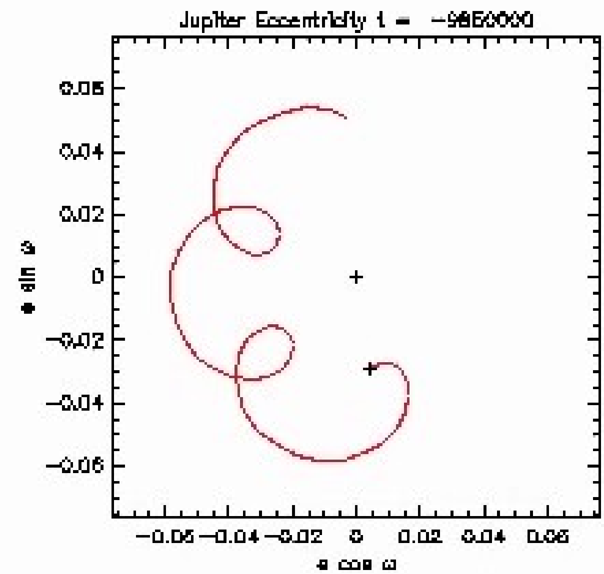


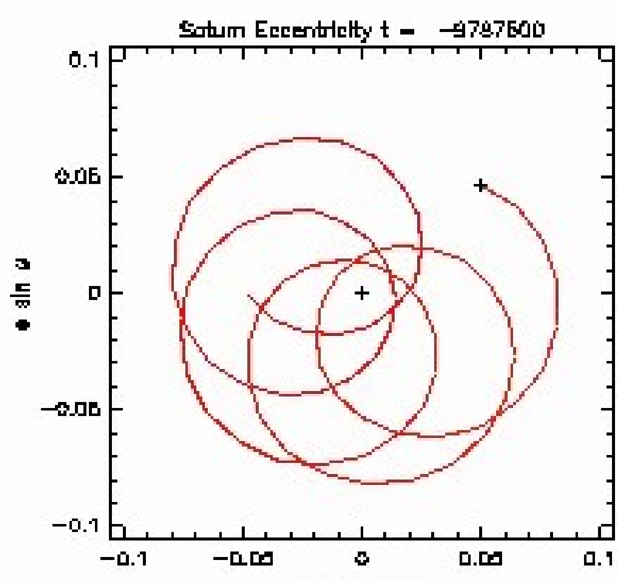

Fig. 7.10. Plots of the eccentricity and inclination (in degrees) of Jupiter, Saturn, Uranus, and Neptune over a period of 10 million years centred on AD 1900, according to the secular theory of Brouwer & van Woerkom (1950).



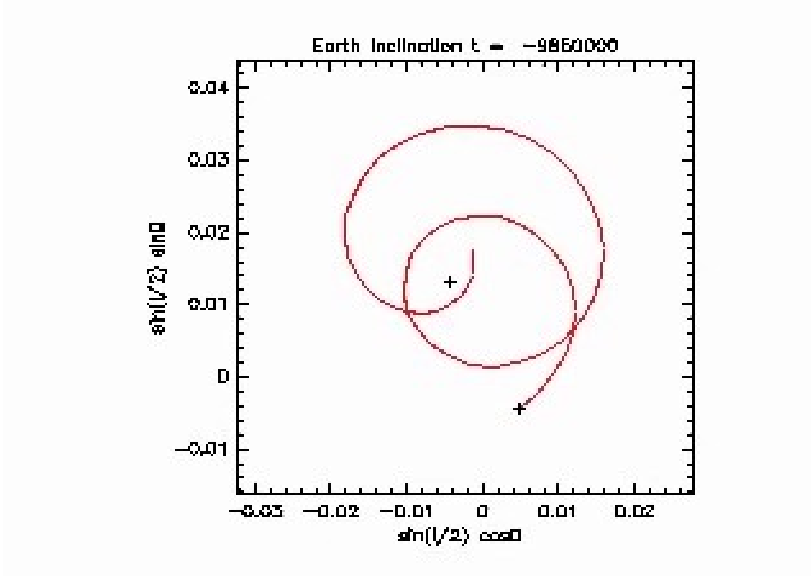
Earth Eccentricity t = -940000
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
e sin ω
-0.04 -0.02 0 0.02 0.04 0.06
e cos ω

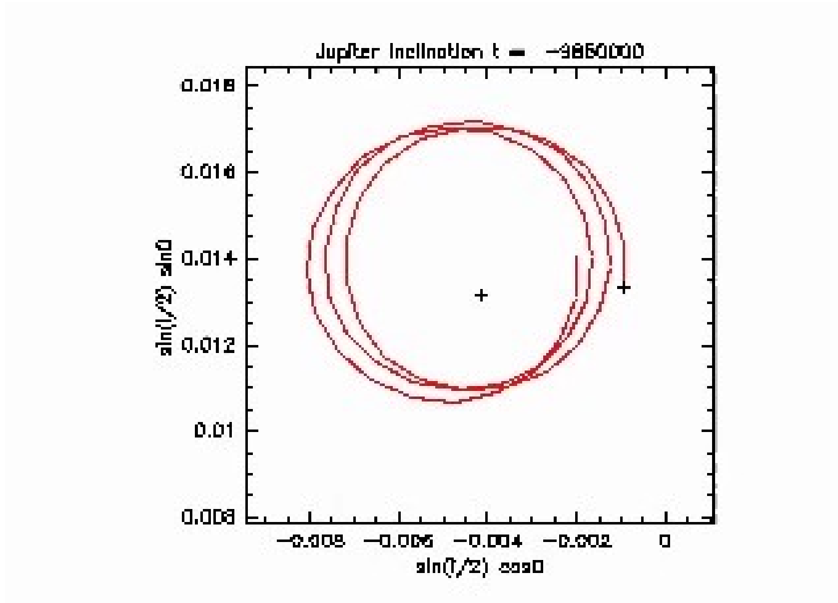



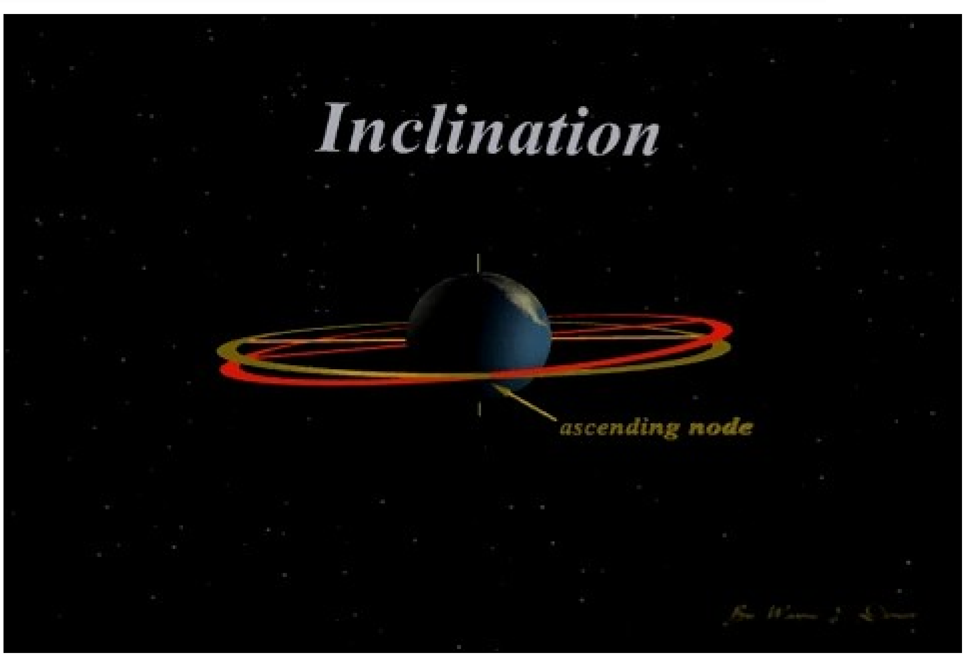
Jupiter Eccentricity t = -9860000
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
e sin ω
-0.06 -0.04 -0.02 0 0.02 0.04 0.06
e cos ω

