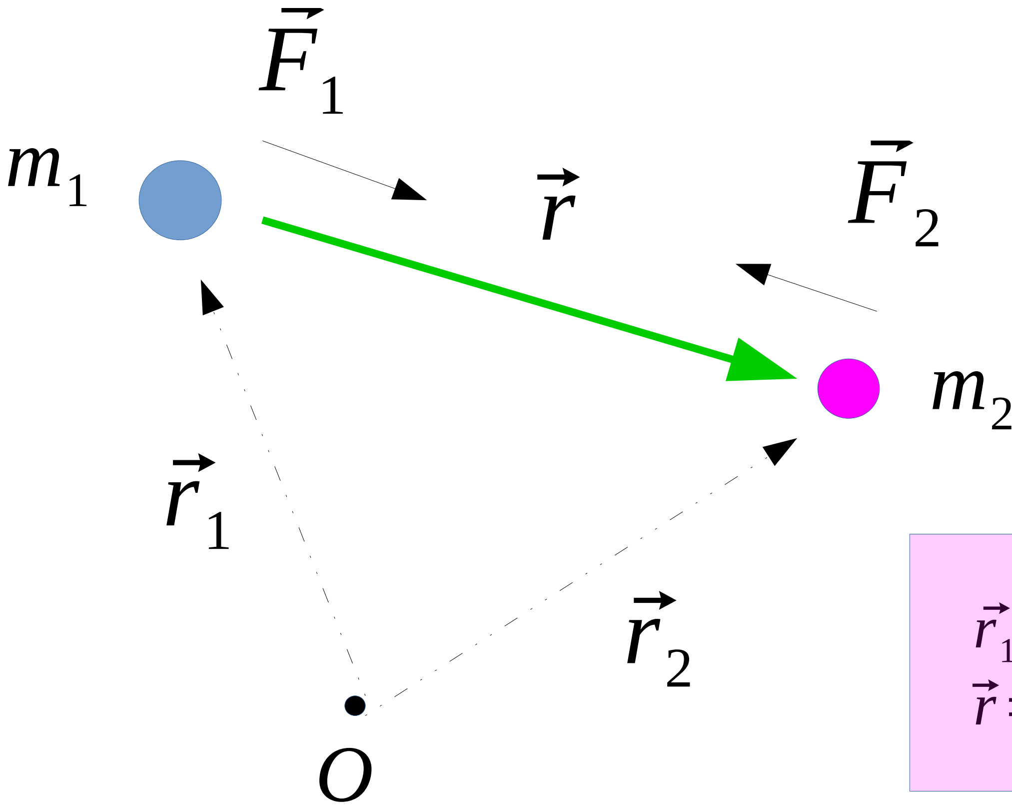
ORBITAL ELMENTS

- **Relative motion**
- **Kepler's equation**
- **Transformation from orbital elements to cartesian coordinates and viceversa.**

Relative motion equation for 2 bodies



$$\vec{r}_1+\vec{r}=\vec{r}_2$$
$$\vec{r}=\vec{r}_2-\vec{r}_1$$

$$\vec{F}_1=\frac{G\,m_1 m_2}{r^3}\vec{r}=m_1\ddot{\vec{r}}_1 \quad Force\ \ on\ \ m_1$$

$$\vec{F}_2=-\frac{G\,m_1 m_2}{r^3}\vec{r}=m_2\ddot{\vec{r}}_2 \quad Force\ \ on\ \ m_2$$

$$m_1\ddot{\vec{r}}_1+m_2\ddot{\vec{r}}_2=\vec{a}_{cm}=0 \qquad No\ \ external\ \ forces.$$

$$\ddot{\vec{r}}=\ddot{\vec{r}}_2-\ddot{\vec{r}}_1=\left(-\frac{G\,m_1 m_2}{r^3 m_2}-\frac{G\,m_1 m_2}{r^3 m_1}\right)\vec{r}=-\frac{G\left(m_1+m_2\right)}{r^3}\vec{r}$$

Orbital elements: definition

Relative motion equation:

$$\frac{d^2\vec{r}}{dt^2}+\mu\frac{\vec{r}}{r^3}=0$$

$$\mu=G(m_1+m_2)$$

The system is isolated, so 3-1 degree of freedom. This implies that the motion occurs on a 2D plane.

Integral of motion: angular momentum

$$\vec{h}=\vec{r}\times\dot{\vec{r}}$$

The relative equation of motion in polar coordinates reads

$$\ddot{\vec{r}}=(\ddot{r}-r\dot{\theta}^2)\hat{r}+\left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]\hat{\theta}$$

Splitting the equation along the two versors $\hat{r},\hat{\theta}$

$$(\ddot{r}-r\dot{\theta}^2)\hat{r}+\mu\frac{r}{r^3}=0 \quad \Rightarrow \quad \hat{r}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=0 \quad \Rightarrow \quad \hat{\theta}$$

$$\vec{h}=r\hat{r}\times(\dot{r}\hat{r}+r\dot{\theta}\hat{\theta})=0+r^2\dot{\theta}\hat{z}$$ *Constant*

Solution of the two body equation of motion: trajectory

$$\left(\ddot{r} - r\dot{\theta}^2\right) + \frac{\mu}{r^2} = 0 \qquad trajectory: \quad r(\theta)$$

