CAPITOLO 8

- Teoria secolare per 2 pianeti
- Teoria secolare per i pianeti del Sistema Solare
- Teoria secolare per i corpi minori
- Risonanze secolari
- Caos

PROBLEMA a N-CORPI:

$$\frac{d^2\bar{r}_i}{dt^2} = -G(M_0+m_i)\frac{\bar{r}_i}{r_i^3} - G\sum_{\substack{j=1\\ j\neq i}}^{N} m_j\left(\frac{\bar{r}_j-\bar{r}_i}{r_{ij}^3} - \frac{\bar{r}_j}{r_j^3}\right)$$

PER 2 PIANETI SI RIDUCE A:

$$\frac{d^2\bar{r}_1}{dt^2} = -G(M_0+m_1)\frac{\bar{r}_1}{r_1^3} - G m_2\left(\frac{\bar{r}_2-\bar{r}_1}{r_{12}^3} - \frac{r_2}{r_2^3}\right) \ldots$$

$$\ddot{r}_i = \nabla_i\left(U_i + R_i\right)$$ dove

$$U_i = G\frac{(M_0+m_i)}{r_i} \qquad R_i = \frac{G m_j}{|\bar{r}_j-\bar{r}_i|} - G m_j\frac{\bar{r}_i\cdot\bar{r}_j}{r_j^3}$$

con $i = 1,2$

- $\dfrac{G m_j}{|\bar{r}_j-\bar{r}_i|}$ termine diretto
- $- G m_j\dfrac{\bar{r}_i\cdot\bar{r}_j}{r_j^3}$ termine indiretto

Conversione di R in funzione degli elementi orbitali e sua espansione in serie di eccentricità ed inclinazione.

$$R_{12} = G m_2 \sum_{j_1 j_2 l_1 l_2 m_1 m_2} S^{j_1, j_2, l_1, l_2, m_1, m_2}(a_1, a_2, e_1, e_2, i_1, i_2) \cdot$$

$$\cdot \cos\left(j_1 \lambda_1 + j_2 \lambda_2 + l_1 \tilde{\omega}_1 + l_2 \tilde{\omega}_2 + m_1 \Omega_1 + m_2 \Omega_2\right)$$

PER CALCOLARE EVOLUZIONE $\Rightarrow$ EQ. di LAGRANGE

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial \lambda}$$

$$\frac{de}{dt} = -\frac{\sqrt{1-e^2}}{na^2 e}\left(1-\sqrt{1-e^2}\right)\frac{\partial R}{\partial \lambda} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \tilde{\omega}}$$

$$\frac{di}{dt} = -\frac{\tan\left(\frac{i}{2}\right)}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \lambda} + \frac{\partial R}{\partial \tilde{\omega}}\right) - \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i}$$

$$\frac{d\tilde{\omega}}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} + \frac{\tan\left(\frac{i}{2}\right)}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}$$

3)

Si MEDIA R sugli angoli VELOCI λ_1 e λ_2 per costruire TEORIA SECOLARE

$$\bar{R}_j = \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} R \, d\lambda_1 \, d\lambda_2$$

- a_i è costante perché:

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial \bar{R}}{\partial \lambda} = 0$$

Si tengono solo termini al 2° ORDINE in ECCENTRICITÀ e INCLINAZIONE. La funzione di disturbo diventa:

$$\bar{R}_j = n_j a_j^2 \left[\frac{1}{2} A_{jj} e_j^2 + A_{jk} e_j e_k \cos(\tilde{\omega}_1 - \tilde{\omega}_2) + \frac{1}{2} B_{jj} i_j^2 + B_{jk} i_j i_k \cos(\Omega_1 - \Omega_2) \right] \quad j \neq k$$

4) **Dove i coefficienti A dipendono solo dalle masse e semiassi maggiori (costanti nella approx. Secolare)**

$$A_{jj} = \frac{m_j}{4} \frac{m_j}{M_0 + m_j} \alpha_{12} \bar{\alpha}_{12} b^{(1)}_{\frac{3}{2}}(\alpha_{12})$$

$$A_{jk} = -\frac{m_j}{4} \frac{m_k}{M_0 + m_j} \alpha_{12} \bar{\alpha}_{12} b^{(2)}_{\frac{3}{2}}(\alpha_{12})$$

$$B_{jj} = -\frac{m_j}{4} \frac{m_j}{M_0 + m_j} \alpha_{12} \bar{\alpha}_{12} b^{(1)}_{\frac{3}{2}}(\alpha_{12})$$

$$B_{jk} = \frac{m_j}{4} \frac{m_k}{M_0 + m_j} \alpha_{12} \bar{\alpha}_{12} b^{(1)}_{\frac{3}{2}}(\alpha_{12})$$

