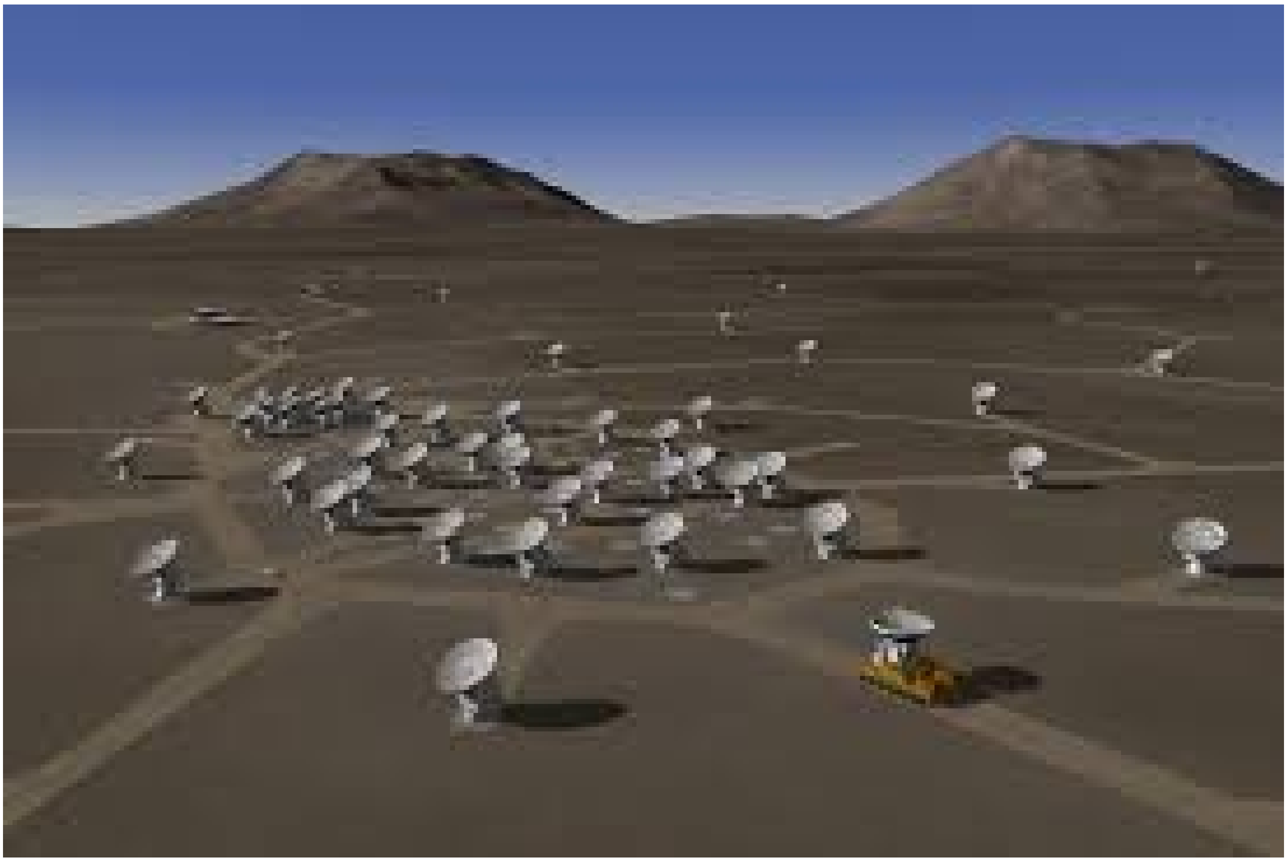
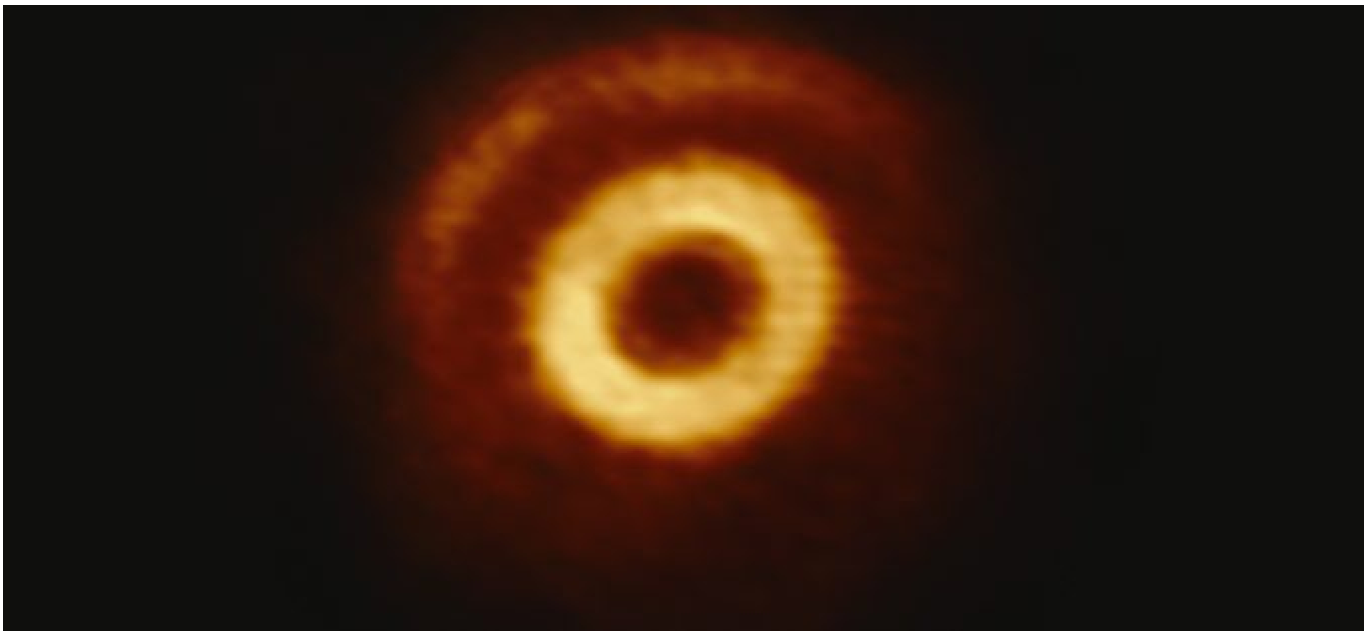
Evolution of circumstellar disks

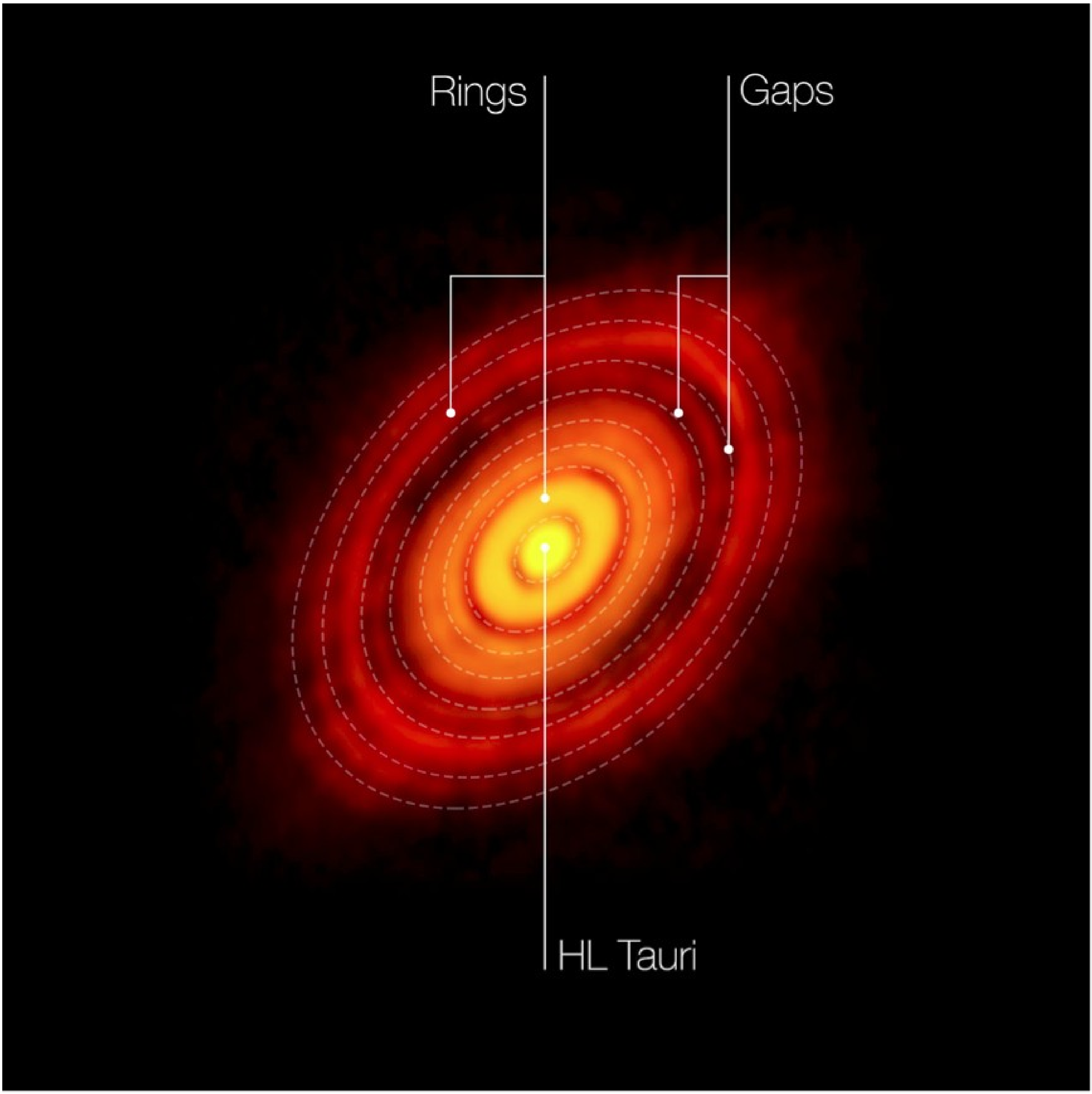
ALMA: radiointerferometer → 66 radiotelescopes observing at millimiter and submillimter wavelength