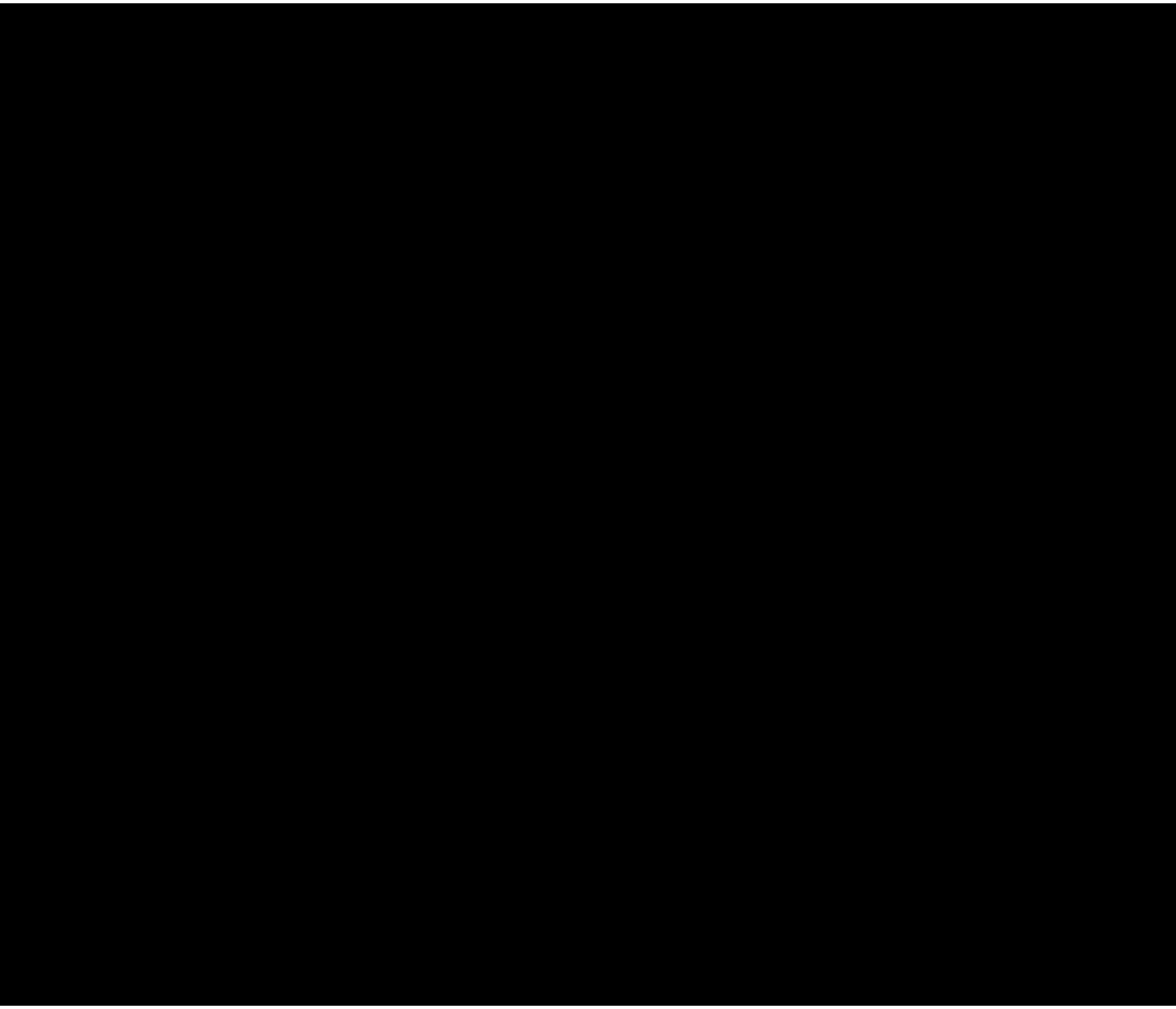
Saturn Eccentricity t = -9797600
0.1
0.05
0
-0.05
-0.1
e sin ω
-0.1 -0.05 0 0.05 0.1
e cos ω


Earth Inclination t = -9860000
0.04
0.03
0.02
0.01
0
−0.01
sin(I/2) sinΩ
−0.03
−0.02
−0.01
0
0.01
0.02
sin(I/2) cosΩ


Jupiter Inclination t = -9860000
0.018
0.016
0.014
0.012
0.01
0.008
sin(I/2) sinΩ
−0.008
−0.006
−0.004
−0.002
0
sin(I/2) cosΩ


Inclination
ascending node

# Transit time variation to discover a second planet from its secular perturbations on the first one.



# Evoluzione secolare dei corpi minori

**Il caso di: 2 pianeti + Corpo minore (m=0)**

FUNZIONE di DISTURBO ⇓

$$\mathcal{R} = n a^2 \left[ \frac{1}{2} A e^2 + \frac{1}{2} B i^2 + \sum_{j=1}^{2} A_j e e_j \cos(\varpi - \varpi_j) + \sum_{j=1}^{2} B_j i i_j \cos(\Omega - \Omega_j) \right]$$

$$A = n \frac{1}{4} \sum_{j=1}^{2} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j)$$

$$A_j = - n \frac{1}{4} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b_{3/2}^{(2)}(\alpha_j)$$

$$B = - n \frac{1}{4} \sum_{j=1}^{2} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j)$$

$$B_j = n \frac{1}{4} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b_{3/2}^{(1)}(\alpha_j)$$

$$\alpha_j = \begin{cases} a_j/a \\ a/a_j \end{cases} \Rightarrow \begin{matrix} a_j < a \\ a_j > a \end{matrix}$$

$$\bar{\alpha}_j = \begin{cases} 1 \\ a/a_j \end{cases} \Rightarrow \begin{matrix} a_j < a \\ a_j > a \end{matrix}$$

2)

$$\dot{h} = A k + \sum_{s=1}^{2} A_s k_s \qquad \dot{p} = B q + \sum_{s=1}^{2} B_s q_s$$

$$\dot{k} = -A h - \sum_{s=1}^{2} A_s h_s \qquad \dot{q} = -B p - \sum_{s=1}^{2} B_s p_s$$

SOLUZIONE:

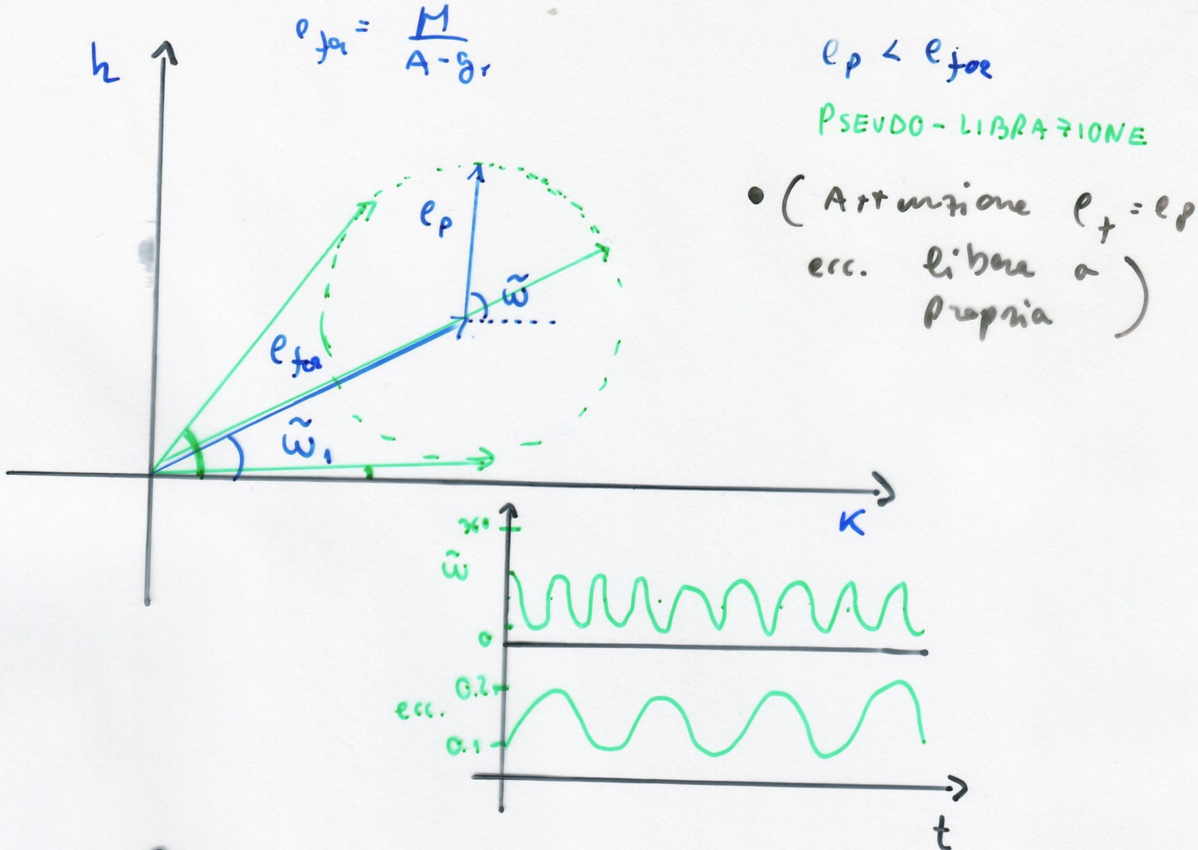
$$\begin{cases} h = e_f \sin(At+\beta) - \sum_{i=1}^{2}\sum_{s=1}^{2} \dfrac{A_s e_{si}}{A-g_i} \sin(g_i t + \beta_i) \\ k = e_f \cdot \cos(At+\beta) - \sum_{i=1}^{2}\sum_{s=1}^{2} \dfrac{A_s e_{si}}{A-g_i} \cos(g_i t + \beta_i) \\ p = i_f \sin(Bt+\gamma) - \sum_{i=1}^{2}\sum_{s=1}^{2} \dfrac{B_s i_{si}}{B-f_i} \sin(f_i t + \gamma_i) \\ q = i_f \cos(Bt+\gamma) - \sum_{i=1}^{2}\sum_{s=1}^{2} \dfrac{B_s i_{si}}{B-f_i} \cos(f_i t + \gamma_i) \end{cases}$$

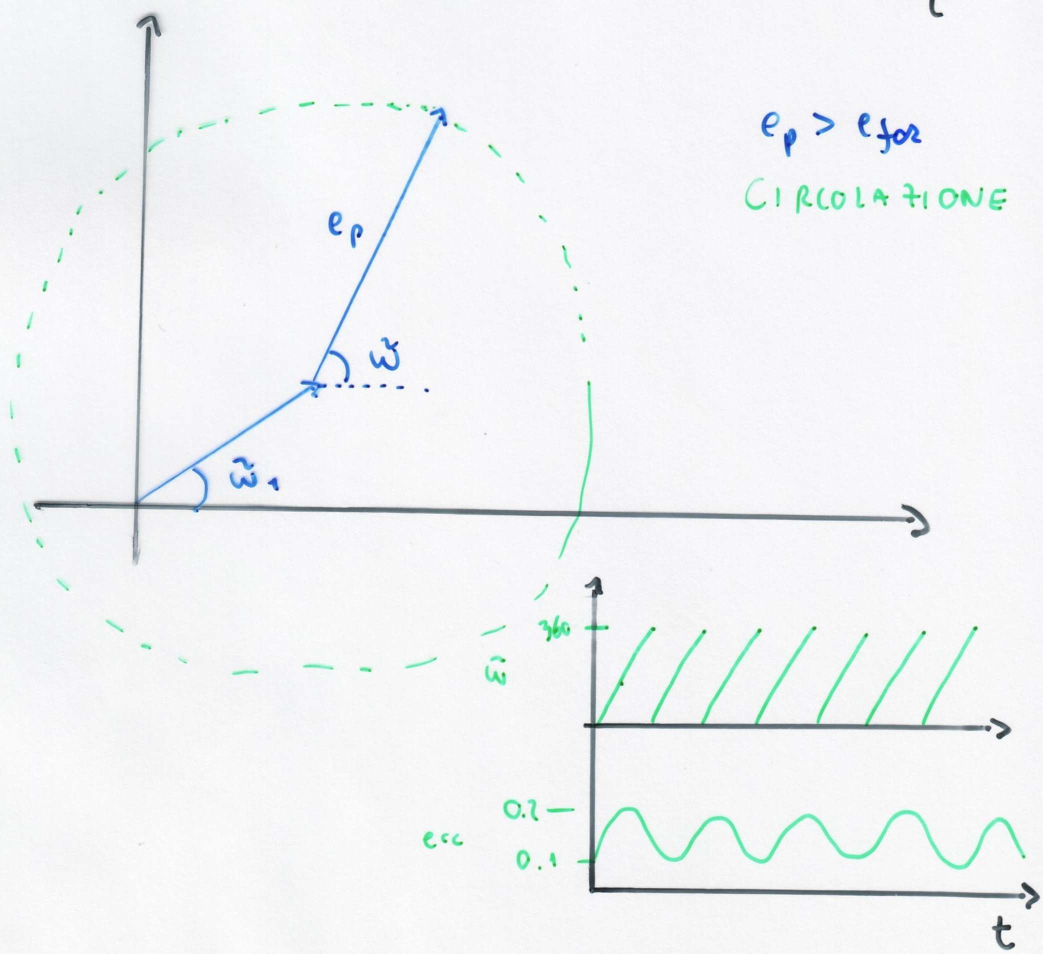
- $e_f, i_f$ => eccentricità e inclinazione libere
- RISONANZE SECOLARI: $A = g_i$, $B = f_i$