- Se $a_1 < a_2$ $\quad \alpha_{12} = \dfrac{a_1}{a_2}$ $\begin{cases} j=1 & \bar{\alpha}_{12} = \alpha_{12} \\ j=2 & \bar{\alpha}_{12} = 1 \end{cases}$

- $$\frac{1}{2} b^{(j)}_s(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(j\psi)\, d\psi}{(1 - 2\alpha\cos\psi + \alpha^2)^s}$$ Coefficienti di LAPLACE

Cap. 6 Murray & Dermott

Calcolo dell'evoluzione secolare degli elementi orbitali.

$$\frac{de_j}{dt} = -\frac{1}{m_j a_j^2 e_j} \frac{\partial \bar{R}_j}{\partial \tilde{\omega}_j}$$

$$\frac{di_j}{dt} = -\frac{1}{m_j a_j^2 i_j} \frac{\partial \bar{R}_j}{\partial \Omega_j}$$

$$\frac{d\tilde{\omega}_j}{dt} = \frac{1}{m_j a_j^2 e_j} \frac{\partial \bar{R}_j}{\partial e_j}$$

$$\frac{d\Omega}{dt} = \frac{1}{m_j a_j^2 i_j} \frac{\partial \bar{R}}{\partial i_j}$$

Attenzione! Nei termini delle equazioni di Lagrange si sono trascurati tutti i termini di ordine maggiore di e_j^2 e i_j^2 (la teoria è al secondo ordine in queste variabili). Ad esempio……

$$\frac{\mathrm{tg}\,\frac{i}{2}}{m a^2 \sqrt{1-e^2}} \cdot \frac{\partial \bar{R}}{\partial \tilde{\omega}}$$ viene trascurata perchè proporzionale a $\frac{i}{2} \cdot \frac{\partial \bar{R}}{\partial \tilde{\omega}}$ e darebbe termini in i^3.

6)

LE EQUAZIONI SONO SINGOLARI SE

$e_j = 0 \quad \sigma \quad i_j = 0$

- PASSAGGIO A VARIABILI NON-SINGOLARI

$h_j = e_j \sin \tilde{\omega}_j \qquad\qquad p_j = i_j \sin \Omega_j$

$k_j = e_j \cos \tilde{\omega}_j \qquad\qquad q_j = i_j \cos \Omega_j$

$$\bar{R}_j = m_j a_j^2 \left[\frac{1}{2} A_{jj} (h_j^2 + k_j^2) + A_{ji} (h_j h_i + k_j k_i) + \frac{1}{2} B_{jj} (p_j^2 + q_j^2) + B_{ji} (p_j p_i + q_j q_i) \right]$$

. $\cos(x-y) = \cos x \cos y + \sin x \sin y$

COME DIVENTANO LE EQUAZIONI per h e k?

$$\dot{h}_1 = \frac{\partial h_1}{\partial e_1}\frac{de_1}{dt} + \frac{\partial h_1}{\partial \tilde{\omega}_1}\cdot\frac{d\tilde{\omega}_1}{dt} = \frac{h_1}{e_1}\dot{e}_1 + K_1\dot{\tilde{\omega}}_1 =$$

$$= -\frac{h_1}{m_1 a_1^2 e_1^2}\frac{\partial \bar{R}_1}{\partial \tilde{\omega}} + \frac{K_1}{m_1 a_1^2 e_1}\frac{\partial \bar{R}}{\partial e_1} =$$

$$= \frac{h_1}{\cancel{m_1 a_1^2}\, e_1^{\cancel{2}}}\left(\cancel{m_1 a_1^2}\, A_{12}\, \cancel{e_1}\, e_2 \sin(\tilde{\omega}_1 - \tilde{\omega}_2)\right) +$$

$$+ \frac{K_1}{\cancel{m_1 a_1^2}\, e_1}\cancel{m_1 a_1^2}\left(A_{11} e_1 + A_{12} e_2 \cos(\hat{\omega}_1 - \hat{\omega}_2)\right) =$$