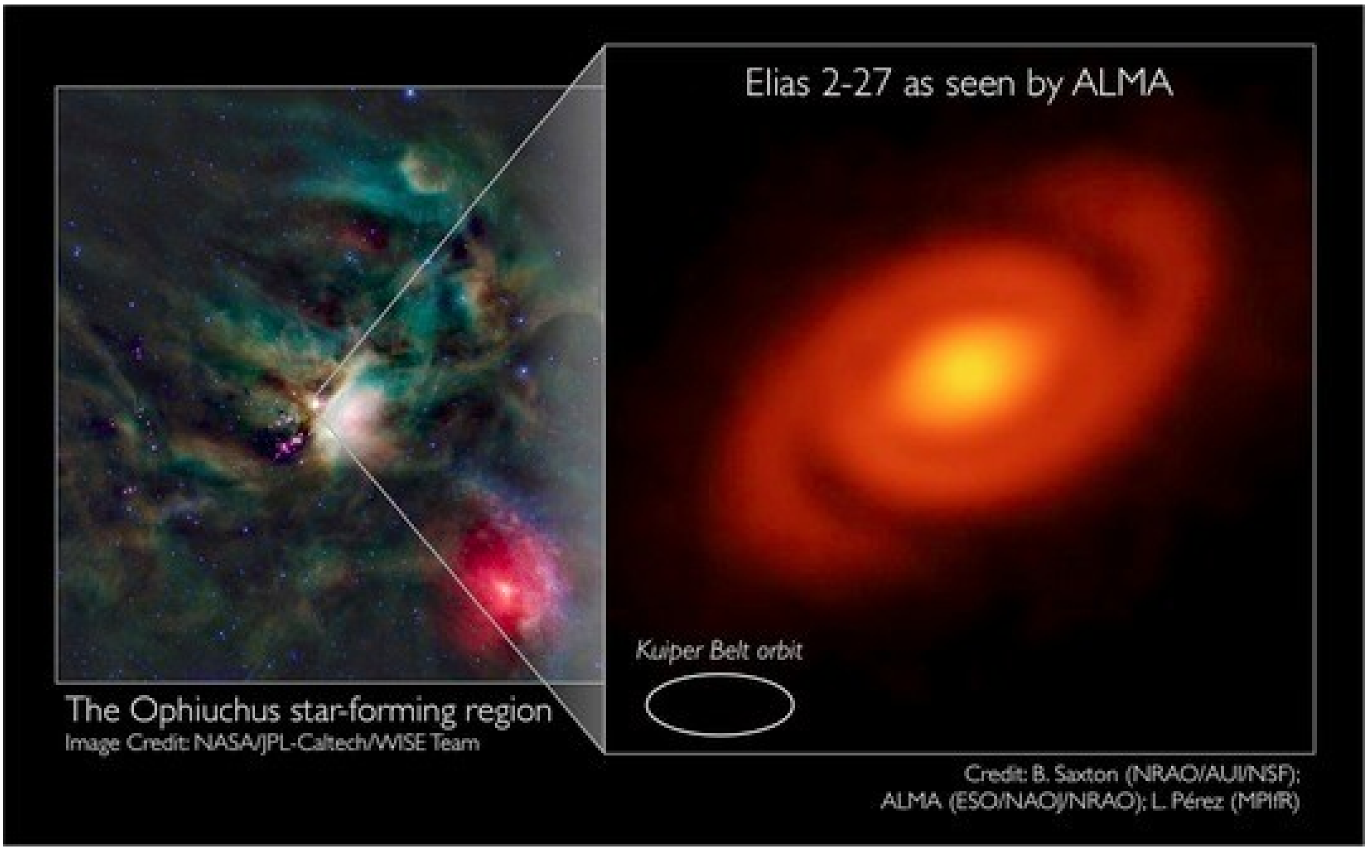


Circumstellar disk in V1247 Orionis

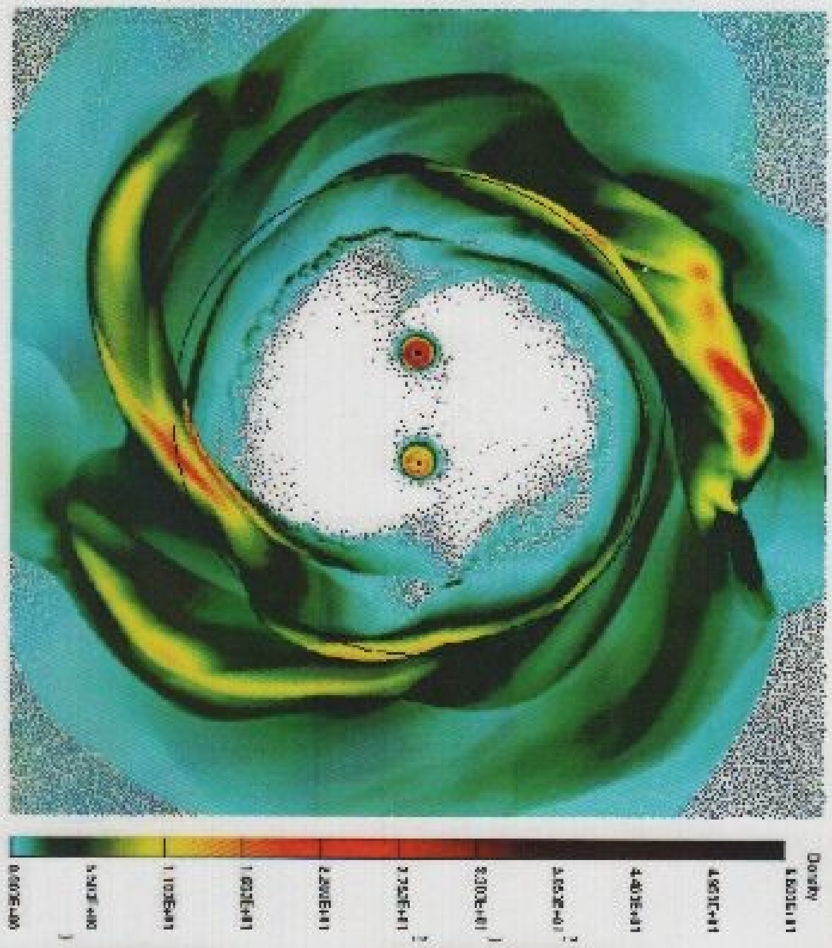
HL Tau, age ~ 100000 yr, type: K9. Disk radius is about 2000 au and the mass is approximately   0.1 solar masses (ALMA images, 0.3- 9.6 mm).



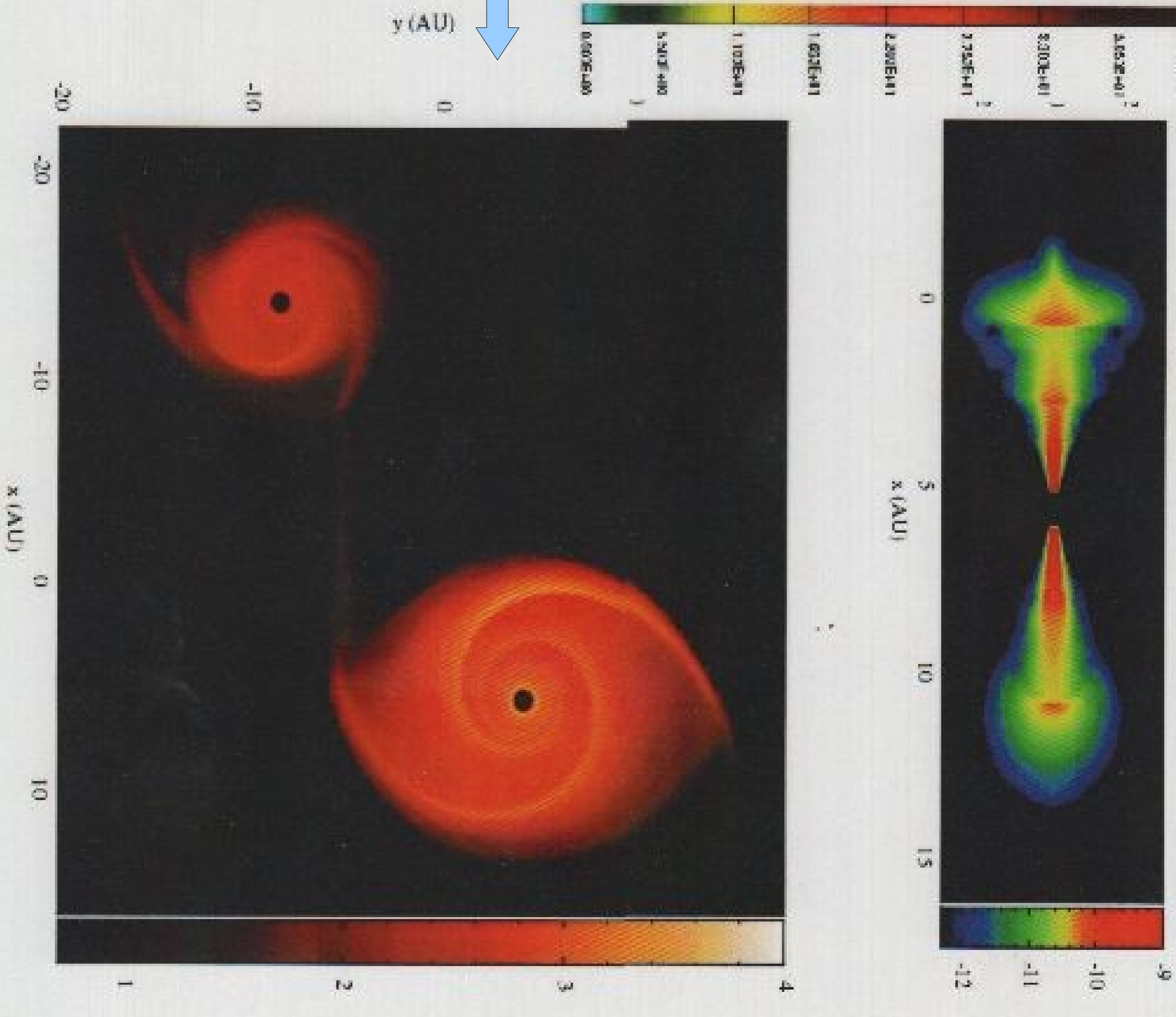
Disk around Elias 2-27 (age: 1 Myr, mass half of that of the sun) in the constellation of Ophiuchus at a distance of about 450 light years from Earth. The disk has spiral arms extending beyond 60 au from the star. Gravitational instability, binary companion or planet(s)?



Numerical modeling of the circumbinary disk in GG Tau (Nelson & Marzari, 2016)



Numerical modeling of a circumstellar disk in a binary star system (Picogna & Marzari, 2013)



Equation of fluid dynamics: Euler equations

1) Continuity equation: mass conservation

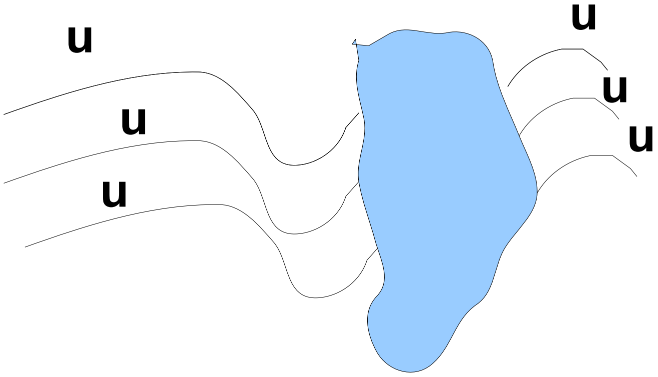
$$\frac{d}{dt}\int_V \rho(\boldsymbol{r},t)\,dV = -\int_S (\rho\boldsymbol{u})\cdot\boldsymbol{n}\,ds$$

Divergence theorem

$$\int_V \frac{\partial\rho}{\partial t}dV = -\int_V \nabla(\rho\boldsymbol{u})\,dV$$

$$\boxed{\frac{\partial\rho}{\partial t}+\nabla(\rho\boldsymbol{u})=0}$$

2) Momentum conservation. In an inertial reference frame the same fluid does not fill the same volume element for an extended interval dt. We must consider a volume co-moving with the fluid otherwise we cannot compute the effects of forces on the same mass element. No flux of mass in/out of the co-moving volume element. It follows the fluid lines



The momentum of the fluid in the volume V is:

$$\int_V \rho(\boldsymbol{r},t)\,\boldsymbol{u}(\boldsymbol{r},t)\,dV$$

Forces acting on the fluid element changing the momentum:

1) Distance forces (gravity, electromagnetism…): The general form is

$$\int_V \rho \boldsymbol{f}\, dv$$

2) Contact forces: forces perpendicular to the volume element like Pressure (we neglect viscous forces).

$$\int_S -P\,\boldsymbol{n}\,dS$$

The equation for the momentum is finally:

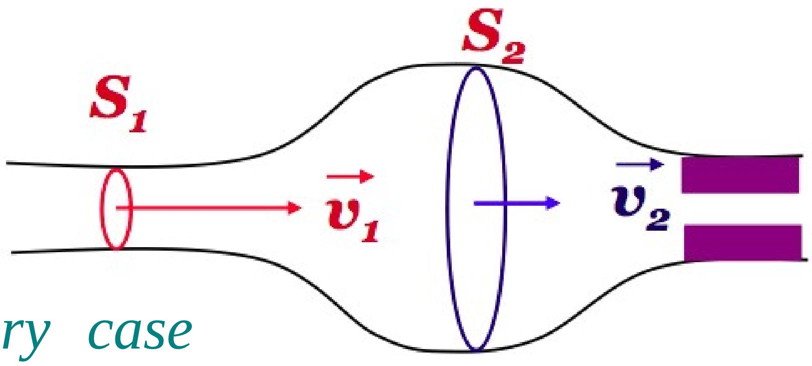
$$\boxed{\frac{d}{dt}\int_V \rho \boldsymbol{u}\,dV = \int_S -P\,\boldsymbol{n}\,dS + \int_V \rho \boldsymbol{f}\,dV}$$

This equation is derived in a co-moving frame. How do we get back to an inertial reference frame where the variables depend on **r** and t? We use the material derivative! Given any physical quantity *f(**r**,t)* of the fluid, ad example the temperature *T(**r**,t), a*t a fixed point ***r*** in space the local derivative is:

$$\frac{\partial f(\boldsymbol{u},t)}{\partial t}$$

If instead we move with the fluid, the variation of f will be given by the local variation + the variation due to the fluid motion. Ad example, in the case of stationary flux, the velocity does not depend on time locally, but as the fluid moves from a region of low cross section to a region of large cross section it changes.

Moving from S1 to S2 the local derivative of the velocity does not change (stationary fluid) but the derivative on a moving fluid element changes. **The material derivative is the rate of change of a quantity (mass, temperature, pressurem energy, momentum..) as experienced by an observer that is moving along with the flow. The observations made by a moving observer are affected by the stationary time-rate-of-change of the property (∂f/∂t), but what is observed also depends on where the observer goes as it floats along with the flow (v·∇f).**



$$\frac{\partial \boldsymbol{u}}{\partial t}=0 \qquad \textit{stationary case}$$

$$\frac{D\boldsymbol{u}}{Dt}=\boldsymbol{u}\cdot\nabla\boldsymbol{u}$$

$$\nabla\boldsymbol{u}=\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

The gradient of a vector is a Jacobian or a 2nd order tensor.