$A$, $B$ => frequenze proprie

# Rappresentazione grafica dell'evoluzione orbitale

Rappresentazione grafica dell'evoluzione orbitale



# Soluzione generale: N pianeti + corpo minore (m=0).

$$h = e_f \sin(At+\beta) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{A_j \cdot M_{jk}}{A - g_k} \sin \tilde{\omega}_k$$

$$k = e_f \cos(At+\beta) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{A_j M_{jk}}{A - g_k} \cos \tilde{\omega}_k$$

$$p = i_f \sin(Bt+\gamma) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{B_j N_{jk}}{B - f_k} \sin \Omega$$

$$q = i_f \cos(Bt+\gamma) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{B_j N_{jk}}{B - f_k} \cos \Omega$$

$e_f, i_f, \beta, \gamma$ Dipendono dalle condizioni iniziali

Ex: 3 corpi di cui 1 a massa nulla
(ex: Giove + Asteroide)

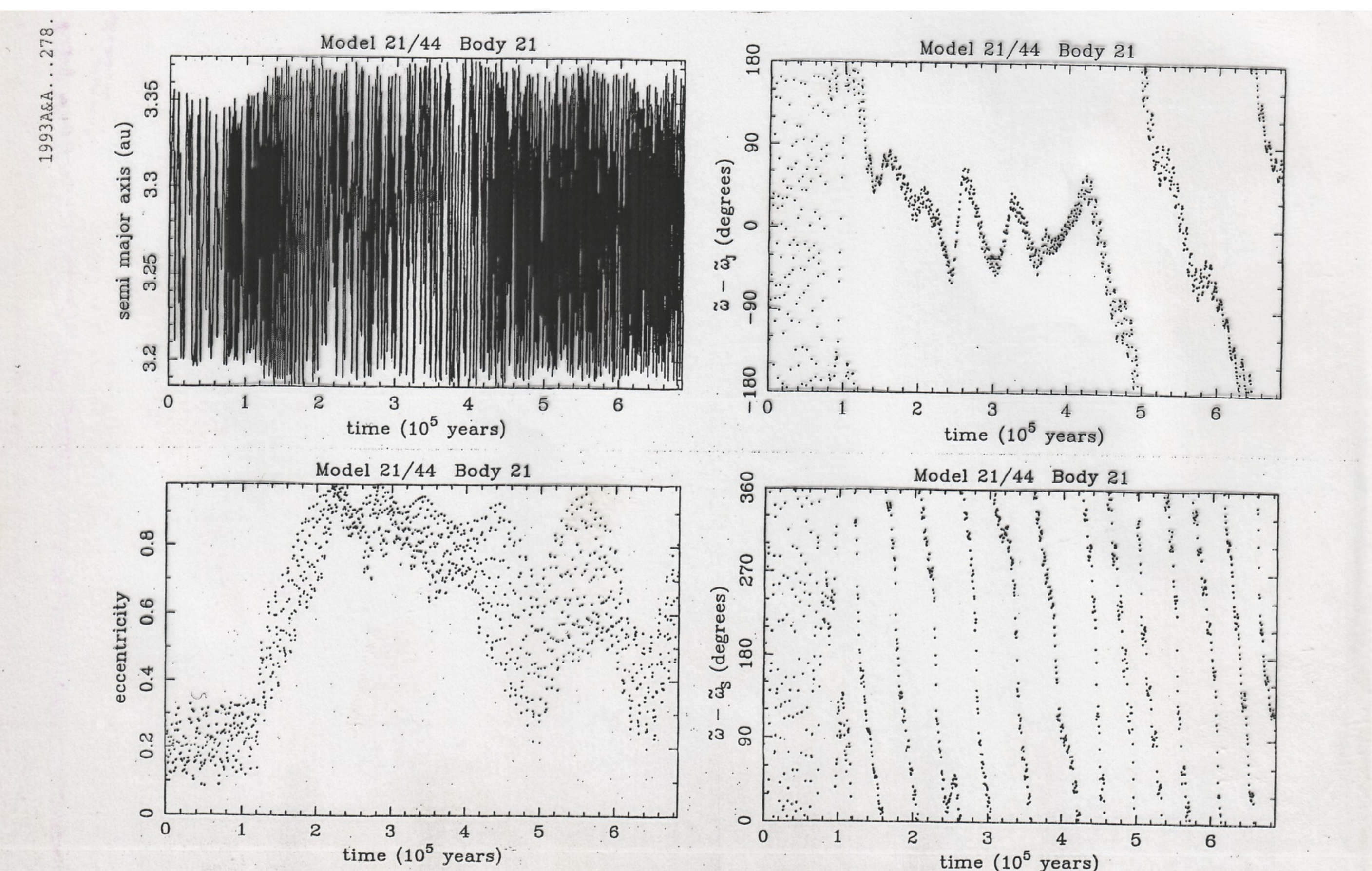
$$h = e \sin \tilde{\omega} = e_f \sin(At+\beta) + \frac{M}{A-g_1} \sin(g_1 t + \varphi)$$

$$e^2 = e_f^2 + \left(\frac{M}{A-g_1}\right)^2 + 2\frac{AM}{A-g_1} \cos\left((A-g_1)t + (\beta-\varphi)\right)$$

•) Se Giove non perturbato $g_1 = 0$ $\quad h = A \sin(At+\beta) + \frac{M}{A} \sin \varphi$

**Si capisce l'origine del caos nelle Kirkwood gaps: risonanze secolari interagiscono con quelle in moto medio causando diffusione nello spazio delle fasi. Ad esempio, all'interno della risonanza in moto medio 2:1 nella fascia asteroidale sono presenti anche le risonanze A=$g_5$ e A=$g_6$**

**Le figure mostrano l'integrazoine numerica dell'orbita di un asteroide nella risonanza 2:1. Anche l'argomento critico della risonanza secolare A=$g_5$ libra causando una crescita caotica dell'eccentricità.**