$$= \frac{e_2}{e_1} h_1 A_{12} \sin(\hat{\omega}_1 - \hat{\omega}_2) + K_1 A_{11} +$$

$$+ K_1 A_{12}\frac{e_2}{e_1}\cos(\hat{\omega}_1 - \hat{\omega}_2) =$$

$$= K_1 A_{11} + \Big(A_{12} e_2 \sin(\hat{\omega}_1)\sin(\hat{\omega}_1 - \hat{\omega}_2) +$$

$$+ e_2 A_{12}\cos(\hat{\omega}_1)\cos(\hat{\omega}_1 - \hat{\omega}_2) =$$

$$= K_1 A_{11} + A_{12} e_2 \Big(\sin(\hat{\omega}_1)\sin(\hat{\omega}_1)\cos(\hat{\omega}_2) +$$

$$- \sin(\cancel{\tilde{\omega}_1)\cos(\tilde{\omega}_1}) \sin(\hat{\omega}_2) + \cos(\hat{\omega}_1)\cos(\hat{\omega}_1)\cos(\hat{\omega}_2) +$$

$$+ \sin(\hat{\omega}_1\cancel{)\cos(\hat{\omega}_1)}\sin(\hat{\omega}_2)\Big) = K_1 A_{11} + K_2 A_{12}$$

8)

$$\dot{h}_1 = A_{11} K_1 + A_{12} K_2 = \frac{1}{m_1 a_1^2} \frac{\partial \bar{R}_1}{\partial K_1}$$

$$\dot{h}_2 = A_{21} K_1 + A_{22} K_2 = \frac{1}{m_2 a_2^2} \frac{\partial \bar{R}_2}{\partial K_2}$$

$$\dot{K}_1 = - A_{11} h_1 - A_{12} h_2 = - \frac{1}{m_1 a_1^2} \frac{\partial \bar{R}_1}{\partial h_1}$$

$$\dot{K}_2 = - A_{21} h_1 - A_{22} h_2 = - \frac{1}{m_2 a_2^2} \frac{\partial \bar{R}_2}{\partial h_2}$$

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = A \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\begin{pmatrix} \dot{K}_1 \\ \dot{K}_2 \end{pmatrix} = -A \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

TEORIA LINEARE (Eq. diff. a coeff. costanti)

SOLUZIONE:

$$h_j = \sum_{i=1}^{2} e_{ji} \sin(g_i t + \beta_i)$$

$$K_j = \sum_{i=1}^{2} e_{ji} \cos(g_i t + \beta_i)$$

g_i = AUTOVALORI di A

e_{ji} = AUTOVETTORI di A

- A dipende solo da a_1, a_2
- β_i dipendono da cond. iniziali

8) PER p, q

$$\begin{pmatrix} \dot p_1 \\ \dot p_2 \end{pmatrix} = B \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix} = -B \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$p_j = \sum_{i=1}^{2} i_{ji} \sin(f_i t + \gamma_i) \qquad q_j = \sum_{i=1}^{2} i_{ji} \cos(f_i t + \gamma_i)$$

MA ...... uno degli autovalori di B è nullo!

$$\begin{vmatrix} B_{11} - f & B_{12} \\ B_{21} & B_{22} - f \end{vmatrix} = 0$$

$\Rightarrow$ l'equazione caratteristica è

$$f^2 - f(B_{11} + B_{22}) + B_{11} B_{22} - B_{12} B_{21} = 0$$

Ma $B_{11} B_{22} - B_{12} B_{21} = 0$ (c'è $b_{\frac{3}{2}}^{(1)}(\alpha_1 2)$ in entrambi i termini)

$$f_1 = 0 \qquad f_2 = B_{11} + B_{22}$$

- CONSEGUENZA della CONSERVAZIONE del MOMENTO ANGOLARE del SISTEMA dei 2 PIANETI.