Auxiliary variable $\quad y = \frac{1}{r}$

$$\dot{r} = -\frac{1}{y^2}\dot{y} = -\frac{1}{y^2}\frac{dy}{d\theta}\dot{\theta} = r^2\dot{\theta}\frac{dy}{d\theta} = -h\frac{dy}{d\theta}$$

$$\ddot{r} = -h\frac{d}{dt}\left(\frac{dy}{d\theta}\right) = -h\frac{d^2y}{d\theta^2}\dot{\theta} = -h^2y^2\frac{d^2y}{d\theta^2}$$

$$-h^2y^2\frac{d^2y}{d\theta^2} - h^2y^3 + \mu y^2 = 0$$

$$\frac{d^2y}{d\theta^2} + y = \frac{\mu}{h^2}$$

Differential equation with constant coefficients

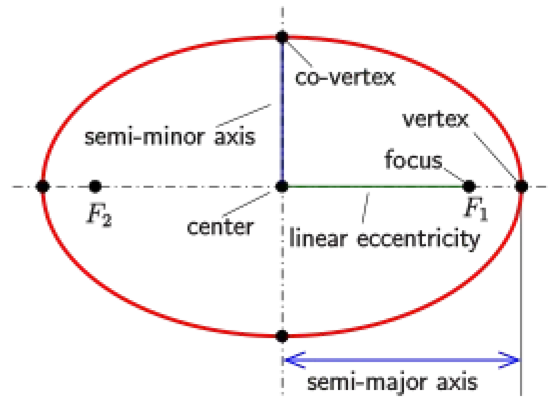
$$y(\theta) = \frac{\mu}{h^2}\left(1 + e\cos\left(\theta - \theta_0\right)\right)$$

$$r(\theta) = \frac{p}{1 + e\cos\left(\theta - \theta_0\right)} \qquad p = \frac{h^2}{\mu}$$

$$Ellipse: \quad p = a(1-e^2) \quad e<1$$

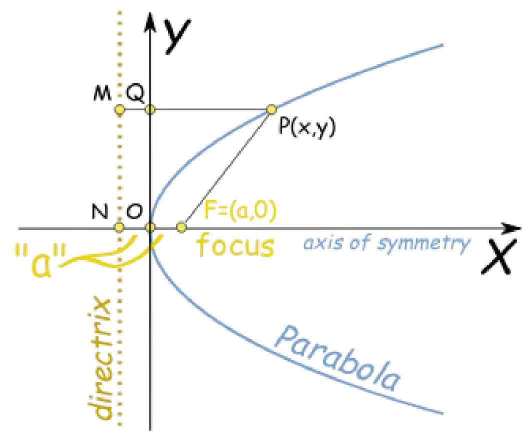
$$\frac{h^2}{\mu} = a(1-e^2) \quad \Rightarrow$$