In general for a fluid the material derivative is:

$$\boxed{\frac{D\boldsymbol{u}}{Dt}=\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}}$$

$$(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_x=u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}$$
$$(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_y=u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}$$
$$(\boldsymbol{u}\cdot\nabla\boldsymbol{u})_z=u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}$$

With the material derivative we can transform the equation for the momentum written for a moving fluid element to an equation written in a fixed reference frame with given coordinates.

$$\frac{d}{dt}\int_V \rho\boldsymbol{u}\,dV = -\int_S P\,\boldsymbol{n}\,dS + \int_V \rho\boldsymbol{f}\,dV$$

$$\frac{d}{dt}\int_V \rho\boldsymbol{u}\,dV \Rightarrow \frac{D}{Dt}\int_V \rho\boldsymbol{u}\,dV = \int_V \frac{D}{Dt}(\rho\boldsymbol{u}\,dV) = \int_V \frac{D\boldsymbol{u}}{Dt}\rho\,dV$$

The mass of the fluid element does not change with time so ρ dV is constant in time and the material derivative applies only to **u**

$$\int_V \frac{D\boldsymbol{u}}{Dt}\rho\,dV = \int_V \rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\nabla\boldsymbol{u}\right)dV$$

$$\int_V \rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\nabla\boldsymbol{u}\right)dV = \int_S -P\,\boldsymbol{n}\,dS + \int_V \rho\boldsymbol{f}\,dV = \int_V (-\nabla P + \rho\boldsymbol{f})\,dV$$

$$\boxed{\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\nabla\boldsymbol{u}\right) = -\nabla P + \rho\boldsymbol{f}}$$

3+1 equations for **u** + ρ but P is unknown. An additional equation is needed. For a gas, P is related to the density ρ and temperature T.

$$PV = nRT = \frac{m}{M}RT \quad\Rightarrow\quad P = \frac{m}{V}\frac{RT}{M} = \rho\frac{RT}{M}$$

The relativistic formulation of Euler equations is more compact. Starting from the energy-momentum tensor (in normalized units i.e. c=1, otherwise ρ + p/c^2)

$T^{\mu\nu}=(\rho+p)U^{\mu}U^{\nu}+p\,\eta^{\mu\nu}$ Conservation of this tensor implies

$$\partial_{\nu}T^{\mu\nu}=T^{\mu\nu}{}_{,\nu}=0$$

This equation translates back into Euler's equations. Let's see for example the first conservation equation. We adopt a non-relativistic limit where

$$U^{\mu}=(1,v^{i})\quad |v^{i}|\ll 1\quad p\ll\rho$$

$$\partial_{\nu}T^{\mu\nu}=\partial_{\nu}(\rho+p)U^{\mu}U^{\nu}+(\rho+p)(U^{\nu}\partial_{\nu}U^{\mu}+U^{\mu}\partial_{\nu}U^{\nu})+\partial_{\nu}p\,\eta^{\mu\nu}=0$$

This equation can be contracted by multipling both members by U_{μ}

$$U_{\mu}\partial_{\nu}T^{\mu\nu}=\partial_{\nu}(\rho+p)U_{\mu}U^{\mu}U^{\nu}+(\rho+p)$$
$$(U_{\mu}\partial_{\nu}U^{\mu}U^{\nu}+U_{\mu}U^{\mu}\partial_{\nu}U^{\nu})+U_{\mu}\partial_{\nu}p\,\eta^{\mu\nu}=0$$

Taking into account that $U_{\nu}U^{\nu}=-1$ and that

$$U_{\nu}\partial_{\mu}U^{\nu}=\frac{1}{2}\partial_{\mu}(U_{\nu}U^{\nu})=\frac{1}{2}\partial_{\mu}(-1)=0$$ since

$$\frac{1}{2}\partial_{\mu}(U_{\nu}U^{\nu})=\frac{1}{2}(U_{\nu}\partial_{\mu}U^{\nu}+U^{\nu}\partial_{\mu}(\eta_{\nu\sigma}U^{\sigma}))=$$
$$\frac{1}{2}(U_{\nu}\partial_{\mu}U^{\nu}+\eta_{\nu\sigma}U^{\nu}\partial_{\mu}U^{\sigma})=\frac{1}{2}(U_{\nu}\partial_{\mu}U^{\nu}+U_{\nu}\partial_{\mu}U^{\nu})=2\frac{1}{2}U_{\nu}\partial_{\mu}U^{\nu}$$

$$U_{\nu}\partial_{\mu}T^{\mu\nu}=-\partial_{\nu}\rho U^{\nu}-\partial_{\nu}p\,U^{\nu}-\rho\,\partial_{\nu}U^{\nu}-p\,\partial_{\nu}U^{\nu}+U^{\nu}\partial_{\nu}p=$$
$$-\partial_{\mu}(\rho U^{\mu})-p\,\partial_{\nu}U^{\nu}\approx-\partial_{\nu}(\rho U^{\nu})=0$$

we get that

$$\partial_{\nu}(\rho U^{\nu})=\partial_{t}\rho+\nabla(\rho\,\mathbf{v})=0$$ In non.relativistic limit.

The other equation comes from the projection of the derivative of T along a direction perpendicular to U. In GR the partial derivative must be changed to Covariant derivative.

Energy equation

$$\frac{D}{Dt}\int_V \left(\frac{1}{2}\boldsymbol{u}^2+U\right)\rho dV=\int_V \epsilon\rho dV-\int_S \boldsymbol{F}\cdot\boldsymbol{n}\, dS+\int_V \boldsymbol{f}\cdot\boldsymbol{u}\,\rho dV+\int_S \boldsymbol{u}\cdot(-P\,\boldsymbol{n})\, dS$$

Where U is the internal energy, ε a local energy source and F the flux of energy from one region to another. In differential form the equation becomes

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\boldsymbol{u}^2+U\right)=\epsilon\rho-\nabla\cdot\boldsymbol{F}+\rho\boldsymbol{f}\cdot\boldsymbol{u}-\boldsymbol{u}\cdot\nabla P-P\nabla\cdot\boldsymbol{u}$$

The ‘mechanical’ work is given by

$$\boldsymbol{u}\cdot\rho\frac{D\boldsymbol{u}}{Dt}=\boldsymbol{u}\cdot(-\nabla P+\rho\boldsymbol{f})\Rightarrow$$

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\boldsymbol{u}^2\right)=-\boldsymbol{u}\cdot\nabla P+\rho\boldsymbol{f}\cdot\boldsymbol{u}$$

The thermal part related to gas compression is from the first principle

$$d\,U=-P\,dV+\delta Q$$

Application of Euler equations to an engineering case: the shape of the de Laval nozzle.

Derivation of Bernoulli's equation for the case of a stationary (the fluid properties do not change with time) and inviscid fluid.

If stationary $\frac{\partial \boldsymbol{u}}{\partial t}=0 \quad .... \quad \frac{\partial \rho}{\partial t}=0$

$$\nabla(\rho \boldsymbol{u})=0$$
$$\rho(\boldsymbol{u}\nabla \boldsymbol{u})=-\nabla P+\rho \boldsymbol{f}$$

From calculus: $(\boldsymbol{u}\nabla)\boldsymbol{u}=\nabla\left(\frac{1}{2}\boldsymbol{u}^2\right)-\boldsymbol{u}\times\nabla\times\boldsymbol{u}$

$$\nabla\left(\frac{1}{2}\boldsymbol{u}^2\right)-\boldsymbol{u}\times\nabla\times\boldsymbol{u}=-\frac{1}{\rho}\nabla P+\boldsymbol{f}$$

We focus on a barotropic flow where the pressure is only function of the density P(ρ). We also introduce the state function enthalpy defined as:

$$h=\int\frac{dP}{\rho}=\int\frac{dP}{d\rho}\frac{d\rho}{\rho} \quad \Rightarrow \quad dh=\frac{dP}{\rho}$$

$$dP=\left(\frac{\partial P}{\partial x}\right)dx+\left(\frac{\partial P}{\partial y}\right)dy+\left(\frac{\partial P}{\partial z}\right)dz \quad \Rightarrow \quad \frac{\nabla P}{\rho}=\nabla h$$

Introducing enthalpy in the previous equation, assuming that the force can be expressed as the gradient of a potential V, we get:

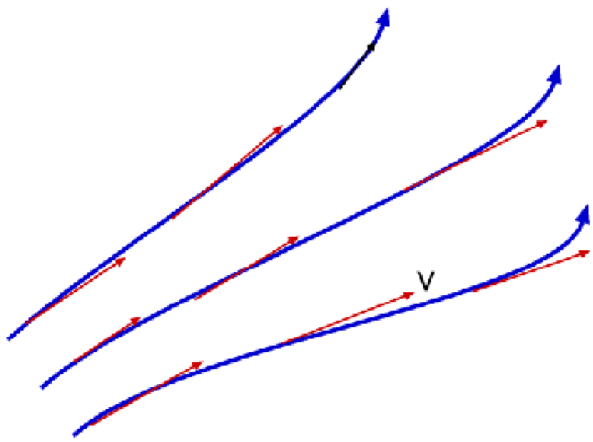
$$\boldsymbol{u}\times\nabla\times\boldsymbol{u}=\nabla\left(\frac{1}{2}\boldsymbol{u}^2\right)+\nabla h+\nabla V=\nabla\left(\frac{1}{2}\boldsymbol{u}^2+h+V\right)$$

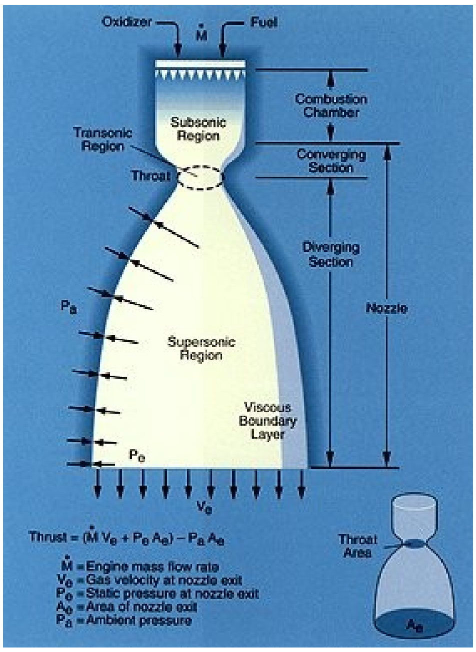
If we multiply both sides of the equation by **u**

$$\boldsymbol{u}\cdot(\boldsymbol{u}\times\nabla\times\boldsymbol{u})=0=\boldsymbol{u}\cdot\nabla\left(\frac{1}{2}\boldsymbol{u}^2+h+V\right)$$

The velocity **u** is tangent to the fluid motion, as a consequence the quantity

$\frac{1}{2}\boldsymbol{u}^2+h+V$ **is constant along the streamlines. Bernoulli theorem!**





Design of the nozzle: why the cross section is diverging when the flux becomes supersonic?

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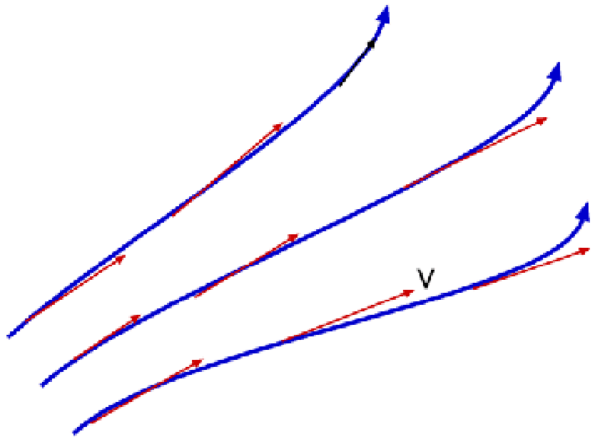

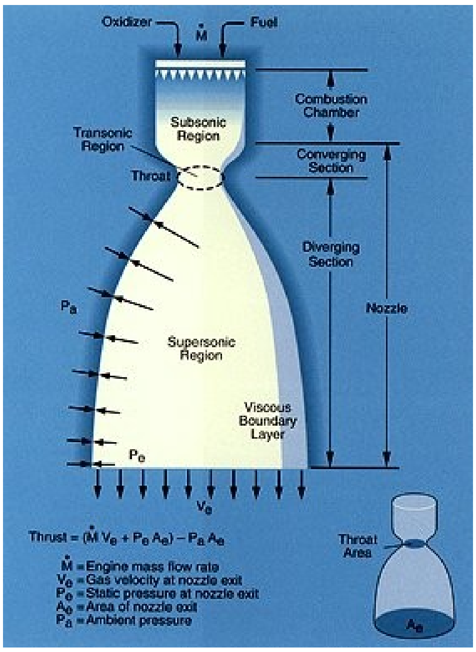

Design of the nozzle: why the cross section is diverging when the flux becomes supersonic?