Soluzione per Giove e Saturno

$$m_1/m_c = 9.54786 \times 10^{-4} \qquad m_2/m_c = 2.85837 \times 10^{-4}$$

$$a_1 = 5.202545 \text{ AU} \qquad a_2 = 9.554841 \text{ AU}$$

$$n_1 = 30.3374 \,^\circ\text{y}^{-1} \qquad n_2 = 12.1890 \,^\circ\text{y}^{-1}$$

$$e_1 = 0.0474622 \qquad e_2 = 0.0575481$$

$$\varpi_1 = 13.983865\,^\circ \qquad \varpi_2 = 88.719425\,^\circ$$

$$I_1 = 1.30667\,^\circ \qquad I_2 = 2.48795\,^\circ$$

$$\Omega_1 = 100.0381\,^\circ \qquad \Omega_2 = 113.1334\,^\circ$$

$$\alpha = a_1/a_2 = 0.544493$$

$$b_{3/2}^{(1)} = 3.17296 \qquad b_{3/2}^{(2)} = 2.07110$$

$$\mathbf{A} = \begin{pmatrix} +0.00203738 & -0.00132987 \\ -0.00328007 & +0.00502513 \end{pmatrix} \,^\circ\text{y}^{-1}$$

$$\mathbf{B} = \begin{pmatrix} -0.00203738 & +0.00203738 \\ +0.00502513 & -0.00502513 \end{pmatrix} \,^\circ\text{y}^{-1}$$

Frequenze:

$g_1 = 9.63435 \times 10^{-4}\ {}^\circ\mathrm{y}^{-1}$ $\qquad$ $g_2 = 6.09908 \times 10^{-3}\ {}^\circ\mathrm{y}^{-1}$

$f_1 = 0$ $\qquad$ $f_2 = -7.06251 \times 10^{-3}\ {}^\circ\mathrm{y}^{-1}$

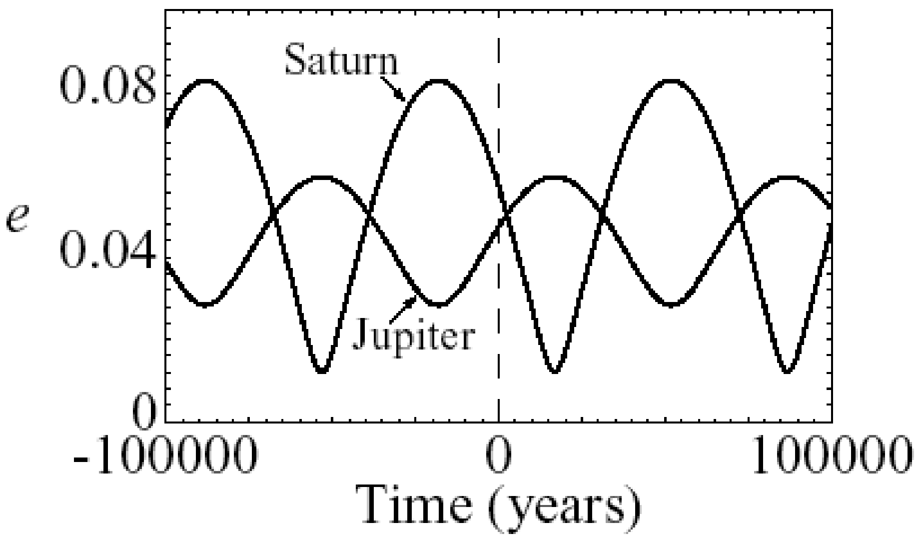
Autovalori:

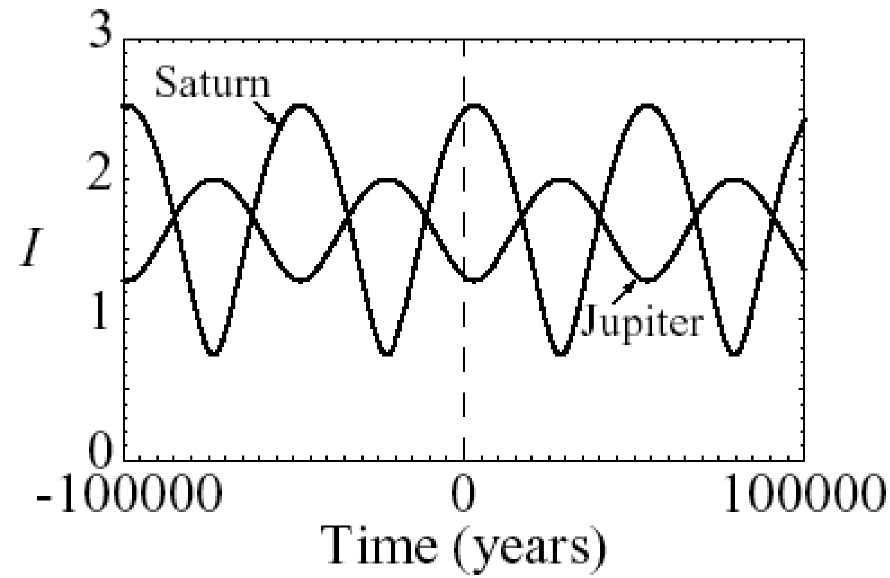
$$\begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix} = \begin{pmatrix} -0.0438821 \\ -0.0354375 \end{pmatrix} \qquad \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix} = \begin{pmatrix} 0.0155788 \\ -0.047581 \end{pmatrix}$$