$$h = \sqrt{\mu a(1-e^2)}$$

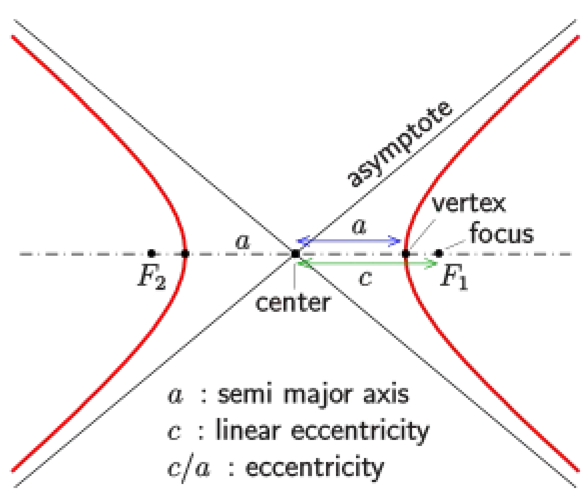


$$Parabola: \quad p = 2q \quad q \text{ min dist}$$

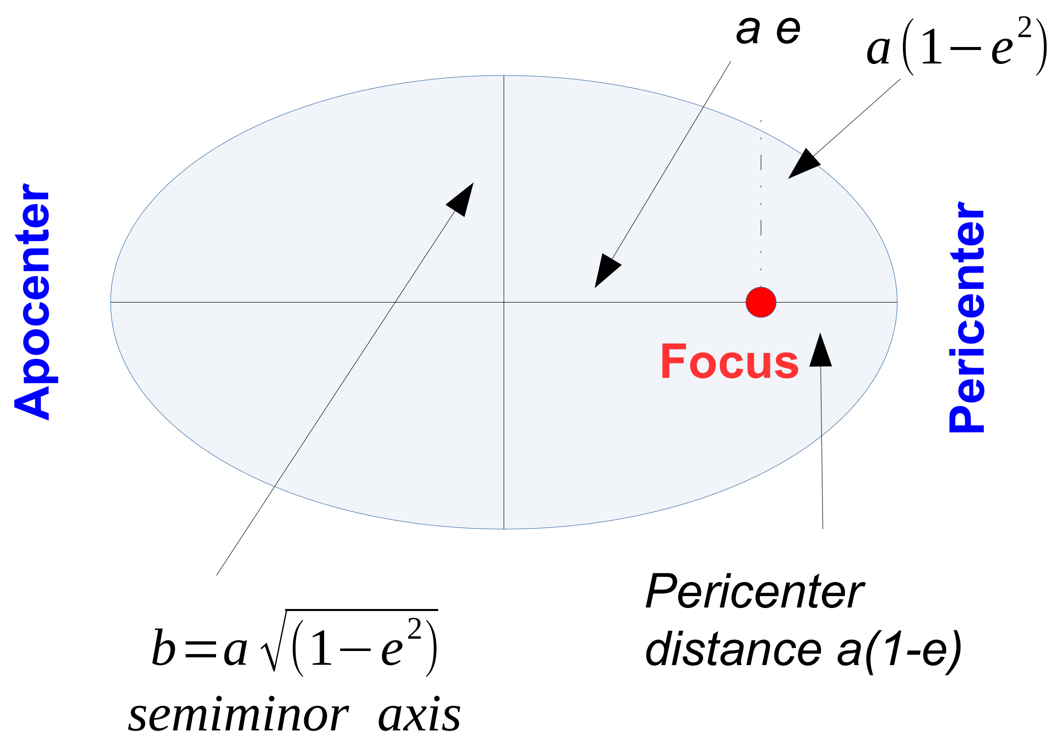
$$e = 1$$

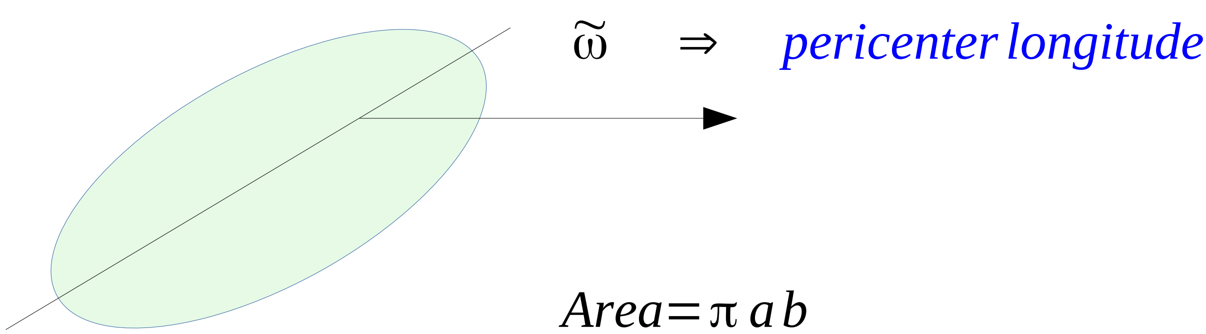


$$Hyperbola: \quad p = a(e^2-1) \quad e>1$$



Elliptical motion in more details:





$$Area=\pi a b$$

$$Perimeter=4\pi E(e)$$

$$E(e)=\int_0^{\frac{\pi}{2}} \sqrt{(1-e^2\sin^2\theta)}\, d\theta$$

complete elliptical integral of the 2^{nd} *kind*

Time dependence

$$V_{ar}=\frac{h}{2} \Rightarrow V_{ar}\cdot T=\frac{hT}{2}=\pi a b=\pi a^2\sqrt{(1-e^2)}$$

$$h=\sqrt{(\mu a(1-e^2))} \Rightarrow \frac{h^2}{\mu}=a(1-e^2)$$

$$T=2\pi\sqrt{\left(\frac{a^3}{\mu}\right)} \qquad \textit{Periodo}$$

$$n=\frac{2\pi}{T}=\sqrt{\frac{\mu}{a^3}}=\sqrt{\frac{G(m_1+m_2)}{a^3}} \qquad \textit{Mean motion}$$

$M = n(t - t_0)$ Mean Anomaly **$f=\theta$ True anomaly**

Average angle determining the exact position only in the case of circular motion.

Energy integral: second constant of motion:

$$\frac{1}{2}v^2-\frac{\mu}{r}=C=-\frac{\mu}{2a}$$

$$\dot{\vec{r}}\cdot\ddot{\vec{r}}+\mu\frac{\dot{\vec{r}}\cdot\vec{r}}{r^3}=0$$

$$\frac{d}{dt}\left(\frac{\dot{\vec{r}}\cdot\dot{\vec{r}}}{2}-\frac{\mu}{r}\right)$$

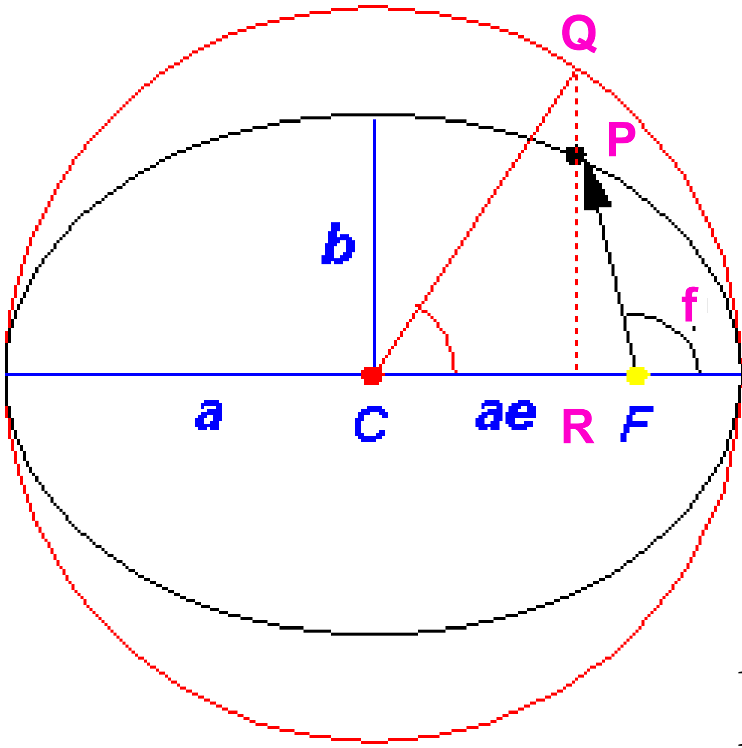
The system has 2 integrals of motion:

Angular momentum (relative motion) **h**

Energy (relative motion) **E**

3-2 = 1 degree of freedom

Kepler equation: full time dependence



U is the eccentric anomaly

GOAL: compute r,f as a function of U and then derive a time equation for U

$$FR = r\cos f = a\cos U - ae$$
$$PR = r\sin f = \sqrt{1-e^2}\,a\sin U$$

$$r = a(1 - e\cos U)$$

$$tg\left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} tg\left(\frac{U}{2}\right)$$

$$tg\left(\frac{f}{2}\right) \quad \Rightarrow \quad \cos f = 1 - 2\sin^2\frac{f}{2} \quad r\cos f = r\left(1 - 2\sin^2\frac{f}{2}\right)$$

$$2r\sin^2\frac{f}{2} = r - r\cos f = a(1 - e\,cosU) - a\cos U + ae = a(1+e)(1-cosU)$$

$$\cos f = 2\cos^2\frac{f}{2} - 1 \quad 2r\cos^2\frac{f}{2} = a(1-e)(1+\cos U)$$

How to compute U as a function ot t?

$$\mu\left(\frac{2}{r}-\frac{1}{a}\right)=v^2=\dot{r}^2+(r\dot{f})^2 \qquad since \quad E=-\frac{\mu}{2a}$$

$$\dot{r}=\frac{na}{\sqrt{1-e^2}}$$

$$r\dot{f}=\frac{na}{\sqrt{1-e^2}}(1+e\cos f)$$

From Murray & Dermott, pg 31

$$\dot{r}=\frac{na}{r}\sqrt{a^2e^2-(r-a)^2}=\frac{na^2e\sin U}{a(1-e\cos U)}$$

$$\dot{r}=ae\sin U\frac{dU}{dt}$$

$$\frac{dU}{dt}=\frac{n}{(1-e\,cosU)}$$

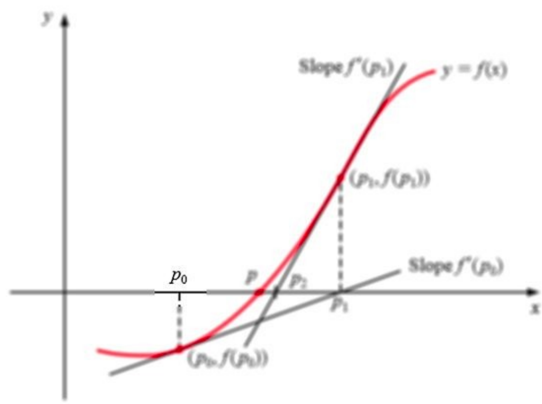
$$U-e\sin U=M=n(t-t_0)$$

Kepler equation: it gives the solution for the time evolution along the trajectory r(θ).

Iterative solution of Kepler equation (for low eccentricity orbits).

$$U_0 = M$$
$$U_1 = U_0 + e\sin U_0$$
$$U_2 = U_0 + e\sin U_1$$
$$\ldots\ldots$$
$$\ldots\ldots$$
$$U_{i+1} = U_0 + e\sin U_i$$

For high eccentricity orbits, Newton-Raphson method grants faster convergence.

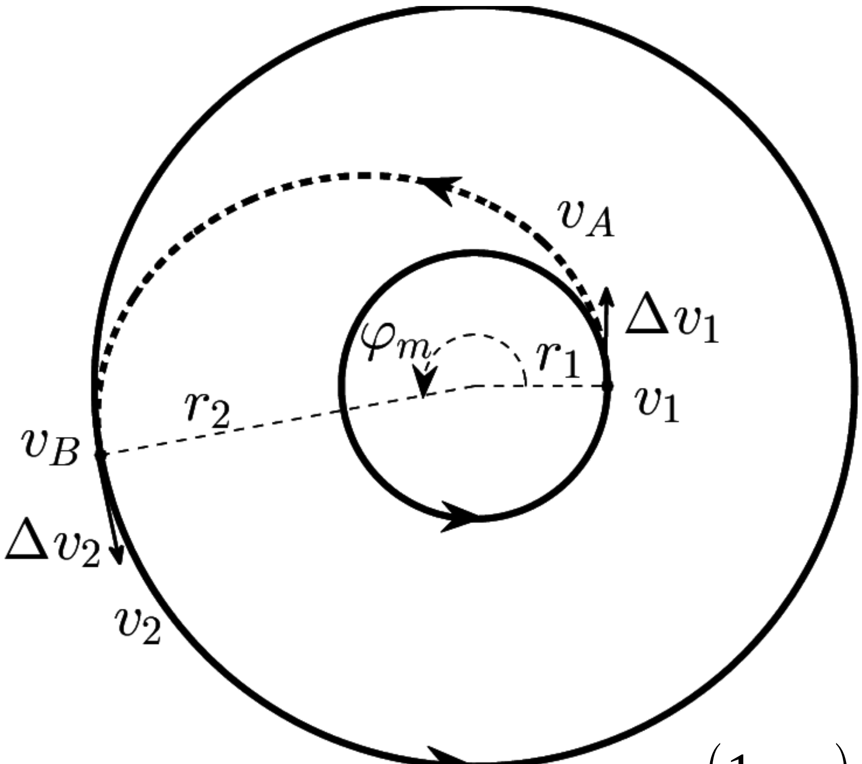


$$f(U) = U - e\sin U - M = 0$$
$$f'(U) = 1 - e\cos U > 0 \quad \textit{single solution}$$

$$U_0 = U$$
$$f(U_0 + \delta) = f(U_0) + f'(U_0)\cdot\delta + \ldots = 0$$
$$U_1 = U_0 + \delta \qquad \delta = -\frac{f(U_0)}{f'(U_0)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$U_{i+1} = U_i + \delta = U_i - \frac{f(U_i)}{f'(U_i)}$$