Let's assume the flux through the nozzle is stationary and barotropic. We also neglect gravity (too small to significantly affect the high velocity gas). The cross section of the nozzle is called A(x) and it is measured at different section distant x from the top. The equations governing the flux are:

$$\frac{1}{2}u^2 + h = const$$
$$\rho u A = const$$

The first equation is derived respect to x:

$$\frac{1}{2} 2u\frac{du}{dx} + \frac{dh}{dx} = 0 \quad \Rightarrow \quad u\,du + dh = 0$$

$$\frac{dh}{dx} = \frac{dP}{dx}\cdot\frac{1}{\rho} = \frac{dP}{d\rho}\cdot\frac{d\rho}{dx}\frac{1}{\rho}$$

$$u\cdot du + \frac{1}{\rho}\frac{dP}{d\rho}d\rho = 0$$

The sound speed in barotropic approximation is:

$$c_s^2 = \frac{dP}{d\rho}$$

$$u \cdot du + \frac{c_s^2}{\rho} d\rho = 0 \quad \Rightarrow \quad \frac{d\rho}{\rho} = -\frac{u^2}{c_s^2}\frac{du}{u} = -M^2 \frac{du}{u}$$

Where M is the Mach number.

$$\frac{d\rho}{\rho} = -M^2 \frac{du}{u}$$

$$\rho u A = constant$$

From the second equation we can derive the following relation between the differentials:

$$\frac{1}{\rho u A}\frac{d}{dx}(\rho u A) = \frac{d\rho}{dx}\frac{uA}{\rho u A} + \frac{du}{dx}\frac{\rho A}{\rho u A} + \frac{dA}{dx}\frac{u\rho}{\rho u A}$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \quad \Rightarrow \quad \frac{d\rho}{\rho} = -\frac{du}{u} - \frac{dA}{A}$$

Combining the equations we finally get to:

$$-\frac{dA}{A} = (1 - M^2)\frac{du}{u}$$

- **If M <<1 (subsonic flow), A must be reduced to increase u**
- **If M > 1 (supersonic flow) A must be increased to increase u**

The optimal design of the nozzle has to converge the flux until the velocity becomes supersonic. From then on, the nozzle has to be divergent.

$$u \cdot du + \frac{c_s^2}{\rho} d\rho = 0 \quad \Rightarrow \quad \frac{d\rho}{\rho} = -\frac{u^2}{c_s^2}\frac{du}{u} = -M^2\frac{du}{u}$$

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Parker's solution for an unmagnetized solar wind

Simplifications: spherically symmetric, isothermal and non-magnetic non-rotating flux.

The steady state equation for the radial component of the **momentum** is:

$$\rho u \frac{du}{dr} = -\frac{dP}{dr} - \rho \frac{GM}{r^2}$$

Isothermal approximation →

$$\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = c_s^2 \frac{d\rho}{dr}$$

$$u\frac{du}{dr} = -c_s^2 \frac{d\rho}{dr}\frac{1}{\rho} - \frac{GM}{r^2}$$

From the mass conservation equation

$$\frac{d\rho}{dr}\frac{1}{\rho} = -\frac{du}{dr}\frac{1}{u} - \frac{dA}{dr}\frac{1}{A}$$

It is a symmetrical spherical flux so the area is $4\pi r^2$

$$c_s^2 \frac{d\rho}{dr}\frac{1}{\rho} = -c_s^2 \frac{du}{dr}\frac{1}{u} - 2c_s^2 \frac{1}{r}$$

$$u\frac{du}{dr} = -c_s^2 \frac{du}{dr}\frac{1}{u} - \frac{2c_s^2}{r} - \frac{GM}{r^2}$$

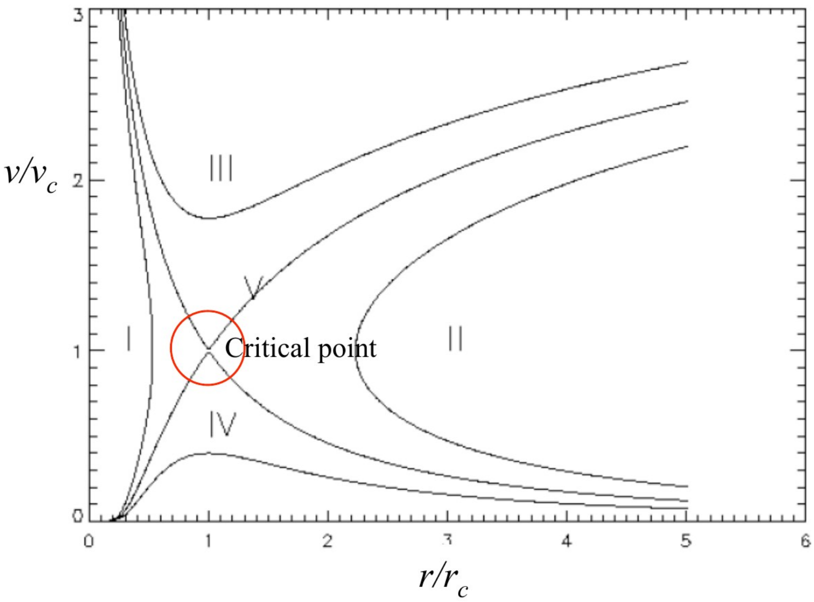
$$\frac{u^2}{u}\frac{du}{dr} - c_s^2 \frac{du}{dr}\frac{1}{u} = \frac{2c_s^2}{r} - \frac{GM}{r^2} \quad \Rightarrow$$

$$\boxed{\frac{(u^2 - c_s^2)}{u}\frac{du}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2}}$$

The equation is similar to that of de Laval nozzle. The solution can be written as (C is an integration constant):

$$\left(\frac{u}{c_s}\right)^2 - \ln\left(\frac{u}{c_s}\right)^2 = 4\ln\left(\frac{r}{r_c}\right) + \frac{4r_c}{r} + C \quad where \quad r_c = \frac{GM}{2}c_s^2$$

- Solution I and II are double valued. Solution II also doesn't connect to the solar surface.
- Solution III is too large (supersonic) close to the Sun - not observed.
- Solution IV is called the solar breeze solution.
- Solution V is the solar wind solution (confirmed in 1960 by Mariner II). It passes through the critical point at $r = r_c$ and $v = v_c$.



Solution V has C=-3

At the critical point the flux from subsonic becomes supersonic (as in the nozzle) and $v_c = c_s$

Accretion disks

Cylindrical coordinates are typically used: (r, ϕ, z)

The disk is considered thin and axis-symmetric $\frac{\partial}{\partial \phi} = 0$

$\boldsymbol{u} \cdot \boldsymbol{e}_z = 0$

The fluid velocity is $\boldsymbol{u} = (u, r\Omega, 0)$

In cylindrical coordinates the r-component of the momentum equation is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} - r\Omega^2 = -\frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{\partial \Psi}{\partial r}$$

Where the gravitational potential is

$$\Psi = -\frac{GM}{(r^2 + z^2)^{1/2}}$$

Under the simplification conditions that $\frac{\partial P}{\partial r} \ll 1$ $u = 0$

$$r\Omega^2 = \frac{GM}{r^2} \quad \Rightarrow \quad \Omega = \sqrt{\frac{GM}{r^3}}$$

The usual Keplerian rotation.

1) Euler equation for u in cylindrical coordinates.

Vectorial form

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P - \rho \nabla \Psi$$

r-component

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi^2}{r} + u_z \frac{\partial u_r}{\partial z}\right) = -\frac{\partial P}{\partial r} - \rho \frac{\partial \Psi}{\partial r}$$

φ-component

$$\rho\left(\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\phi u_r}{r} + u_z \frac{\partial u_\phi}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \phi} - \rho \frac{\partial \Psi}{\partial \phi}$$

z-component

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \rho \frac{\partial \Psi}{\partial z}$$

2) Scale height of a disk in isothermal approximation

In hydrostatic equilibrium, $u_z = 0$ so the z-component of the Euler equations read:

$$\frac{1}{\rho}\frac{\partial P}{\partial z}=-\frac{\partial \Psi}{\partial z}=\frac{\partial}{\partial z}\left[\frac{GM}{\left(r^2+z^2\right)^{1/2}}\right]\sim-\frac{GM\,z}{r^3}$$

In **isothermal approximation (fixed temperature profile),** the state equation is:

$$P=\frac{R\rho T}{\mu}$$

$$c_s^2=\frac{dP}{d\rho}=\frac{RT}{\mu} \qquad \mu \text{ is the mean molecular weight}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z}=\frac{1}{\rho}c_s^{\,2}\frac{\partial \rho}{\partial z}=-\frac{GM\,z}{r^3} \quad \Rightarrow \quad \ln(\rho)=-\frac{1}{2}\frac{\Omega^2}{c_s^2}z^2$$

$$\rho(z)=e^{-\frac{1}{2}\frac{\Omega^2}{c_s^2}z^2}=e^{-\frac{1}{2}\frac{z^2}{H^2}} \qquad \text{where} \qquad H=\frac{c_s}{\Omega}$$

$$H=\frac{c_s}{\Omega}=\frac{c_s}{\Omega r}r=h\,r$$

The vertical disk scale-height h is around 0.05 so the isothermal sound speed is smaller than the keplerian velocity

3) Equation for the superficial density of a disk

$$\Sigma(r,t)=\int_{-\infty}^{\infty}\rho(r,t,z)\,dz \quad\Rightarrow\quad \Sigma=\rho_0\sqrt{2\pi}\;h$$

$$Shear \qquad A=r\frac{d\Omega}{dr}=-\frac{3}{2}\sqrt{(GM)}\,r^{-5/2}$$

Specific angular momentum (*per unit of mass*)

$$\vec{J}=r\cdot\vec{e}_\phi\cdot\vec{u}=r\begin{Bmatrix}0\\1\\0\end{Bmatrix}\cdot\begin{Bmatrix}u\\r\Omega\\0\end{Bmatrix}=r^2\Omega$$

The ϕ *component of the momentum conservation equation must include a viscosity term*

$$\rho\left\{\frac{\partial u_\phi}{\partial t}+u\frac{\partial u_\phi}{\partial r}+\frac{u\,u_\phi}{r}\right\}=f_\phi\rho$$

where f *is the viscous force per mass unit*

$$\frac{\partial J}{\partial t}+u\frac{\partial J}{\partial r}=\frac{DJ}{Dt}=r f_\phi$$

$$u\frac{\partial j}{\partial r}=u\frac{\partial(r u_\phi)}{\partial r}=u u_\phi+u r\frac{\partial u_{phi}}{\partial r}$$

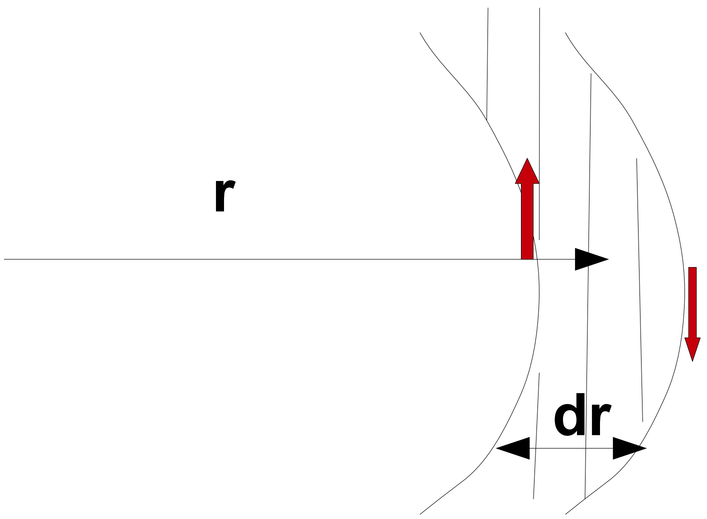
4)

4) Viscosity

Friction between adiacent surfaces. The component of viscosity tangent to the motion is

$$\mu A(r) = \rho \nu A(r)$$

This is a force per unit length. The larger is the shear or the gas density, the higher is the viscosity



Let's consider a ring of gas with width *dr*. It feels the friction from the gas at the outer and inner edges of the ring.

The force at the inner edge tends to accelerate it while that at the outer edge slows it down due to the Keplerian motion.

$$F_{inn} = 2\pi\left[r\nu|A|\int_{-\infty}^{\infty}\rho\,dx\right]_{r-dr/2} = 2\pi\left(r\nu|A|\Sigma\right)_{r-dr/2}$$

$$F_{out} = 2\pi\left(r\nu|A|\Sigma\right)_{r+dr/2}$$

The torque acting on the ring, assuming that the forces act at center of the ring r, is

$$T = F_{inn}r_{inn} - F_{out}r_{out}$$

Be careful!, the sign of d Ω /dt is negative.