# La sovrapposizione tra risonanze secolari e in moto medio avviene anche per i Toriani (risonanza 1:1).



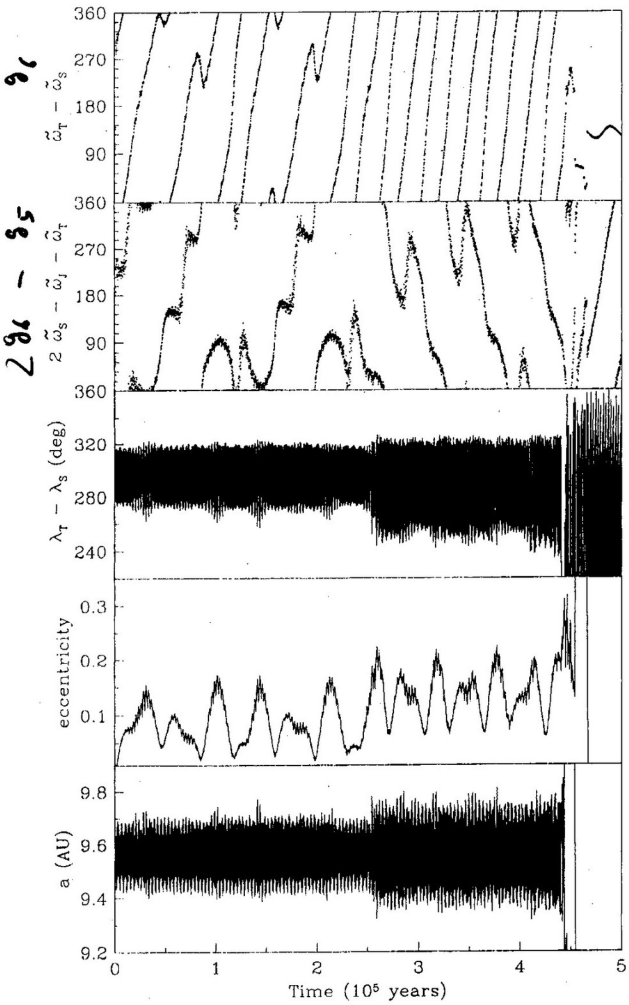



**FIG. 3.** Orbital evolution of a L5 Saturn Trojan. The crossing of the mixed secular resonance with critical argument $2\tilde{\omega}_S - \tilde{\omega}_J - \tilde{\omega}_T$ is shown by the change of the direction of circulation after about $2.5 \times 10^5$ year. Ejection occurs near $4.5 \times 10^5$ year. Indices T, J, S refer, respectively, to Trojan, Jupiter, Saturn.

amplitude of both the semimajor axis and the critical argument of the Trojan resonance $\lambda_T - \lambda_S$, and in eccentricity. This jump occurs at the same time as the inversion in the circulation direction. After the crossing, instability builds up and the eccentricity grows until the body is ejected out of the resonance into a chaotic orbit, eventually, after a close encounter with Saturn. The inversion of circulation direction during the resonance crossing is very similar to that shown in Fig. 1b for the $\nu_{16}$ secular resonance and Jupiter Trojans. However, for the mixed resonance we cannot plot the $h$, $k$ variables (same as $p$, $q$ variables, but for resonances involving the perihelia) to show in more detail the resonance crossing. We lack, in fact, the definition of the $h$, $k$ variables for the mixed resonance, since we need a perturbative scheme to derive the correct action variable related to the critical argument. However, the resonance crossing is evident in Fig. 3 and is well supported by the related changes in the orbital elements. Regarding the short-period modulations superimposed onto the circulation trends of the secular arguments in Fig. 3, they are related to the changes in the precession rate of $\tilde{\omega}_T$. These are caused by the oscillations in the orbital eccentricity. For most of our low-amplitude libration orbits we found that this kind of behavior leads to escape from both L4 and L5 orbits on a time scale of the order of $10^5$ years. Since it was suggested by De la Barre *et al.* (1996) that the $\nu_6$ secular resonance can also play an important role in destabilizing Saturn Trojans, we also plot in Fig. 3 the corresponding critical argument. However, in this case, it does not show any particular behavior that may be related to the instability of the orbit.

The passage through the $2\tilde{\omega}_S - \tilde{\omega}_J - \tilde{\omega}_T$ mixed secular resonance may occur after a longer time scales. Figure 4 illustrates