Quadratic convergence, much faster than the iterative method

Hohmann transfer: it uses the lowest possible amount of energy in transferring probe from circular orbit of radius r_1 to circular orbit of radius r_2.



$$a(1-e)=r_1$$
$$a(1+e)=r_2$$
$$a(1-e)+a(1+e)=2a=r_1+r_2$$
$$a=\frac{r_1+r_2}{2} \qquad e=1-\frac{r_1}{a}$$

Computation of the Δv needed to perform the orbital transfers.

$$\frac{1}{2}v^2-\frac{\mu}{r}=-\frac{\mu}{2a} \quad \Rightarrow v^2=\left(\frac{2}{r}-\frac{1}{a}\right)\mu$$
$$v=\sqrt{\mu\left(\frac{2}{r}-\frac{1}{a}\right)}$$

$$v_{C1}=\sqrt{\mu\left(\frac{2}{r}-\frac{1}{a}\right)}=\sqrt{\frac{\mu}{r_1}} \qquad V_{C2}=\sqrt{\frac{\mu}{r_2}}$$

$$V_{PT}=\sqrt{\mu\left(\frac{2}{a(1-e)}\right)-\frac{1}{a}}=\sqrt{\frac{\mu}{a}\left(\frac{(1+e)}{(1-e)}\right)} \qquad V_{AT}=\sqrt{\frac{\mu}{a}\left(\frac{(1-e)}{(1+e)}\right)}$$

$$\Delta v_1=V_{PT}-v_{C1}=\sqrt{\frac{\mu}{a}\left(\frac{(1+e)}{(1-e)}\right)}-\sqrt{\frac{\mu}{r_1}}$$

$$\Delta v_2=v_{C2}-V_{AT}=\sqrt{\frac{\mu}{r_2}}-\sqrt{\frac{\mu}{a}\left(\frac{(1-e)}{(1+e)}\right)}$$

Flight transfer time is equal to half the period of the transfer orbit

$$T_{of}=\pi\sqrt{\frac{a^3}{\mu}}$$

ESEMPI:

Satellite con $e=0$ e $r = 1.2769 \times 10^4$ Km.
Calcolare ΔV minima per raddoppiare altitudine.

$r_1 = 1.2769 \times 10^4$ Km $(= 2 r_\oplus)$

$r_2 = 3 r_\oplus = 1.9154 \times 10^4$ Km

$a = \frac{5}{2} r_\oplus = 1.5961 \times 10^4$ Km $\quad e = 0.2$

$V_{c1} = \sqrt{\frac{\mu}{r_1}} = 5.595 \times 10^3$ m/s $\quad V_{P1} = 6.129 \times 10^3 \frac{m}{s}$

$\Delta V_1 = 0.534 \times 10^3$ m/s

$V_{c2} = 4.569 \times 10^3$ m/s $\quad V_{A2} = 4.086 \times 10^3 \frac{m}{s}$

$\Delta V_2 = 0.482 \times 10^3$ m/s

- $\Delta V_{tot} = 1.016 \times 10^3 \frac{m}{s}$

TRASFERIMENTO di Hohmann su MARTE

$a = \frac{1+1.5}{2} = 1.25$ AU $\quad e = 0.2$

$T_{OF} = \frac{T_{ab}}{2} = 255$ giorni ~ 8.5 mesi

Transformation from orbital elements to cartesian coordinates and viceversa: the planar case

$$a, e, M \Rightarrow \vec{x}, \vec{v}$$

$$U - e\sin U = M \quad M \Rightarrow U$$

$$r = a(1 - e\cos U) \quad tg\left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} tg\left(\frac{U}{2}\right) \Rightarrow r, f$$

$$x = r\cos f = a(\cos U - e) \quad y = r\sin f = a\sqrt{(1-e^2)}\sin U$$

$$v_x = -\frac{na\sin U}{1 - e\cos U} \quad v_y = \frac{na\sqrt{(1-e^2)}\cos U}{1 - e\cos U}$$