5)

$$T = -2\pi\left(r^2\nu\Sigma r\frac{d\Omega}{dr}\right)_{r-dr/2} + 2\pi\left(r^2\nu\Sigma r\frac{d\Omega}{dr}\right)_{r+dr/2} =$$

$$= 2\pi\frac{d}{dr}\left(r^2\nu\Sigma r\frac{d\Omega}{dr}\right)\cdot dr$$

The ring mass is: $M_{ring} = 2\pi r\Sigma dr$

T for **mass unit** is

$$T = r f_\phi = 2\pi\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)\frac{dr}{2\pi r\Sigma dr} =$$

$$= \frac{1}{r\Sigma}\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)$$

The equation for the specific angular momentum is finally:

$$\boxed{\frac{DJ}{dt} = \frac{1}{r\Sigma}\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)}$$

6)

From the mass conservation we can derive an additional equation:

$$\frac{D\rho}{Dt}+\rho\,\nabla\cdot\vec{u}=0$$

Which, once integrated in z, gives an equation for the superficial density Σ

$$\frac{D\Sigma}{Dt}+\Sigma\nabla\cdot\vec{u}=\frac{\partial\Sigma}{\partial t}+\nabla\cdot(u\cdot\Sigma)=0$$

In cylindrical coordinates

$$\Sigma\nabla\cdot\vec{u}=\frac{1}{r}\Sigma\frac{\partial}{\partial r}(r\,u)+\frac{1}{r}\Sigma\frac{\partial u_\phi}{\partial\phi}+\Sigma\frac{\partial u_z}{\partial z} \qquad u\nabla\Sigma=\frac{1}{r}u\frac{\partial}{\partial r}(r\Sigma)$$

The last two terms are =0 because of the axisymmetric and thin disk approximations.

$$\boxed{\frac{\partial\Sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\,u\,\Sigma)=0}$$

The Keplerian frequency Ω is function only of r and so is also J by definition

$$J=r^2\,\Omega$$

$$\boxed{\frac{DJ}{Dt}=\frac{dJ}{dr}\cdot u}$$

Where u is the component along r of the fluid velocity vector u. We are looking for a stationary solution for J → its time partial derivative = 0

$$\frac{DJ}{dt}=\frac{1}{r\Sigma}\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)$$

$$\frac{DJ}{Dt}=\frac{dJ}{dr}\cdot u$$

$$\frac{\partial\Sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r u\Sigma)=0$$

$$J=r^2\Omega$$

$$\Omega=\sqrt{\frac{GM}{r^3}}$$

$$u\frac{dJ}{dr}=\frac{1}{r\Sigma}\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)$$

$$u r\Sigma=\frac{1}{\frac{dJ}{dr}}\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)$$

$$\frac{\partial\Sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(\left(\frac{dJ}{dr}\right)^{-1}\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)\right)=0$$

$$\frac{\partial\Sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(\left(\frac{d(r^2\Omega)}{dr}\right)^{-1}\frac{d}{dr}\left(r^3\nu\Sigma\frac{d\Omega}{dr}\right)\right)=0$$

$$\frac{\partial\Sigma}{\partial t}=\frac{3}{r}\frac{\partial}{\partial r}\left(r^{1/2}\frac{\partial}{\partial r}[\nu\Sigma r^{1/2}]\right)$$

5) Viscous mass accretion rate on the star:

The conservation of mass in an annulus leads to:

$$\frac{\partial \Sigma}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\,u\,\Sigma)=0 \;\Rightarrow\; 2\pi r\frac{\partial \Sigma}{\partial t}=2\pi\frac{\partial}{\partial r}(-r\,u\,\Sigma)$$

For the angular momentum:

$$L=r\,v_{\phi}\,M=r\cdot(r\,\Omega)\cdot(2\pi r\,dr\,\Sigma)$$

Since $F_L=(r\,M\,v_{\theta}\,dr)\frac{1}{dt}=r\,M\,v_{\theta}\cdot u$ we get

(u is negative)

$$\frac{\partial}{\partial t}(2\pi r\,dr\,\Sigma\cdot r^2\Omega)=-2\pi(r+dr)u(r+dr)\Sigma(r+dr)(r+dr)^2\Omega(r+dr)$$
$$+2\pi r\,u(r)\Sigma(r)r^2\Omega(r)$$

$$2\pi r\,dr\frac{\partial}{\partial t}(\Sigma\cdot r^2\Omega)=-2\pi\frac{\partial}{\partial r}(r\,\Sigma\,u\cdot r^2\Omega)dr$$

In the change of angular momentum due to the mass flux between adjacent rings also the viscous force contributes.

$$2\pi r\frac{\partial}{\partial t}(\Sigma\cdot(r^2\Omega))+2\pi\frac{\partial}{\partial r}(r\,\Sigma\,u\cdot r^2\Omega)=$$
$$=2\pi\frac{\partial}{\partial r}\left(r^2\nu\,\Sigma\,r\frac{d\Omega}{dr}\right)$$

See pg. 5 without division by the mass.

We look for a **stationary** solution to the differential equation where Σ is constant with time.

$$\frac{\partial}{\partial t}\Rightarrow 0$$

$$\frac{\partial}{\partial r}(r\Sigma u\cdot r^2\Omega)=\frac{\partial}{\partial r}(r^2\nu\Sigma r\frac{d\Omega}{dr})$$

The mass accretion rate from a ring to an inner one is given by:

$$\boxed{\dot{M}=-2\pi r\,dr\,\Sigma/dt=-2\pi r\Sigma u} \Rightarrow -\frac{\partial}{\partial r}(\dot{M}r^2\Omega)=\frac{\partial}{\partial r}(2\pi\nu\Sigma r^3\frac{d\Omega}{dr})$$

Integrating in r the previous equation we get:

$$-\int_{r_s}^{r}\frac{\partial}{\partial r}\left(\dot{M}(r')r'^2\Omega(r')\right)dr'=\int_{r_s}^{r}\frac{\partial}{\partial r}\left(2\pi\nu\Sigma(r')r'^3\frac{d\Omega(r')}{dr'}\right)dr'$$

Where r_s is the star radius. It is assumed for simplicity that the inner limit of the disk is the star surface. In this case:

$$-\dot{M}r^2\Omega=2\pi\nu\Sigma r^3\frac{d\Omega}{dr}+const$$

In the constant all values are computed on the star surface where $\frac{d\Omega}{dr}=0$ ..since the star rotates as a rigid body (approx…).

$$const=-\dot{M}r_s^2\Omega(r_s)$$

At the end

$$\dot{M}\, r^2 \sqrt{\left(\frac{GM}{r^3}\right)} = -2\pi\Sigma\nu\frac{3}{2}r^2\sqrt{\left(\frac{GM}{r^3}\right)} + \dot{M}\, r_s^2\sqrt{\left(\frac{GM}{r_s^3}\right)} =$$

$$-3\pi\Sigma\nu r^2\sqrt{\left(\frac{GM}{r^3}\right)} + \dot{M}\sqrt{\left(\frac{r_s}{r}\right)}\sqrt{\left(\frac{GM}{r^3}\right)}\, r^2$$

$$\dot{M}\left(1-\sqrt{\left(\frac{r_s}{r}\right)}\right) = 3\pi\nu\Sigma$$

For $r >> r_s$ (in the majority of the disk apart very close to the inner border) the following is obtained:

$$\dot{M} = 3\pi\nu\Sigma$$

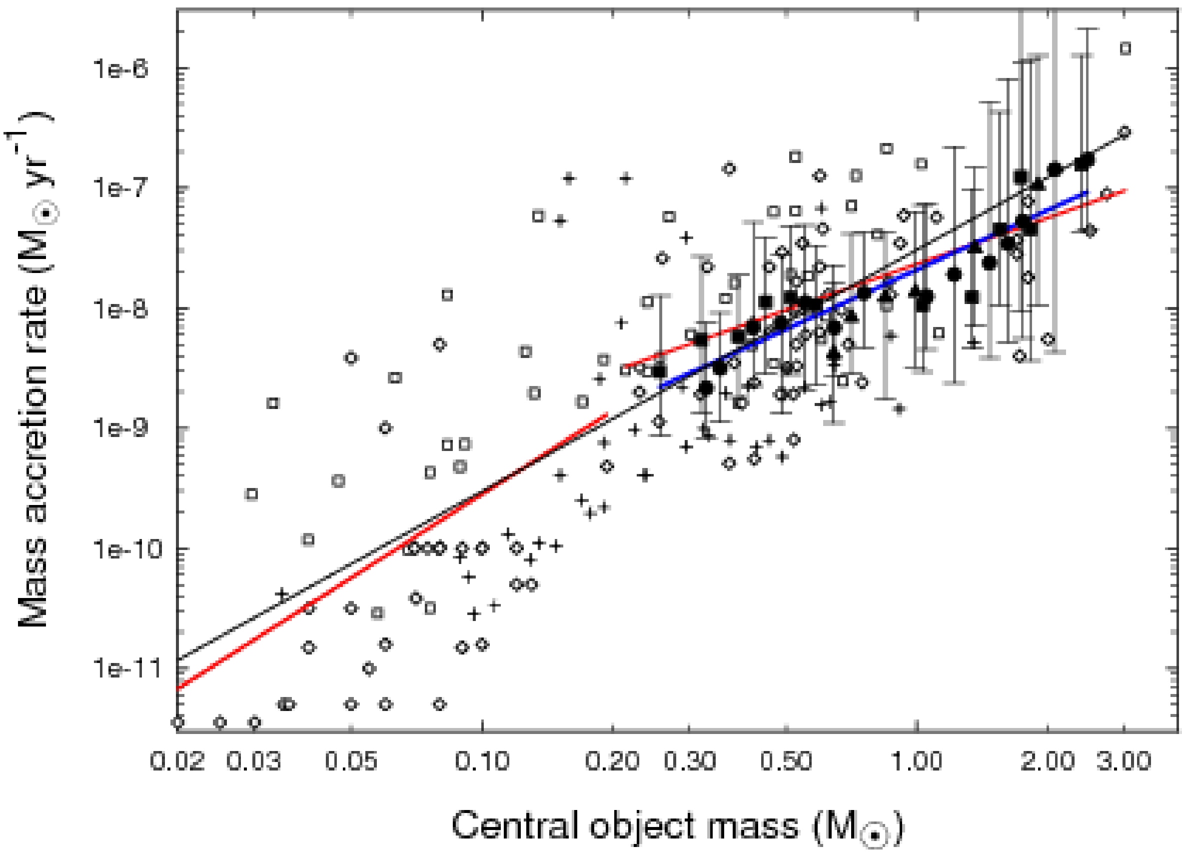
This value is derived assuming a stationary state, however in general Σ depends on t.

The Shakura-Sunayev alpha viscosity:

$$\nu = \alpha\, c_s h$$

Alpha is a constant all over the disk and an adimensional quantiy. It ranges from 0.1 to 0.00001.