$$\begin{pmatrix} I_{11} \\ I_{21} \end{pmatrix} = \begin{pmatrix} 0.0285689 \\ 0.0285689 \end{pmatrix} \qquad \begin{pmatrix} I_{12} \\ I_{22} \end{pmatrix} = \begin{pmatrix} -0.00629766 \\ 0.015533 \end{pmatrix}$$

Coefficienti costanti

$\beta_1 = -146.892^\circ$ $\quad$ $\beta_2 = -53.3565^\circ$ $\quad$ $\gamma_1 = 105.74^\circ$ $\quad$ $\gamma_2 = 126.825^\circ$





$$e_1^2(t) = (e_{11}^2 + e_{12}^2) - 2e_{11}e_{12}\cos((g_1 - g_2)t + \beta_1 - \beta_2)$$

$$e_2^2(t) = (e_{21}^2 + e_{22}^2) + 2e_{21}e_{22}\cos((g_1 - g_2)t + \beta_1 - \beta_2)$$

Le eccentricità dei 2 pianeti oscillano in antifase con una frequenza pari alla differenza tra g1 e g2.

Equzioni secolari per tutti i pianeti del Sistema Solare

$$h_j = e_j \sin \tilde{\omega}_j = \sum_{k=1}^{N} M_{j,k} \sin(g_k t + \beta_k)$$

$$k_j = e_j \cos(\tilde{\omega}_j) = \sum_{k=1}^{N} M_{j,k} \cos(g_k t + \beta_k)$$

$$p_j = \sin\left(\frac{i_j}{2}\right) \sin \Omega_j = \sum_{k=1}^{N} N_{j,k} \sin(s_k t + \delta_k)$$

$$q_j = \sin\left(\frac{i_j}{2}\right) \cos \Omega_j = \sum_{k=1}^{N} N_{j,k} \cos(s_k t + \delta_k)$$

SOLUZIONE di LAGRANGE-LAPLACE

- Per la CONSERVAZIONE del MOMENTO ANGOLARE non tutti gli s_k sono INDIPENDENTI:

$$L_x = \sum_{j=1}^{N} \mu_j \sqrt{G(M_0 + m_j) a_j (1 - e_j^2)} \quad \sin i_j \cos \Omega_j$$

$$L_y = \sum_{j=1}^{N} \mu_j \cdot h_j \cdot \sin i_j \sin \Omega_j$$

$$\mu_j = \frac{M_0 m_j}{M_0 + m_j} \qquad h_j = \sqrt{G(M_0 + m_j) a_j (1 - e_j^2)}$$

Da non confondere con le h, k....

L_x e L_y SONO COSTANTI

Per un dato $\bar{J}$

$$\operatorname{tg} \Omega_{\bar{J}} = \frac{L_y - \sum_{J \neq \bar{J}} \mu_J h_J \sin i_J \sin \Omega_J}{L_x - \sum_{J \neq \bar{J}} \mu_J h_J \sin i_J \cos \Omega_J}$$

Allora $\Omega_{\bar{J}}$ può essere calcolato sapendo gli altri $\Omega_{J \neq \bar{J}}$. Questo riduce a $N-1$ le frequenze del sistema. Per convenzione si pone $S_5 = 0$

- LA TEORIA SECOLARE viene di solito calcolata nel PIANO INVARIANTE $\Rightarrow$ piano $\perp$ al vettore momento angolare $\vec{L}$ totale del Sistema Solare.