$$\frac{dU}{dt} = \frac{n}{1 - e\cos U}$$

$$\vec{x}, \vec{v} \Rightarrow a, e, M$$

$$r = \sqrt{(x^2 + y^2 + z^2)} \quad v = \sqrt{(v_x^2 + v_y^2 + v_z^2)}$$

$$C = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$$

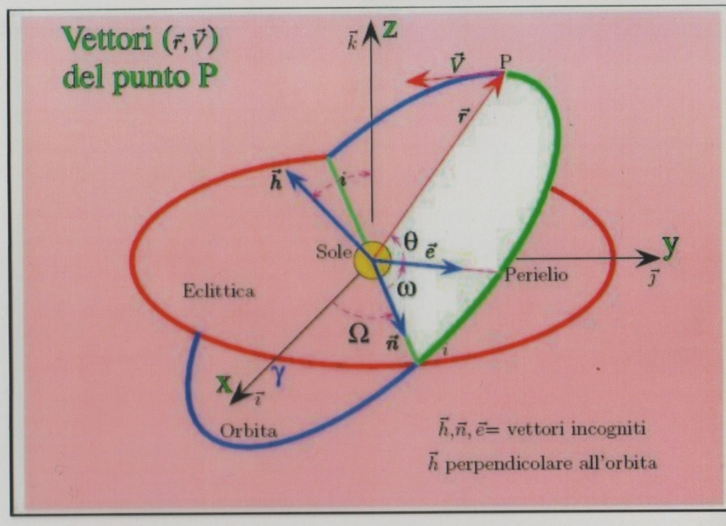
$$h = |r \times v| = \sqrt{\mu a(1-e^2)}$$

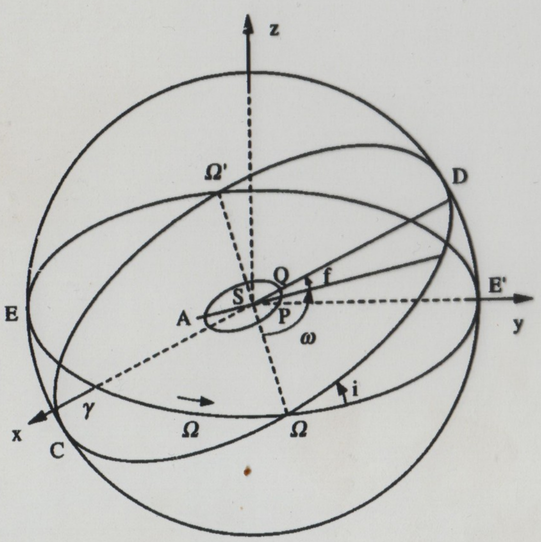
$$r = \frac{a(1-e^2)}{1 + e\cos f} \quad \dot{r} = \frac{n a e \sin f}{\sqrt{(1-e^2)}} = \frac{\vec{v}\cdot\vec{r}}{r}$$

$$tg\left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} tg\left(\frac{U}{2}\right) \quad U - e\sin U = M$$

Elliptical orbit in 3D

Orbital elements





A semimajor axis,
e eccentricity,
i inclination,
M mean anomaly,
ω pericenter argument,
Ω node longitude

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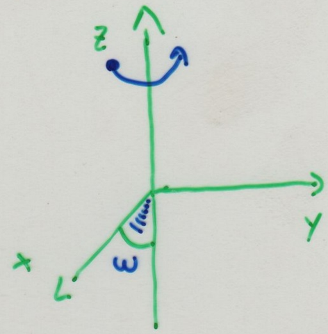
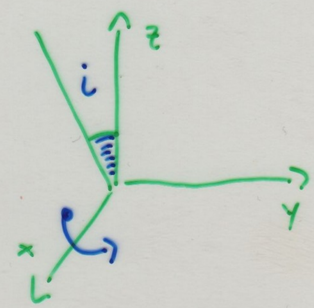
ORBITA NELLO SPAZIO 3D

$$M_1 = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Attorno asse Z



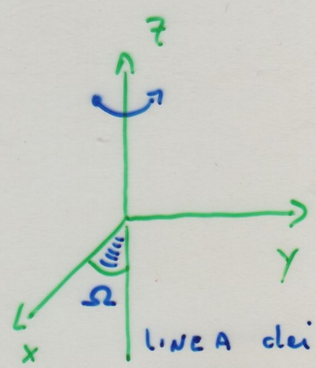
ω = argomento pericentro

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}$$ Attorno asse X



i = inclinazione

$$M_3 = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Attorno asse Z



Ω = longitudine nodo

$$\tilde{\omega} = \omega + \Omega$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M_3 M_2 M_1 \begin{pmatrix} r\cos f \\ r\sin f \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} r\cos\Omega\cos(\omega+f) - r\sin\Omega\sin(\omega+f)\cos i \\ r\sin\Omega\cos(\omega+f) + r\cos\Omega\sin(\omega+f)\cos i \\ r\sin(\omega+f)\sin i \end{pmatrix}$$

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D-M

11-b

TRASFORMAZIONE da ELEMENTI KEPLERIANI a COORDINATE CARTESIANE.

$$a, e, i, \omega, \Omega, M \Rightarrow \bar{x}, \bar{V}$$

1) Eq. Keplera $U - e \sin U = M \rightarrow f$

2) $x = a(\cos U - e)$

$y = a\sqrt{1-e^2}\,\sin U$

3) $V_x = -\dfrac{n a \sin U}{1 - e\cos U}$

$V_y = \dfrac{n a \sqrt{1-e^2}\,\cos U}{1 - e\cos U}$

$$a, e, M \rightarrow (x, y, 0)\ (V_x, V_y, 0)$$

4)

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M_3(\Omega) \cdot M_1(i) \cdot M_3(\omega) \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} V_X \\ V_Y \\ V_Z \end{pmatrix} = \ldots\ldots \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}, \quad \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix}$$

10 - b

SAPENDO x, y, z TROVARE GLI ANGOLI ORBITALI
V_x, V_y, V_z

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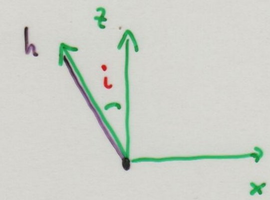
D-M