Observations suggest that the mass accretion rate depends on the mass of the central object. It implies that the heavier is the star the more massive is the disk (higher Σ). $\dot{M} = 3\pi\nu\Sigma$



6) Self-similar solutions:

When $\nu \propto r^{\gamma}$

The equation
$$\frac{\partial \Sigma}{\partial t}=\frac{3}{r}\frac{\partial}{\partial r}\left(r^{1/2}\frac{\partial}{\partial r}\left[\nu \Sigma r^{1/2}\right]\right)$$

admits self-similar solutions of the type:

$$\Sigma(r,t)=\frac{C}{3\pi \nu_1 r^{\gamma}}T^{-(\frac{5}{2}-\gamma)/(2-\gamma)}\exp\left(-\bar{r}\frac{(2-\gamma)}{T}\right)$$

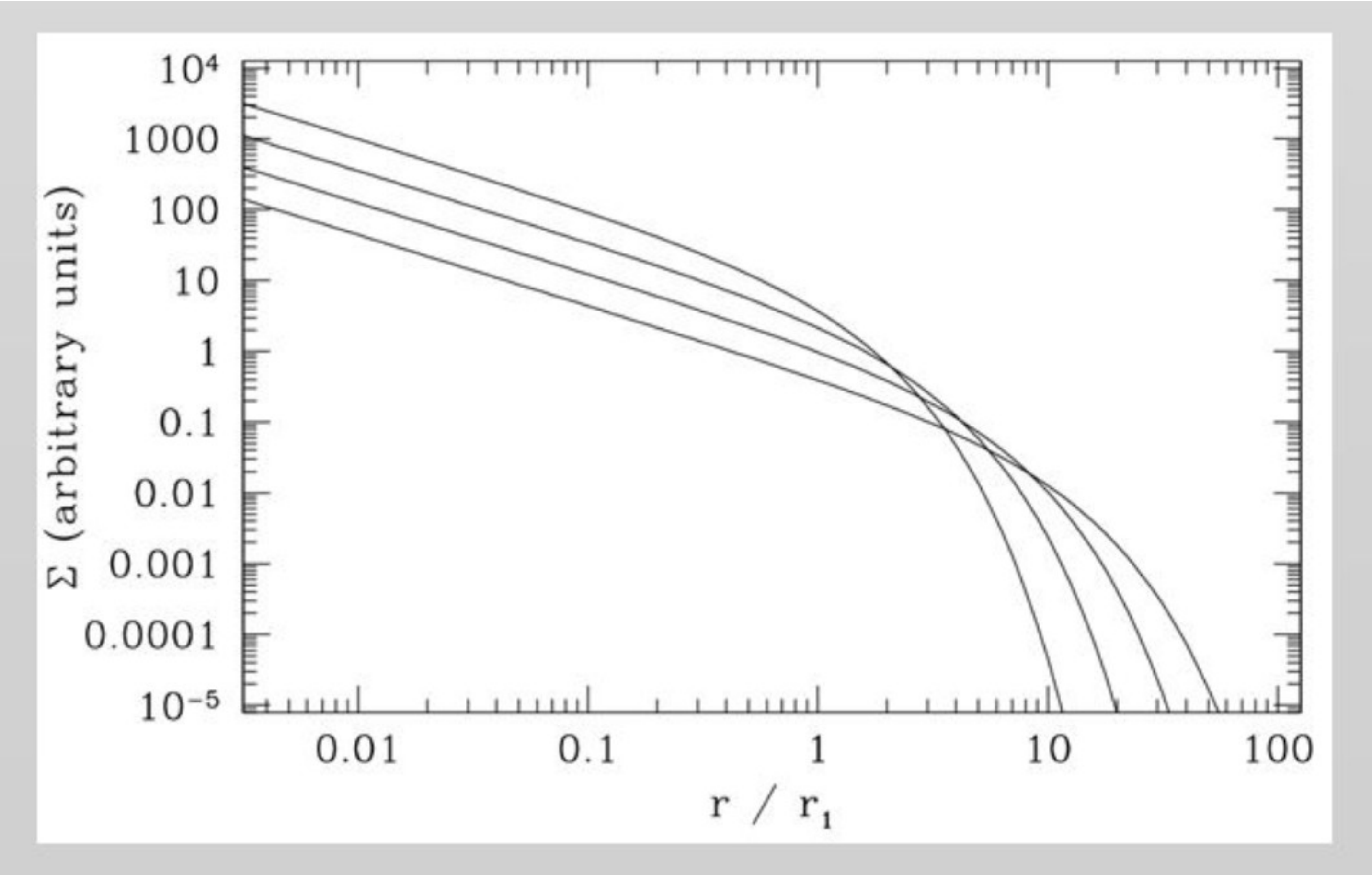
$$T=\frac{t}{t_s}+1 \quad dove \quad t_s=\frac{1}{3(2-\gamma)^2}\frac{R_1}{\nu_1} \qquad \nu=\alpha c_s h \ [1/s]$$

t_s is the **viscous scaling time:** for timescales shorter than t_s the system does not significantly evolve.

$$\bar{r}=\frac{r}{R_1} \quad e \quad \nu_1=\nu(R_1)$$

where **R_1** is a radial scaling factor (1 AU etc…). The constant **C** is given from the initial conditions.

Self-similar: it means that they are scalable and do not depend on the initial conditions.

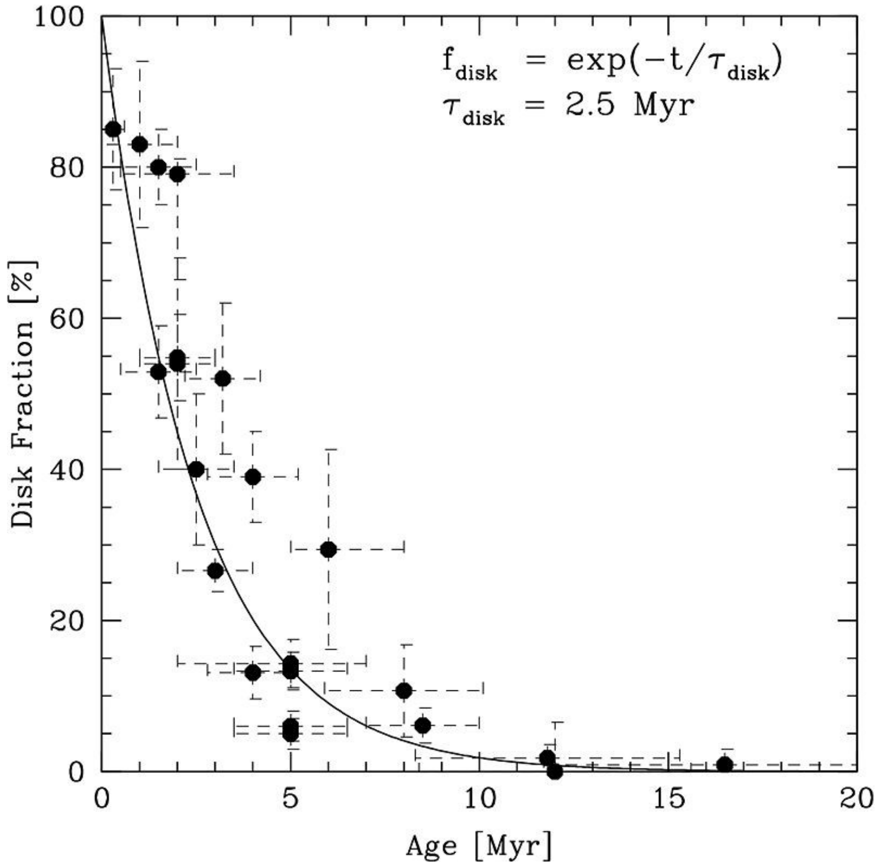


$$\dot{M}(r,t)=C\,T^{-(\frac{5}{2}-\gamma)/(2-\gamma)}\exp\left(-\bar{r}\,\frac{(2-\gamma)}{T}\right)\left[1-\frac{2(2-\gamma)\,r^{(2-\gamma)}}{T}\right]$$

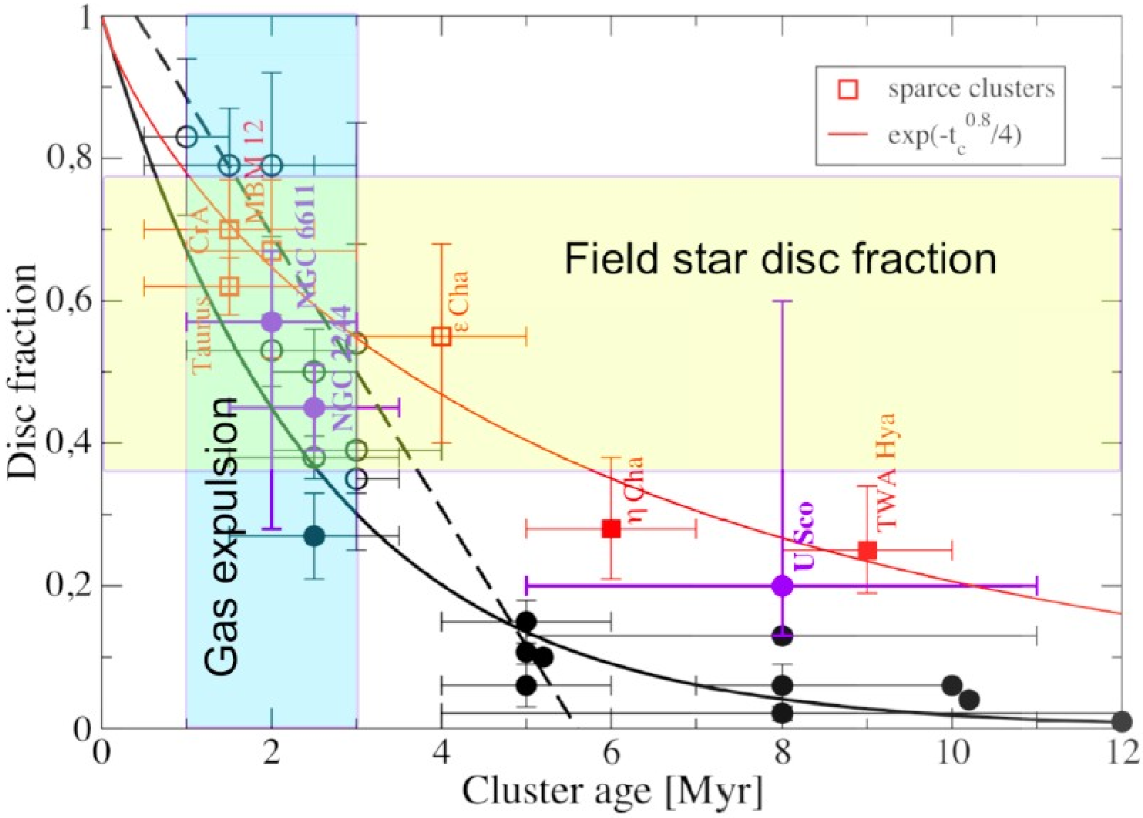
$$When \quad r=r_T=R_1\left[\frac{T}{2(2-\gamma)}\right]^{\frac{1}{(2-\gamma)}}$$

The mass accretion rate changes sign: positive inside, negative outside.

PROBLEM WITH ACCRETION DISKS: the rate of viscous dissipation is too slow! In addition, from observations it looks like disks are not expanding….



Ages of disks estimated in stellar clusters: fraction of stars with disks as a function of the cluster age. However, these ages might be too short since clusters dissipate and only core stars are observed for older clusters. .



..these stars are strongly irradiated by O, B bright stars which form in the core of clusters. For these reason, core stars lose their disks on a shorter timescale. The disk age is then underestimated. Red line more realistic estimate. (Pfalzner et al 2014).

7) Vertical structure of the disk.

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$T \sim 10^4$

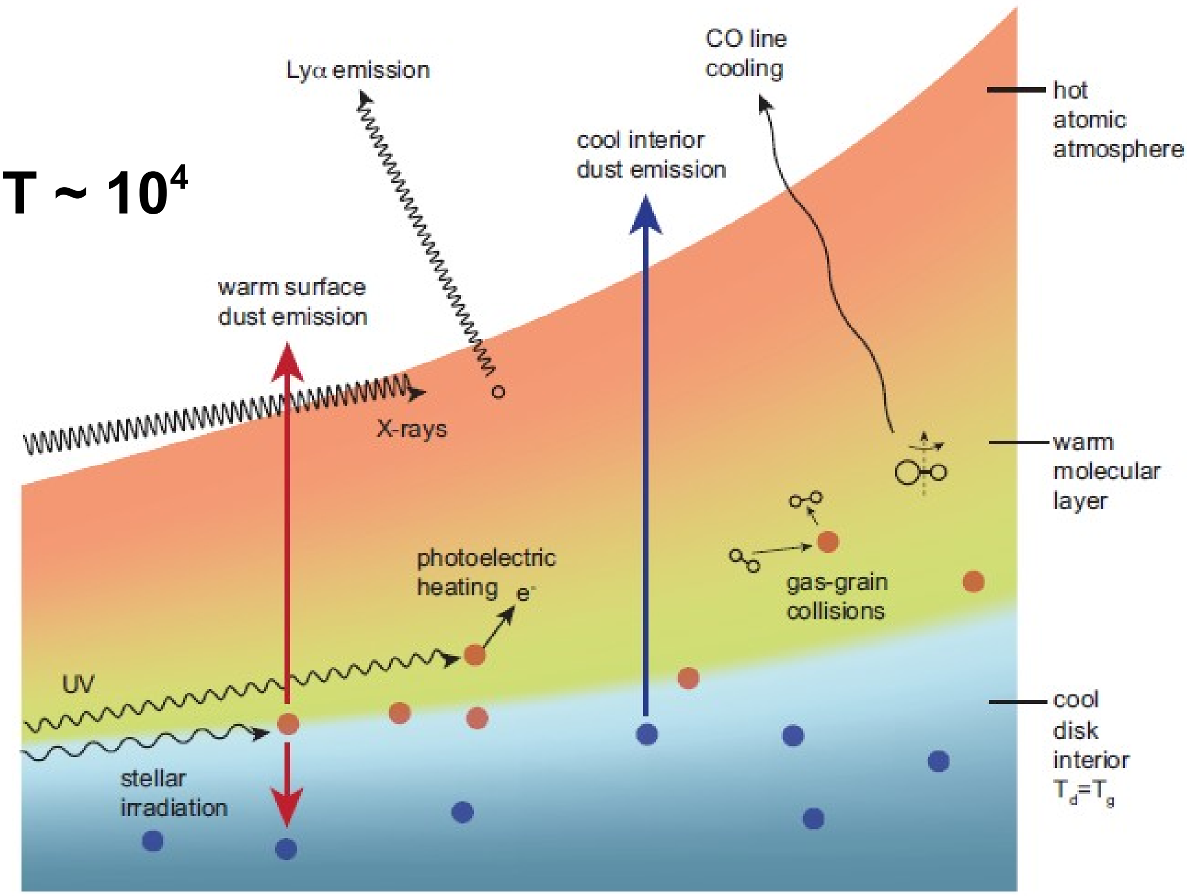


Fig. 5 Illustration of some of the physical processes determining the temperature and emission properties of irradiated protoplanetary disks.