- NB: Nel piano invariante tutti i $N_{J,5} = 0$ altrimenti

$$q_J = \sum_{K=1}^{N} N_{J,K} \cos(s_K t + \delta_K) = N_{J,5} \cdot \cos(\delta_K) + \dots$$

Il termine costante $N_{J,5} \cos(\delta_K)$ darebbe una inclinazione forzata per tutti i J e il piano invariante NON sarebbe $\perp$ a $\vec{L}$

3

k	g_k	$\beta_k(°)$	s_k	$\delta_k(°)$
1	5.5964	112.08	−5.6174	348.60
2	7.4559	200.51	−7.0795	273.25
3	17.3646	305.12	−18.8512	240.20
4	17.9156	335.38	−17.7482	303.75
5	4.2575	30.65	0.0000	107.58
6	28.2455	128.09	−26.3450	307.29
7	3.0868	121.36	−2.9927	320.62
8	0.6726	74.06	−0.6925	203.90

Arcsec/year

• FREQUENTE FONDAMENTALI del SISTEMA SOLARE

$N_{j,k} \times 10^5$ RISPETTO ECLITTICA

	$j \backslash k$	1	2	3	4	5	6	7	8
M	1	39957	30169	1678	72261	13772	−139	−1665	−724
V	2	6716	−4045	−9544	−5759	13772	−60	−959	−663
T	3	4960	−3431	8760	4024	13772	−1404	−866	−650
Ma	4	860	−566	−15421	34689	13772	−4579	−628	−615
G	5	−11	4	0	−1	13772	3153	−485	−584
S	6	−14	6	−2	−13	13772	−7858	−394	−564
U	7	11	−3	0	1	13772	353	8887	543
N	8	0	0	0	0	13772	38	−1062	5790

$M_{j,k} \times 10^5$

	$j \backslash k$	1	2	3	4	5	6	7	8
M	1	185444	−27700	1458	−1428	36353	113	623	7
V	2	6668	20733	−11671	13464	19636	−551	614	11
T	3	4248	16047	9406	−13159	18913	1506	650	12
Ma	4	650	2917	40133	49032	20300	7030	862	20
G	5	−7	−12	−1	0	44187	−15700	1814	58
S	6	−6	−12	−7	−7	32958	48209	1511	57
U	7	2	3	0	0	−37587	−1547	29033	1666
N	8	0	0	0	0	1881	−103	−3697	9118

3-1)