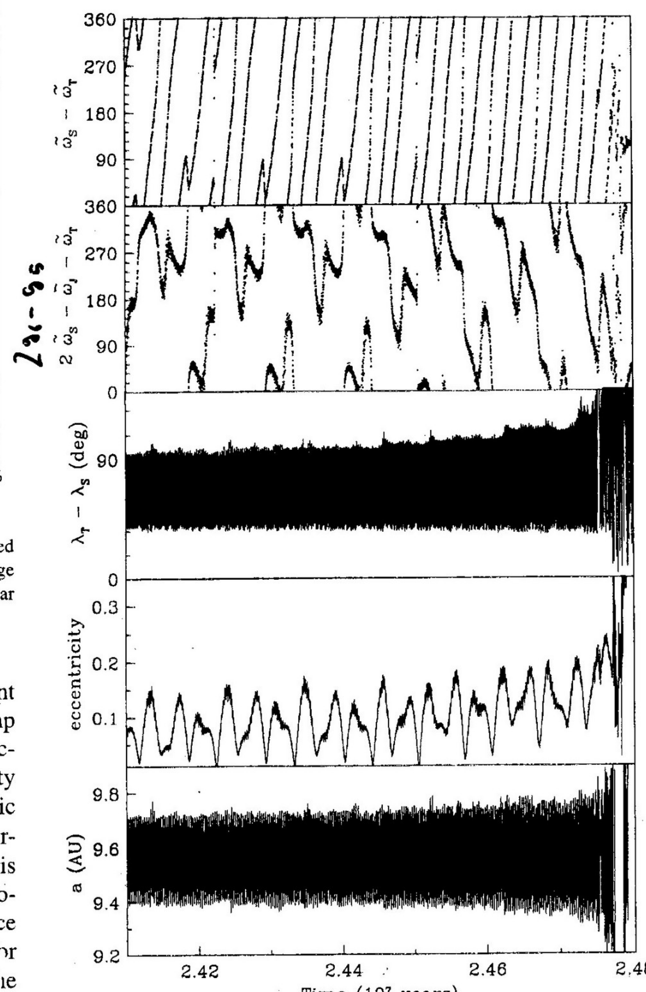



**FIG. 4.** Instability of a Saturn Trojan on a longer time scale ($2.4 \times 10^7$ year) than in Fig. 2.

# Per i Troiani, le risonanze secolari possono ‘pompare’ l’inclinazione a valori elevati.

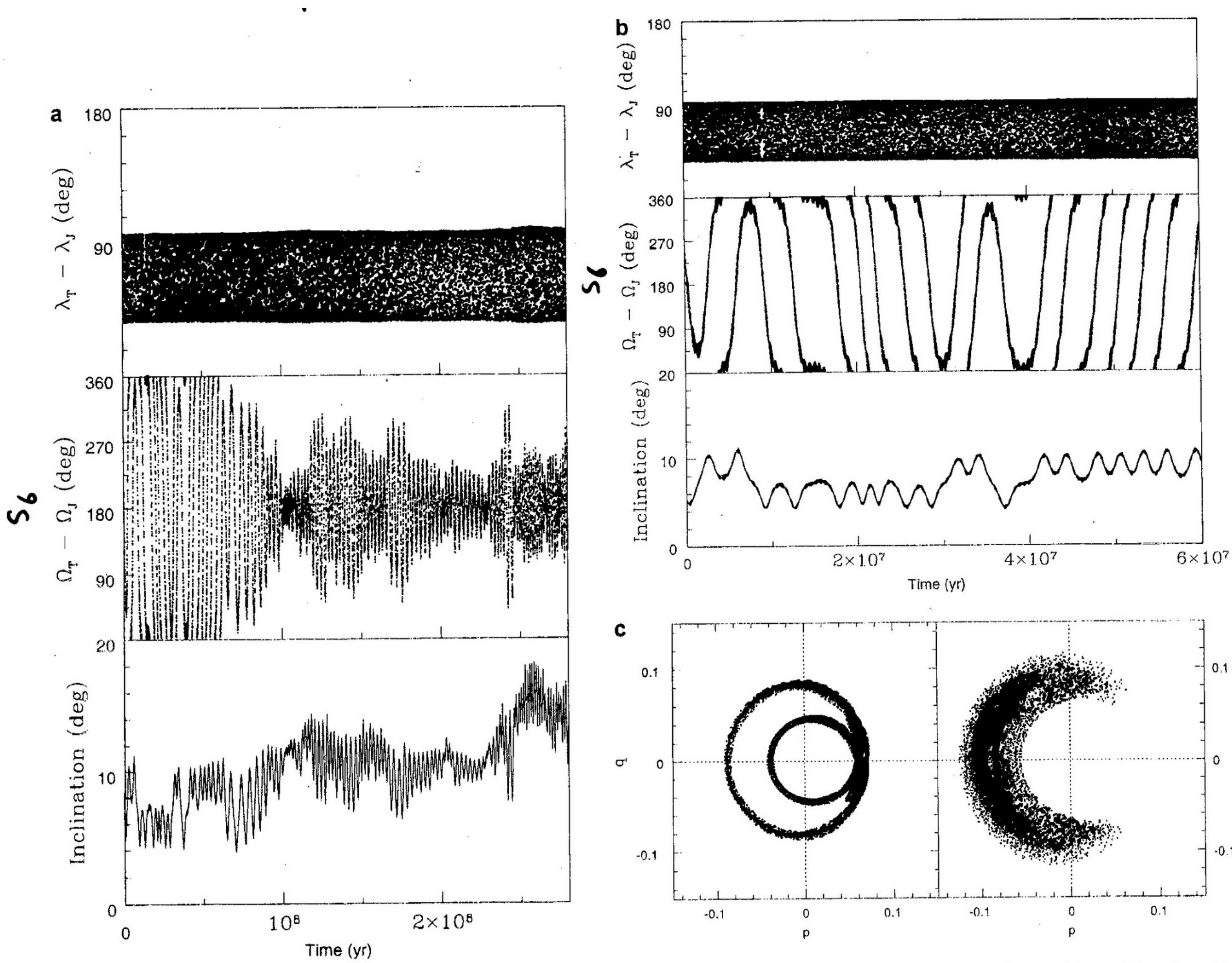
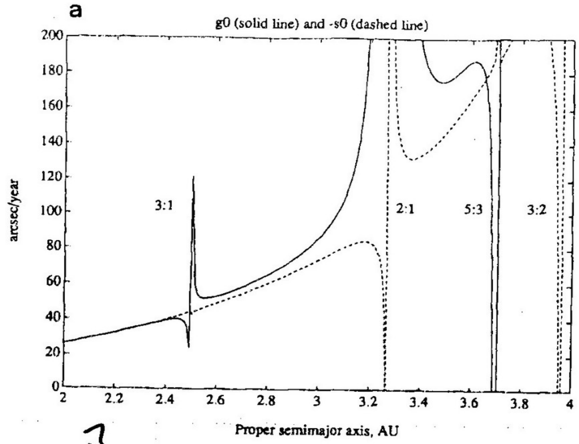


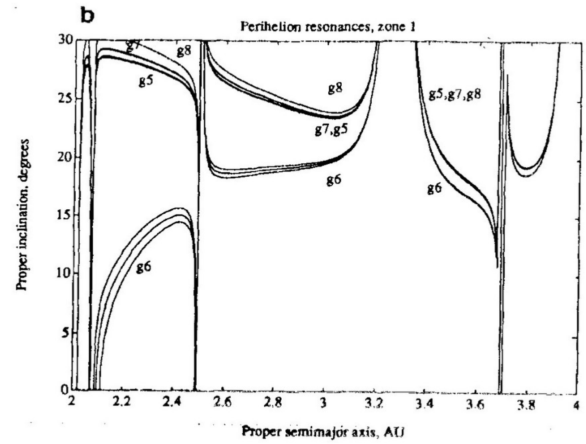

FIG. 1. (a) Evolution of the inclination of a Jupiter L4 Trojan orbit with an initial inclination of 4.5° and libration amplitude of about 59°. The critical argument of the $\nu_{16}$ resonance shows inversions of the circulation direction in the first $0.5 \times 10^8$ year of evolution, typical of resonance crossing; then it enters the resonance and librates for the remaining $2 \times 10^8$ year. (b) Focus on the first $10^8$ year of orbital evolution of the case shown in Fig. 1a. The changes from positive to negative circulation and vice versa are clearly visible and are related to the $\nu_{16}$ resonance crossings. A jump in inclination occurs at every reversion. (c) The variables $p$ and $q$ are plotted for the case in Fig. 1a. On the left the data points cover the time interval $2 \times 10^7$–$5 \times 10^7$ year; the resonance crossing is marked by the change from negative to positive circulation. On the right the variables $p$ and $q$ are shown during the time interval $1 \times 10^8$–$2 \times 10^8$ year. The classical banana shape characterizes the libration of the $\nu_{16}$ critical argument.

first $10^8$ years of evolution (Fig. 1b). Any time a reversion occurs, the body crosses the secular resonance and the inclination jumps. A similar behavior, but for the eccentricity, is known when a mean motion resonance is crossed (e.g., Peale 1986, Marzari *et al.* 1997). Finally, the body enters the resonance and the critical argument starts to librate around 180° with a period of about $1.7 \times 10^6$ years. The resonance crossing and the libration are made more clear in Fig. 1c, where the values of the variables $p$ and $q$ are plotted from $t = 2 \times 10^7$ year to $t = 5 \times 10^7$ year, and from $t = 1 \times 10^8$ year to $t = 2 \times 10^8$ year. The inclination grows from 4.5° to a maximum of 19.5° in $2.8 \times 10^8$ year. The libration amplitude of the Trojan critical argument $\lambda_p - \lambda_j$ grows from ~58.9° to 64° during this evolution.

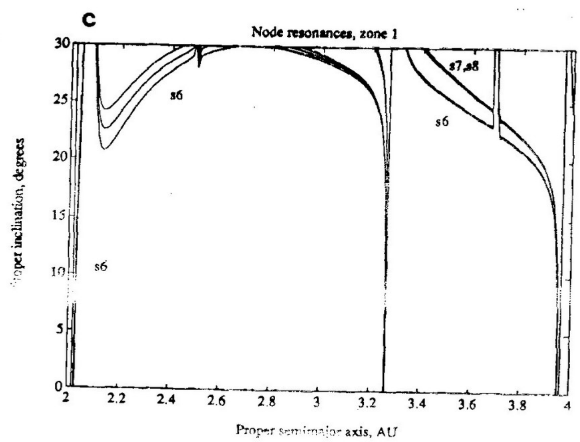
The second case (Fig. 2) has an initial libration amplitude of 58° and an initial inclination of 4°. Its behavior is more chaotic, with frequent changes between circulation and libration. The eccentricity remains low ($\leq 0.05$) while the inclination grows up to about 30°. Shortly after such a high inclination is reached, the libration amplitude increases, the orbit becomes unstable, and it escapes from the Trojan swarm.