8) Boundary layer (border between the star surface and the disk)

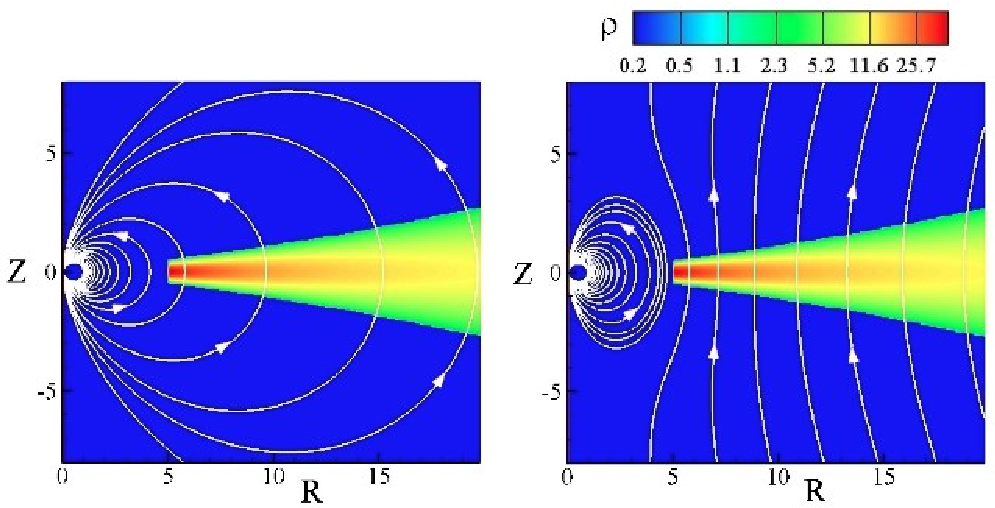
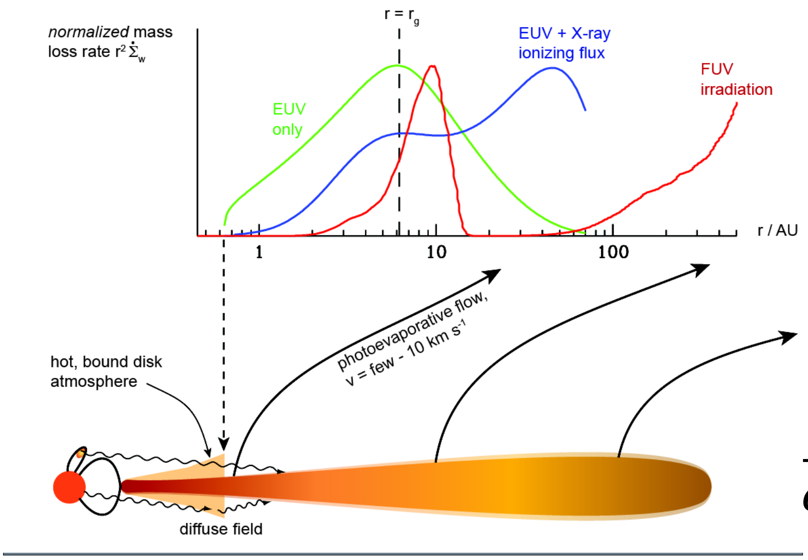


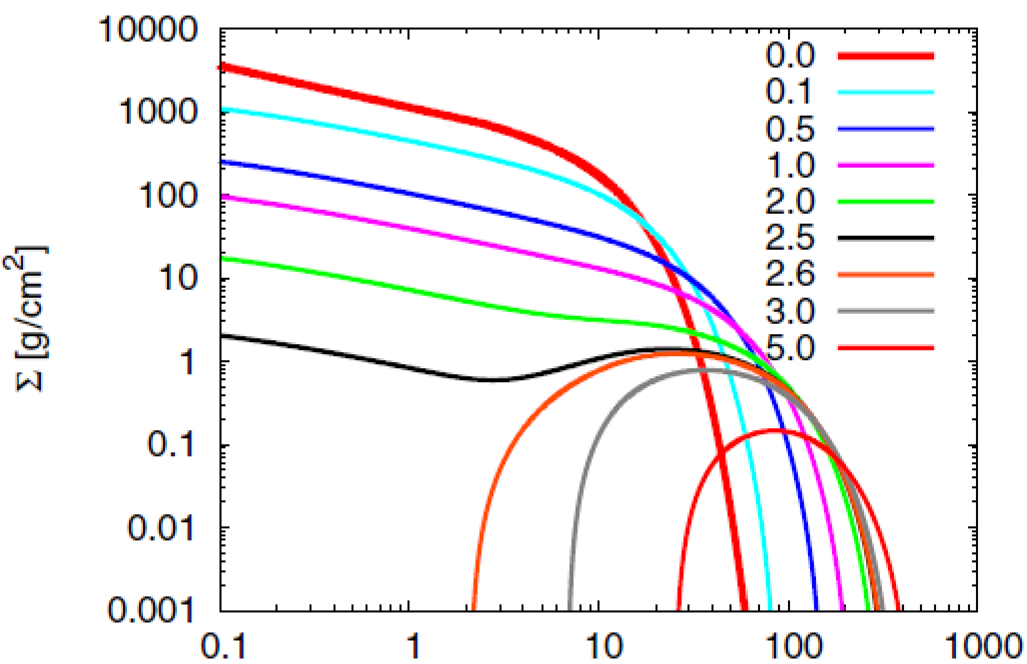
Figure 2. Plot of density ρ and poloidal field lines for a stellar dipole (left), a stellar dip
dipole and an anti-parallel disc-field.

9) Photoevaporation

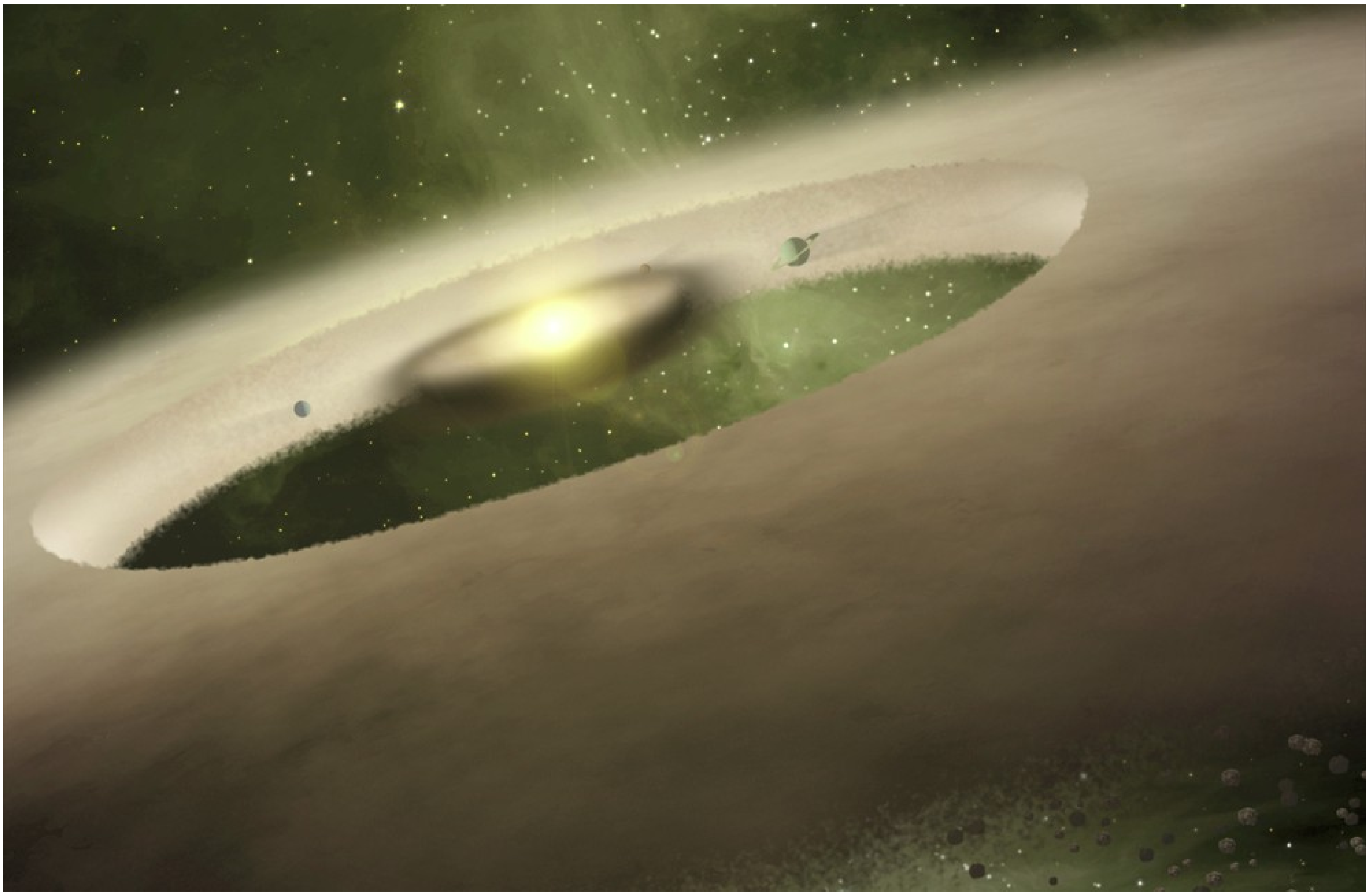


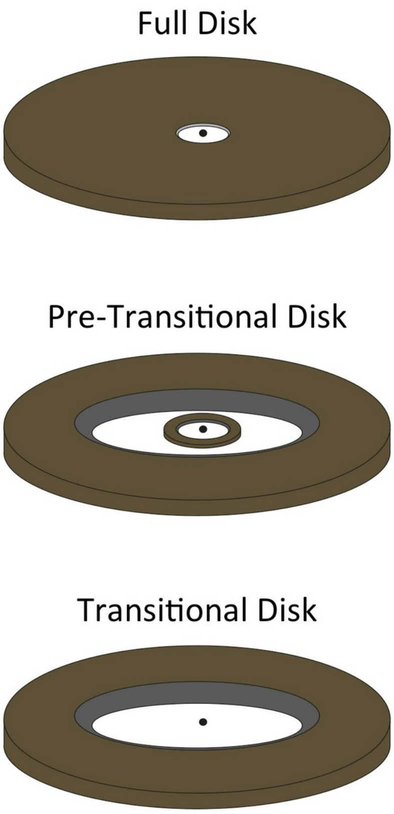
$$\frac{\partial}{\partial t}\Sigma+\frac{1}{r}\frac{\partial}{\partial r}(r\,\Sigma u_r)=-\dot{\Sigma}_{pe}$$

At a given radius (about 1-10 au) the mass inflow due to viscosity is slower than the mass loss due to photoevaporation. The inner disk is not refilled and a hole developes in the inner region which propagates outside.

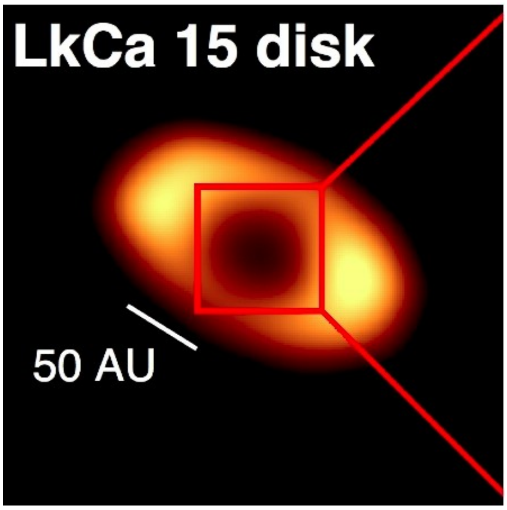


Transition disks.





Planet formation?
Stellar winds?
Photoevaporation?