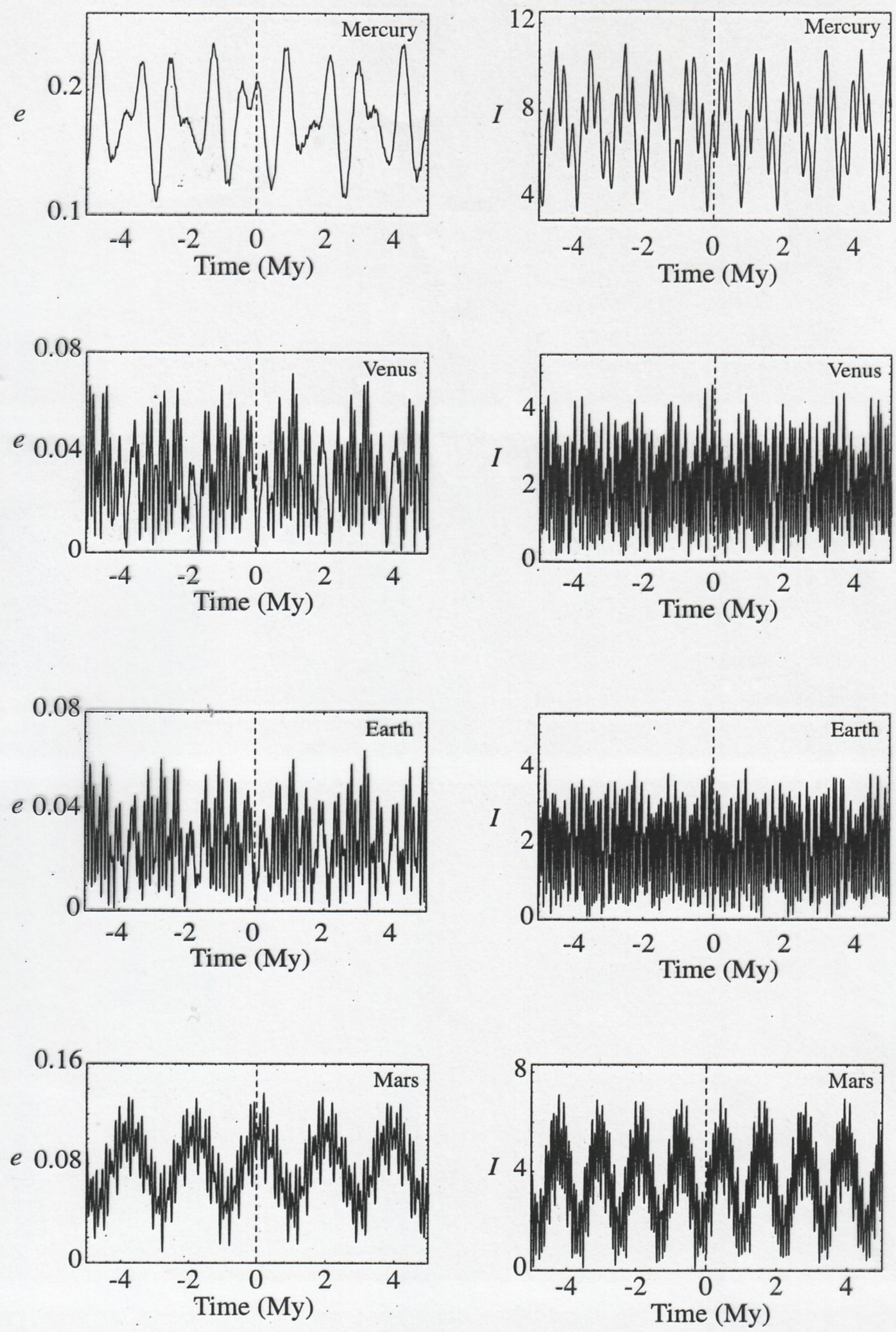


Fig. 7.9. Plots of the eccentricity and inclination (in degrees) of Mercury, Venus, Earth, and Mars over a period of 10 million years centred on AD 1900, according to the secular theory of Brouwer & van Woerkom (1950).

-2)