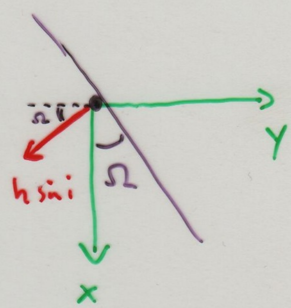
- i

$h \cos i = h_z$



- Ω

$$\begin{cases} h \sin i \sin \Omega = \pm h_x \\ h \sin i \cos \Omega = \mp h_y \end{cases}$$



$$\operatorname{tg} \Omega = -\left(\frac{h_x}{h_y}\right)$$

- ω

$$\begin{cases} \sin(\omega + f) = \dfrac{z}{r \sin i} \\ \cos(\omega + f) = \dfrac{x}{r \cos \Omega} + \dfrac{z}{r} \dfrac{\sin \Omega}{\cos \Omega} \dfrac{\cos i}{\sin i} \end{cases}$$

⇩

$\omega + f \Rightarrow \omega$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M_3 M_2 M_1 \begin{pmatrix} r \cos f \\ r \sin f \\ 0 \end{pmatrix}$$

12-b

TRASFORMAZIONE da COORDINATE CARTESIANE ⇒ ELEMENTI KEPLERIANI

$$\dot{r} = \frac{\bar{v} \cdot \bar{r}}{r}$$

$$\bar{h} = \bar{r} \times \bar{v} = h \hat{h}$$

1) $r = \sqrt{x^2+y^2+z^2}$ $\quad V = \sqrt{V_x^2+V_y^2+V_z^2}$

$C = \frac{1}{2} v^2 - \frac{\bar{\mu}}{r}$ $\quad C = -\frac{\bar{\mu}}{2a}$

$h^2 = \bar{\mu} a(1-e^2)$ ➡ $\boxed{a, e}$

2) $h \cos i = h_z$ $\quad \boxed{i = \arccos\left(\frac{h_z}{h}\right)}$

3) $h \sin i \sin\Omega = \pm h_x$

$h \sin i \cos\Omega = \mp h_y$ $\quad \boxed{tg(\Omega) = -\left(\frac{h_x}{h_y}\right)}$

4) $\cos f = \frac{a(1-e^2)}{r \cdot e} - \frac{1}{e}$

$\sin f = \frac{\dot{r}\sqrt{1-e^2}}{nae}$ $\quad \boxed{tg(f) = \frac{\dot{r} r \sqrt{1-e^2}}{na(a(1-e^2)-r)}}$

5) $\sin(\omega+f) = \frac{z}{r \sin i}$ $\quad \Rightarrow \omega + f$ ➡ $\boxed{\omega}$

$\cos(\omega+f) = \frac{x}{r\cos\Omega} + \frac{z}{r} \frac{\sin\Omega}{\cos\Omega} \frac{\cos i}{\sin i}$

6) $tg\left(\frac{U}{2}\right) = \left(\frac{1-e}{1+e}\right)^{\frac{1}{2}} tg\left(\frac{f}{2}\right) \Rightarrow U \Rightarrow$

$M = U - e \sin U$ ➡ $\boxed{M}$

10) **PROBLEMA 4**: Calcolare l'intervallo tra 2 congiunzioni di Marte e Terra e mostrare che la distanza minima alla congiunzione può variare di un fattore 2.

$G = 6.6743 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$

$K = 0.01720211$

$GM_\odot = K^2 \frac{AU^3}{day^2} \rightarrow 2.9591 \times 10^{-4}$

$a_M = 1.5236631$ AU

$a_T = 1.000..$

$n_M = 9.146 \times 10^{-3} \frac{1}{day}$ $\quad T_M = 686.96$ day

$n_T = 1.720 \times 10^{-2} \frac{1}{day}$ $\quad T_T = 365.25$ day

$\frac{T_M}{T_T} \simeq 1.88$ circa 2

$$n_M \Delta t = n_E \Delta t - 2\pi \Rightarrow \Delta t = \frac{2\pi}{n_E - n_M} \simeq 780 \text{ day}$$