10) Self gravity

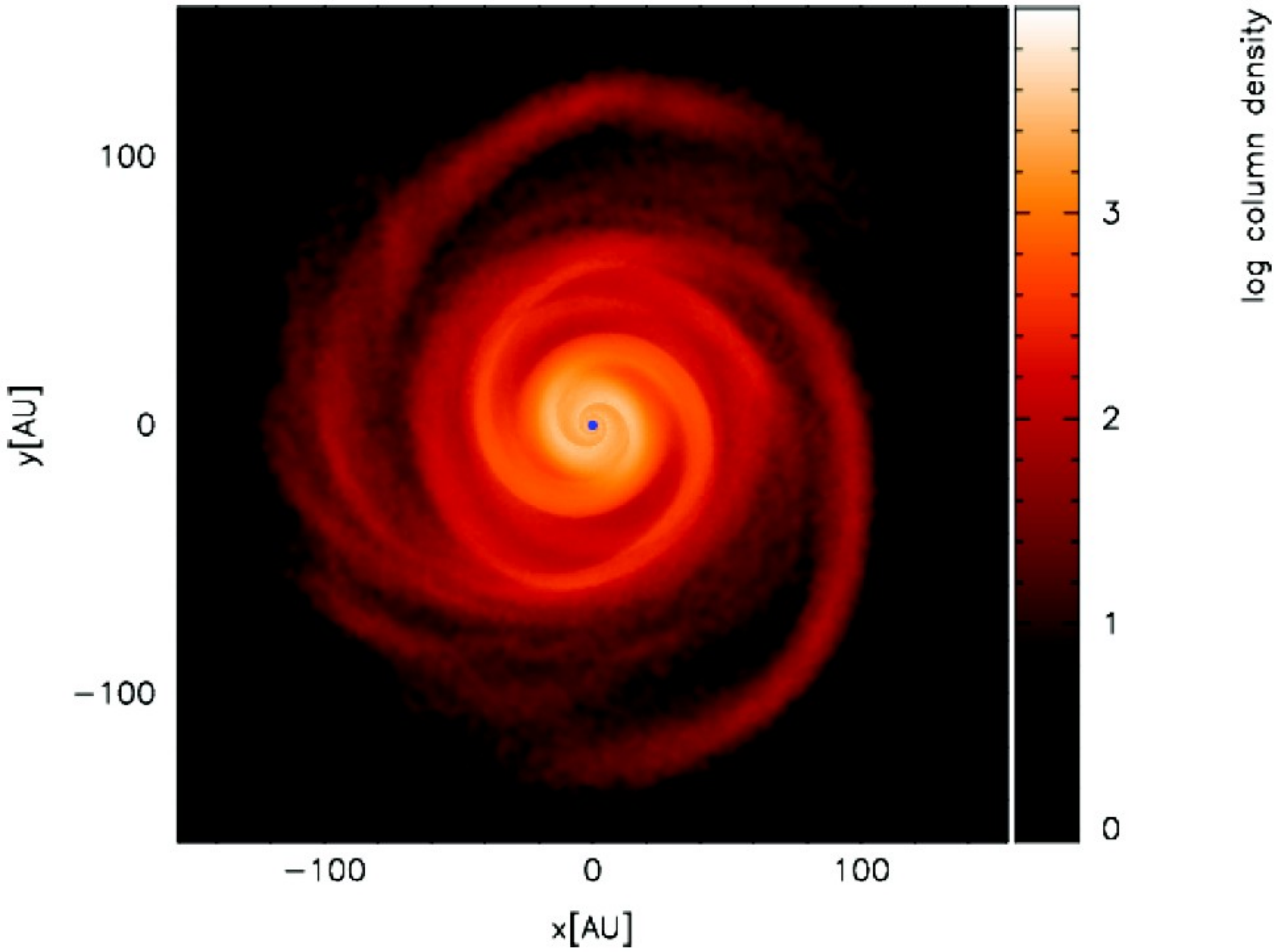
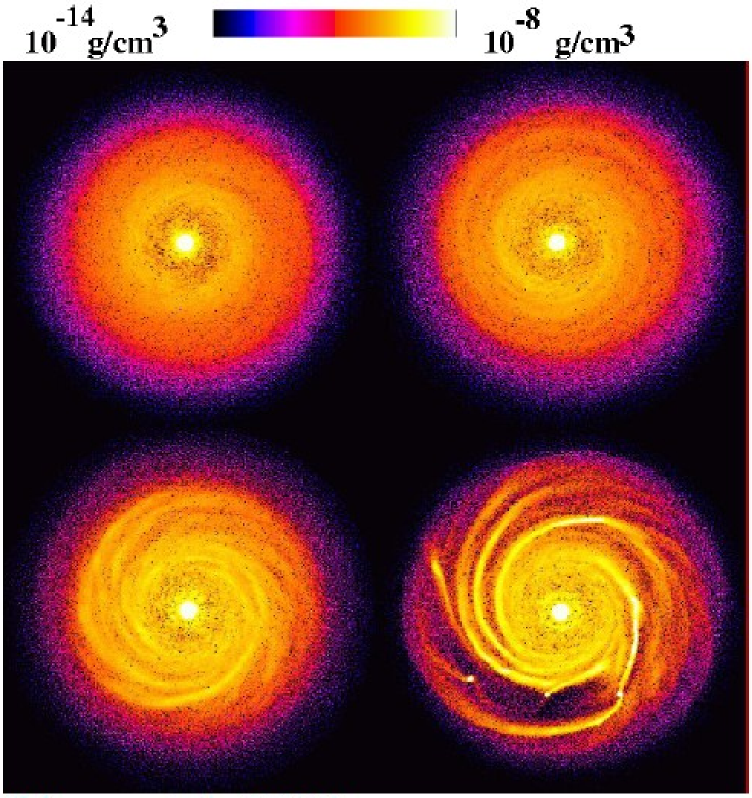


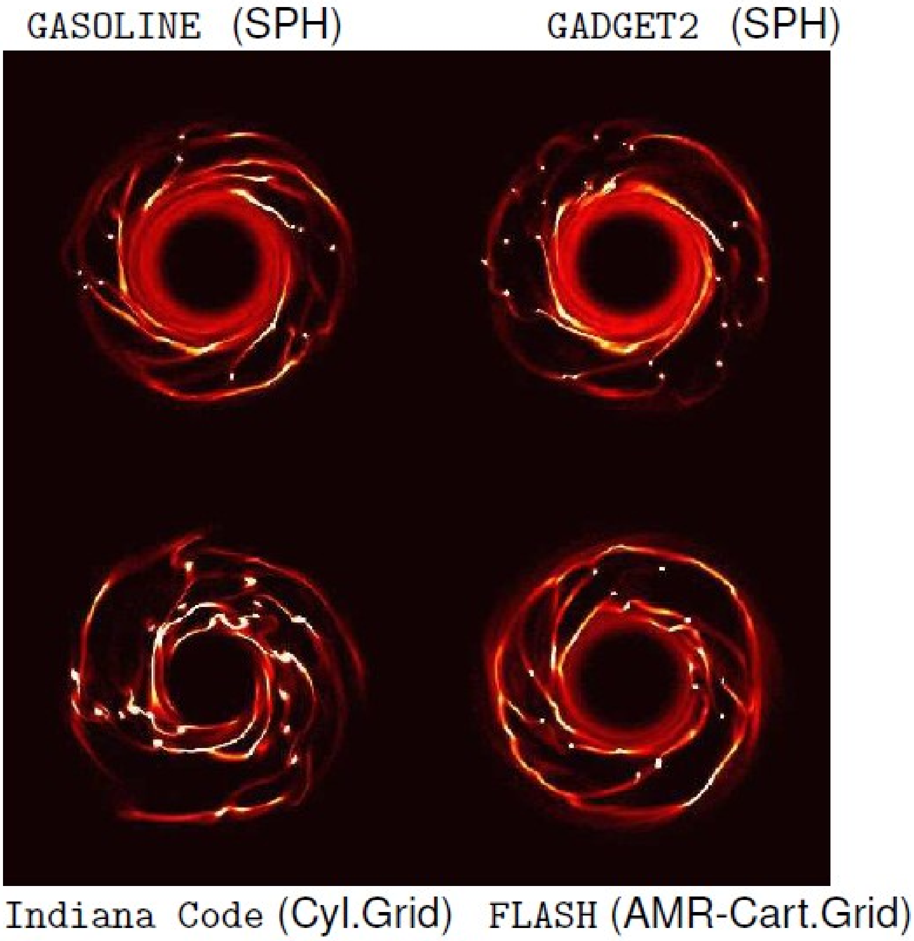
Figure 4: Structure of a massive ($M_{\rm disk} = M_* = M_\odot$) self-gravitating disk in the non-fragmenting regime, based on simulations by Forgan et al. (2010). The simulated disk had an initial surface density profile $\Sigma \propto r^{-3/2}$, and was evolved with an approximate radiative transfer scheme. At this mass ratio, strong low-order spiral structure dominates.

Toomre parameter

$$Q = \frac{c_s \Omega_K}{\pi G \Sigma}$$



(Mayer et al. 2004)



(Durison et al. PPV, 2007)

Planet formation by disk gravitational instability. Only giant planets without a solid core. They form in the outer parts of the disk where the temperature is lower allowing gravitational instability.

Sound waves in a fluid:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \boldsymbol{u}) = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla P$$

We apply a small perturbation to the stationary fluid with ρ_0, P_0 and $\mathbf{u}_0 = 0$.

$$P = P_0 + \Delta P \quad \rho = \rho_0 + \Delta \rho \quad \boldsymbol{u} = \boldsymbol{u_0} + \Delta \boldsymbol{u}$$

$$\frac{\partial (\rho_0 + \Delta \rho)}{\partial t} + \nabla((\rho_0 + \Delta \rho)(\boldsymbol{u_0} + \Delta \boldsymbol{u})) = 0 \Rightarrow$$

$$\boxed{\frac{\partial \Delta \rho}{\partial t} + \rho_0 \nabla(\Delta \boldsymbol{u}) = 0}$$

ρ_0 can be taken out of the nabla since it is constant.

$$\frac{\partial (\boldsymbol{u_0} + \Delta \boldsymbol{u})}{\partial t} + (\boldsymbol{u_0} + \Delta \boldsymbol{u}) \nabla (\boldsymbol{u_0} + \Delta \boldsymbol{u}) = -\frac{1}{(\rho_0 + \Delta \rho)} (P_0 + \Delta P) \Rightarrow$$

$$\frac{\partial \Delta \boldsymbol{u}}{\partial t} + \Delta \boldsymbol{u} \nabla (\Delta \boldsymbol{u}) = -\frac{1}{\rho_0} \nabla (\Delta P) \Rightarrow$$

Neglecting 2nd order terms in $\Delta \mathbf{u}$

$$\boxed{\frac{\partial \Delta \boldsymbol{u}}{\partial t} = -\frac{1}{\rho_0} \nabla (\Delta P)}$$

Assuming a barotropic flux with P(ρ) we get:

$$\nabla(\Delta P)=\nabla\left(\frac{dP}{d\rho}\Delta\rho\right)=\frac{dP}{d\rho}\nabla(\Delta\rho)$$

$$\boxed{\frac{\partial\Delta\boldsymbol{u}}{\partial t}=-\frac{1}{\rho_0}\frac{dP}{d\rho}\nabla(\Delta\rho)}$$

Both P and ρ are constants so the derivative is constant.

$$\frac{\partial}{\partial t}\left(\frac{\partial\Delta\rho}{\partial t}\right)+\frac{\partial}{\partial t}\left(\rho_0\nabla(\Delta\boldsymbol{u})\right)=0$$

1st equation derived respect to t, third multiplied by $\rho_0\nabla$

$$-\rho_0\nabla\left(\frac{\partial\Delta\boldsymbol{u}}{\partial t}\right)=-\rho_0\nabla\left(-\frac{1}{\rho_0}\frac{dP}{d\rho}\nabla(\Delta\rho)\right)$$

Since we can exchange the derivative order in the first term of last equation

$$\rho_0\nabla\left(\frac{\partial\Delta\boldsymbol{u}}{\partial t}\right)=\frac{\partial}{\partial t}\left(\rho_0\nabla(\Delta\boldsymbol{u})\right)$$

We can combine the two equations in a single **wave equation**

$$\boxed{\frac{\partial^2}{\partial t^2}\Delta\rho=\frac{dP}{d\rho}\nabla^2(\Delta\rho)}$$

The solution of the wave equation is a perturbation of the form:

$$\Delta\rho = \Delta\rho_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\cdot t)}$$

Inserting this solution in the wave equation we get

$$\Delta\rho_0 (i\omega)^2 = \frac{dP}{d\rho}(i\boldsymbol{k})^2 \Delta\rho_0$$

...and then the **dispersion relation**

$$\frac{dP}{d\rho} = \frac{\omega^2}{k^2} = c_s$$

...where c_s is the propagation speed of signals in the fluid (sound speed).

$$T^{00} = \frac{1}{c^2}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

EM field, 00 component

$$T^{\mu\nu} = \frac{E}{c^2} \frac{\left(U^{\mu} U^{\nu}\right)}{\gamma^2} \delta\left(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{p}}\right)$$

For single particle