Due to the presence of the $\nu_{16}$ resonance at low inclinations, the scenario for the origin of highly inclined Trojans at L4 and L5 may have been the following. Planetesimals with low inclinations are trapped as Trojans by the mass growth of Jupiter in the early Solar System with a broad distribution of libration amplitudes (see Fig. 10 in Marzari and Scholl, 1998b). Trojans which

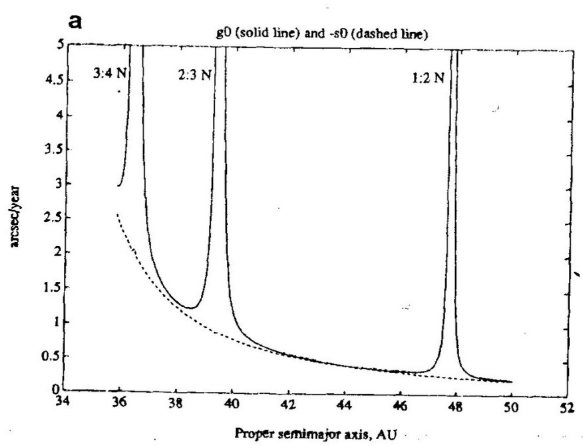


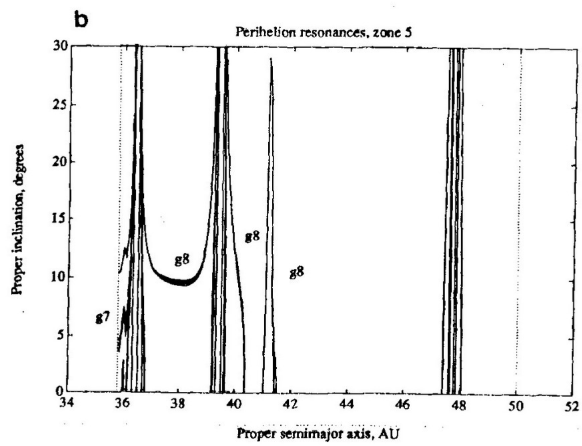



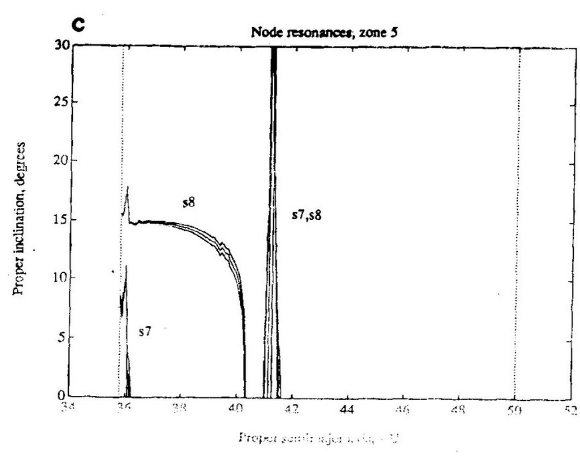

frequenze libere
(o proprie)