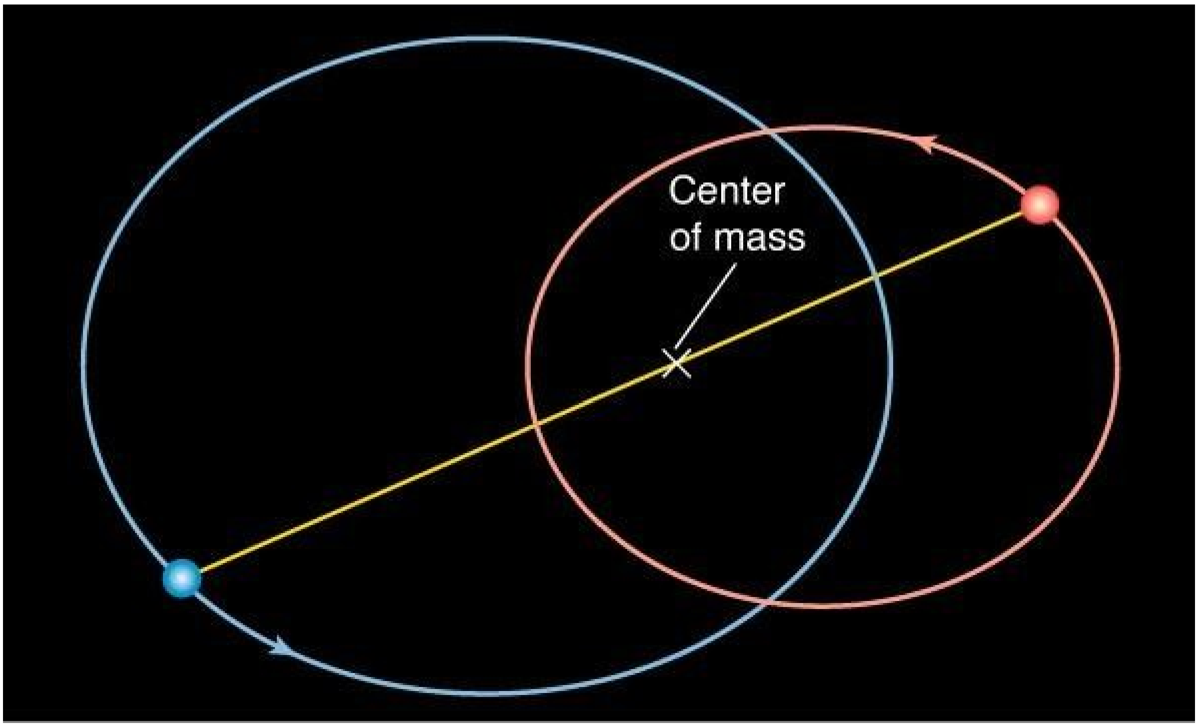
$$D_{min} = -a_T(1+e_T) + a_M(1-e_M) = 0.0365 \text{ AU}$$

af. $\qquad$ per.

$$D_{MAX} = -a_T(1-e_T) + a_M(1+e_M) = 0.0683 \text{ AU}$$

per. $\qquad$ af.

Baricentric orbits.



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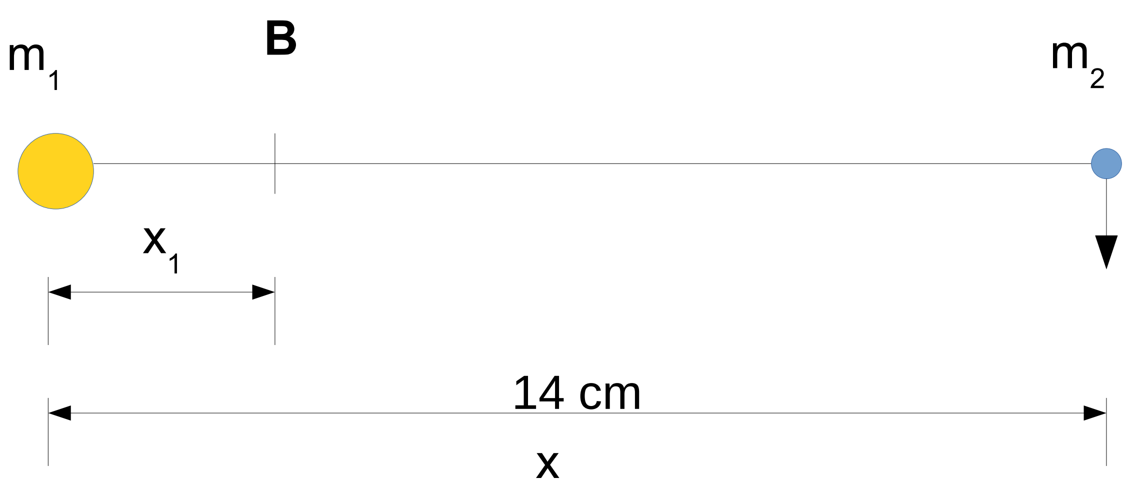

$$-m_1 x_1 + m_2(x - x_1) = 0 \Rightarrow$$

Baricenter position in the baricenter reference frame.

$$x_1 = \frac{m_2}{m_1 + m_2} x \quad \Rightarrow \quad R_1 = \frac{m_2}{m_1 + m_2} R$$

$$R_2 = \frac{m_1}{m_1 + m_2} R$$

R_1 and R_2 are the radial distance from the baricenter of the bodies 1 and 2.

$$R_1^2 \dot{\theta} = \left(\frac{m_2}{m_1 + m_2}\right)^2 R^2 \dot{\theta} = \left(\frac{m_2}{m_1 + m_2}\right)^2 h = h_1$$

$$R_2^2 \dot{\theta} = \left(\frac{m_1}{m_1 + m_2}\right)^2 R^2 \dot{\theta} = \left(\frac{m_1}{m_1 + m_2}\right)^2 h = h_2$$

$$L = m_1 h_1 + m_2 + h_2 = \frac{m_1 m_2}{m_1 + m_2} h$$

$$a_1 = \frac{m_2}{m_1 + m_2} a \qquad n_1 = n_2 = n$$

$$E = \frac{m_1 m_2}{m_1 + m_2} C = -\frac{m_1 m_2}{m_1 + m_2} \frac{G(m_1 + m_2)}{2a} = -\frac{G m_1 m_2}{2a}$$

Poincare' action-angle variables

$$\Lambda = \mu\sqrt{G(M_s + m)a} \qquad \lambda = M + \varpi$$
$$\mathrm{T} = \Lambda\left(1 - \sqrt{1 - e^2}\right) \qquad \gamma = -\varpi$$
$$Z = (1 - \mathrm{T})(1 - \cos i) \qquad \sigma = -\Omega$$

Jacobian coordinates