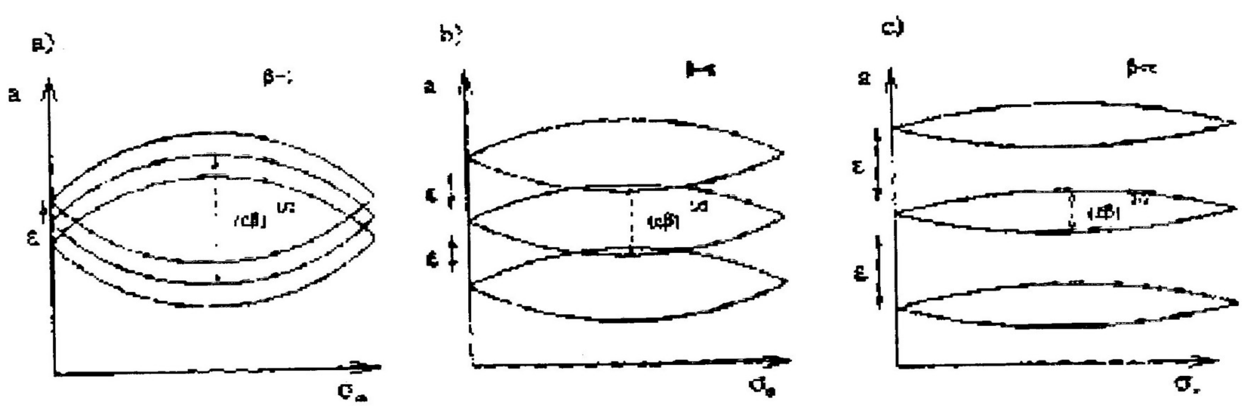

**Mappa delle risonanze secolari nel Sistema Solare**

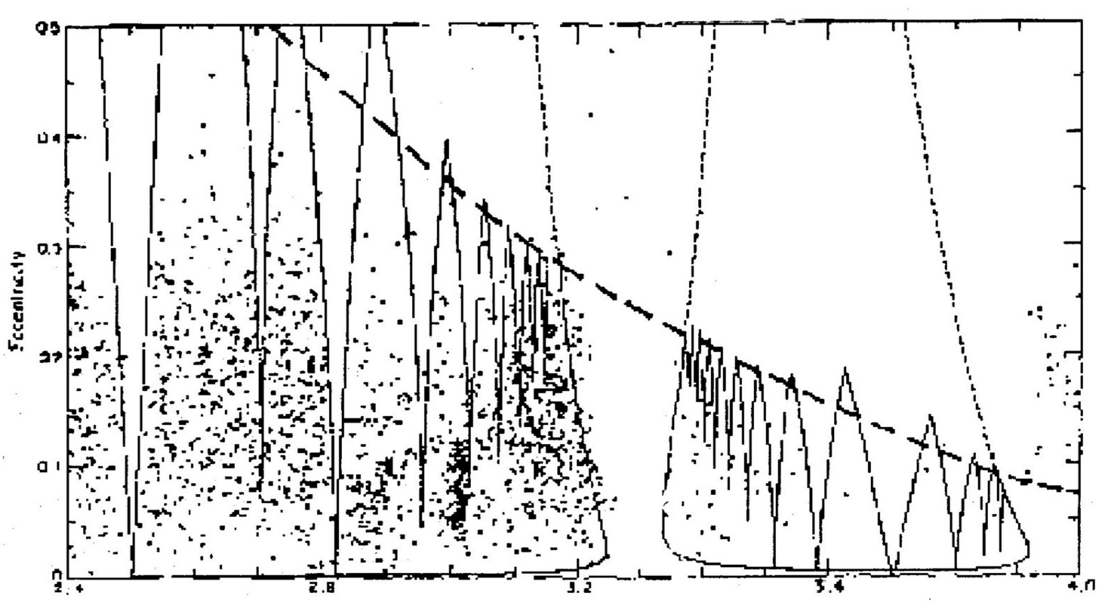


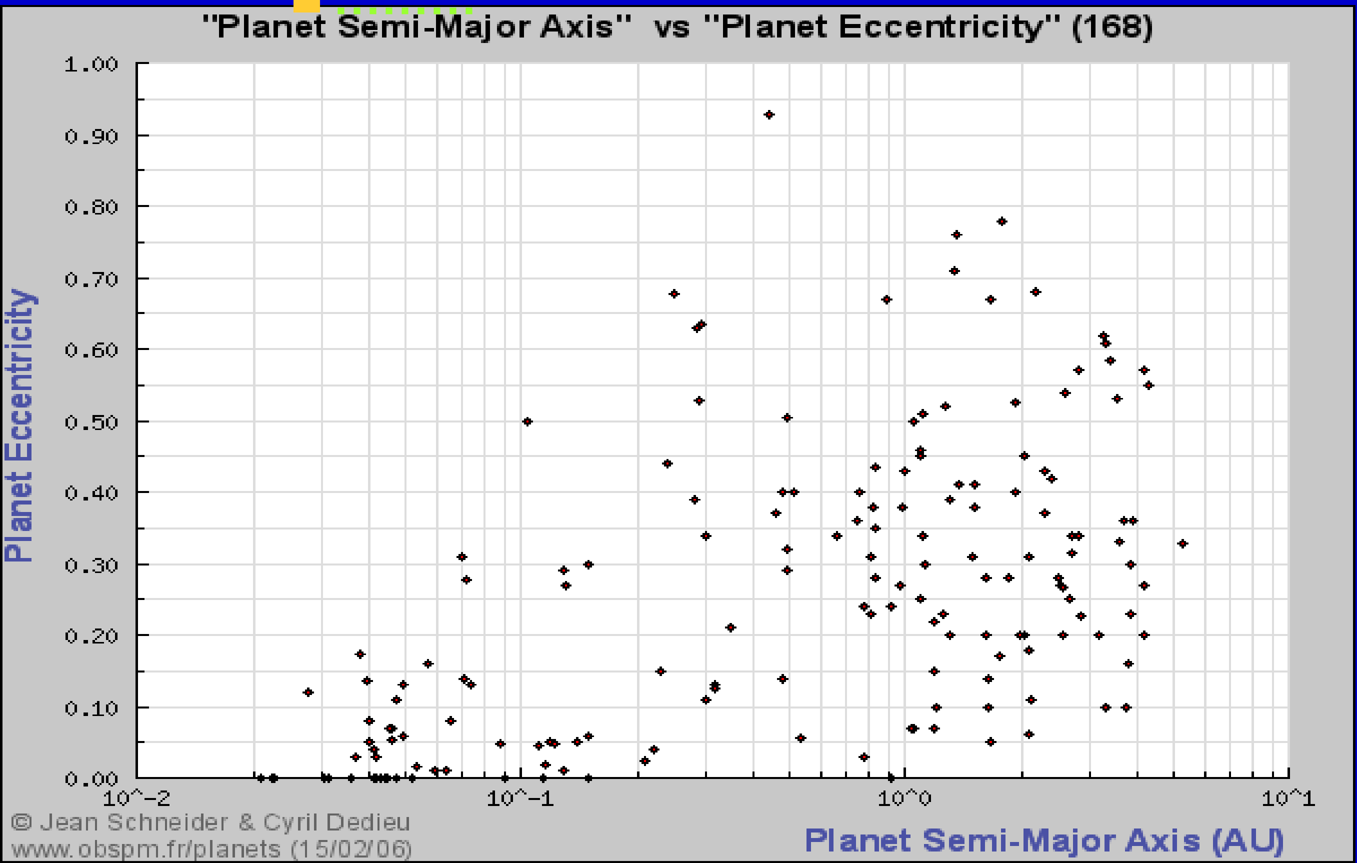
**Il principio di sovrapposizione delle risonanze funziona anche nella parte esterna della fascia asteroidale. Al di là della risonanza 2:1, al crescere dell'eccentricità le numerose risonanze in moto medio adiacenti si allargano portando a moto caotico causato dalla loro sovrapposizione.**



# La teoria di Lagrange-Laplace funziona per piccoli e ed i, mentre in molti sistemi extrasolari i pianeti hanno elevate eccentricità!

- HD74156 $e_1$=0.636 $e_2$=0.583
- HD202206 $e_1$=0.435 $e_2$=0.267
- HD12661 $e_1$=0.350 $e_2$=0.20
- HD128311 $e_1$=0.25 $e_2$=0.17
- Ups And $e_1$=0.012 $e_2$=0.27



# Risonanza apsidale (g1 ~ g2)

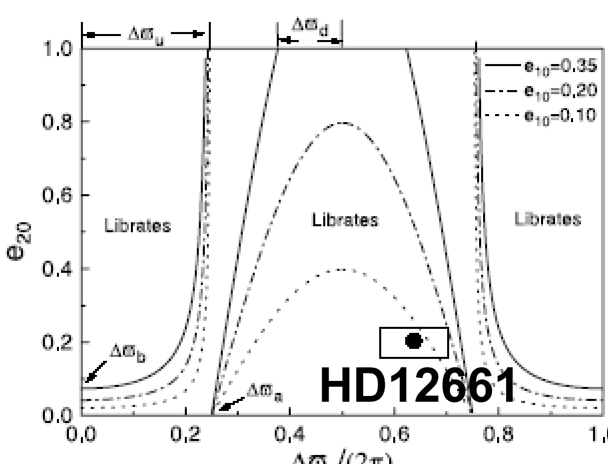
$$e_1 e_2 \cos \Delta\varpi = (e_{11}e_{21} + e_{12}e_{22}) + (e_{11}e_{22} + e_{12}e_{21}) \cos(\psi_1 - \psi_2) \, .$$

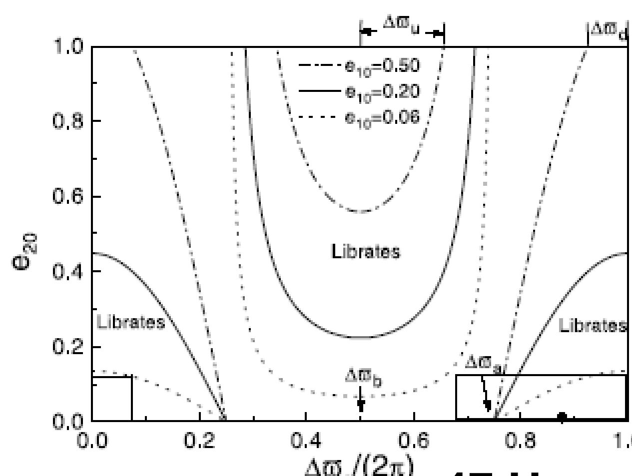
$$\psi_2 - \psi_1 = (g_2 t + \beta_2) - (g_1 t + \beta_1),$$

$$S = \left| \frac{e_{11}e_{22} + e_{12}e_{21}}{e_{11}e_{21} + e_{12}e_{22}} \right| \, .$$

**Se S > 1 risonanza apsidale. Librazione di Δω attorno a 0 o 180 gradi.**

**Perché la risonanza apsidale è importante? Orbite allineate con alta eccentricità e in risonanza apsidale sembrano più stabili di quelle non risonanti (calcolo del coeff. lyapunov).**




**47 Uma**