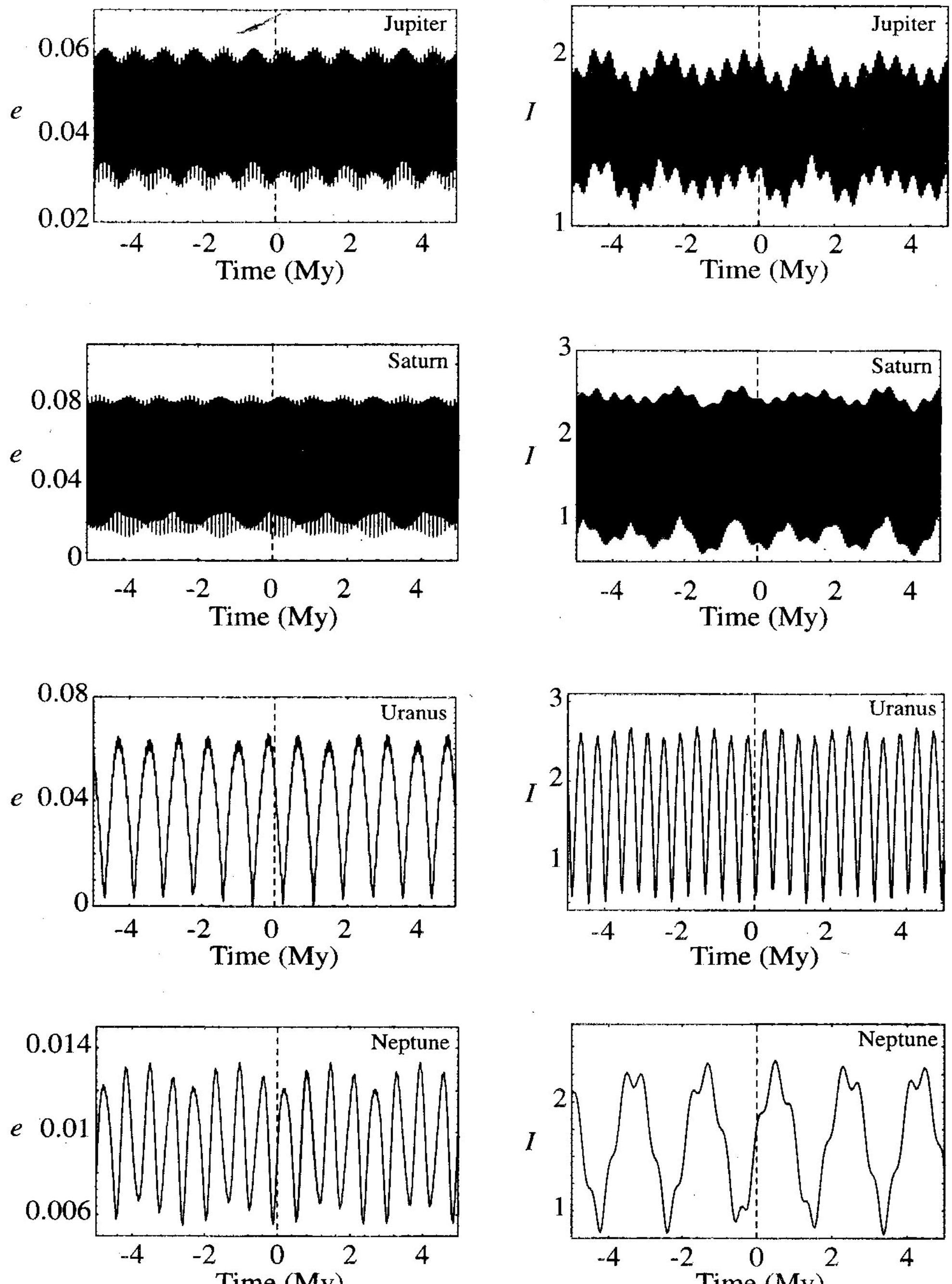


Fig. 7.10. Plots of the eccentricity and inclination (in degrees) of Jupiter, Saturn, Uranus, and Neptune over a period of 10 million years centred on AD 1900, according to the secular theory of Brouwer & van Woerkom (1950).

Evoluzione secolare dei corpi minori

II caso di: 2 pianeti + Corpo minore (m=0)

FUNZIONE di DISTURBO ⇓

$$\mathcal{R} = n a^2 \left[\frac{1}{2} A e^2 + \frac{1}{2} B i^2 + \sum_{j=1}^{2} A_j e e_j \cos(\varpi - \varpi_j) + \sum_{j=1}^{2} B_j i i_j \cos(\Omega - \Omega_j) \right]$$

$$A = n \frac{1}{4} \sum_{j=1}^{2} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b^{(1)}_{3/2}(\alpha_j)$$

$$A_j = - n \frac{1}{4} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b^{(2)}_{3/2}(\alpha_j)$$

$$B = - n \frac{1}{4} \sum_{j=1}^{2} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b^{(1)}_{3/2}(\alpha_j)$$

$$B_j = n \frac{1}{4} \frac{m_j}{M_0} \alpha_j \bar{\alpha}_j b^{(1)}_{3/2}(\alpha_j)$$

$$\alpha_j = \begin{cases} a_j/a & \Rightarrow a_j < a \\ a/a_j & \Rightarrow a_j > a \end{cases}$$

$$\bar{\alpha}_j = \begin{cases} 1 & \Rightarrow a_j < a \\ a/a_j & \Rightarrow a_j > a \end{cases}$$

2)

$$\dot{h} = AK + \sum_{s=1}^{2} A_s K_s \qquad \dot{p} = Bq + \sum_{s=1}^{2} B_s q_s$$

$$\dot{K} = -Ah - \sum_{s=1}^{2} A_s h_s \qquad \dot{q} = -Bp - \sum_{s=1}^{2} B_s p_s$$

SOLUZIONE:

$$\begin{cases} h = e_f \sin(At + \beta) - \sum_{i=1}^{2} \sum_{s=1}^{2} \frac{A_s e_{si}}{A - g_i} \sin(g_i t + \beta_i) \\ k = e_f \cdot \cos(At + \beta) - \sum_{i=1}^{2} \sum_{s=1}^{2} \frac{A_s e_{si}}{A - g_i} \cos(g_i t + \beta_i) \\ p = i_f \sin(Bt + \gamma) - \sum_{i=1}^{2} \sum_{s=1}^{2} \frac{B_s i_{si}}{B - f_i} \sin(f_i t + \gamma_i) \\ q = i_f \cos(Bt + \gamma) - \sum_{i=1}^{2} \sum_{s=1}^{2} \frac{B_s i_{si}}{B - f_i} \cos(f_i t + \gamma_i) \end{cases}$$

- e_f, i_f => eccentricità e inclinazione libere
- RISONANTE SECOLARI: $A = g_i$, $B = f_i$

$\boxed{A\ B \Rightarrow \text{frequenze proprie}}$

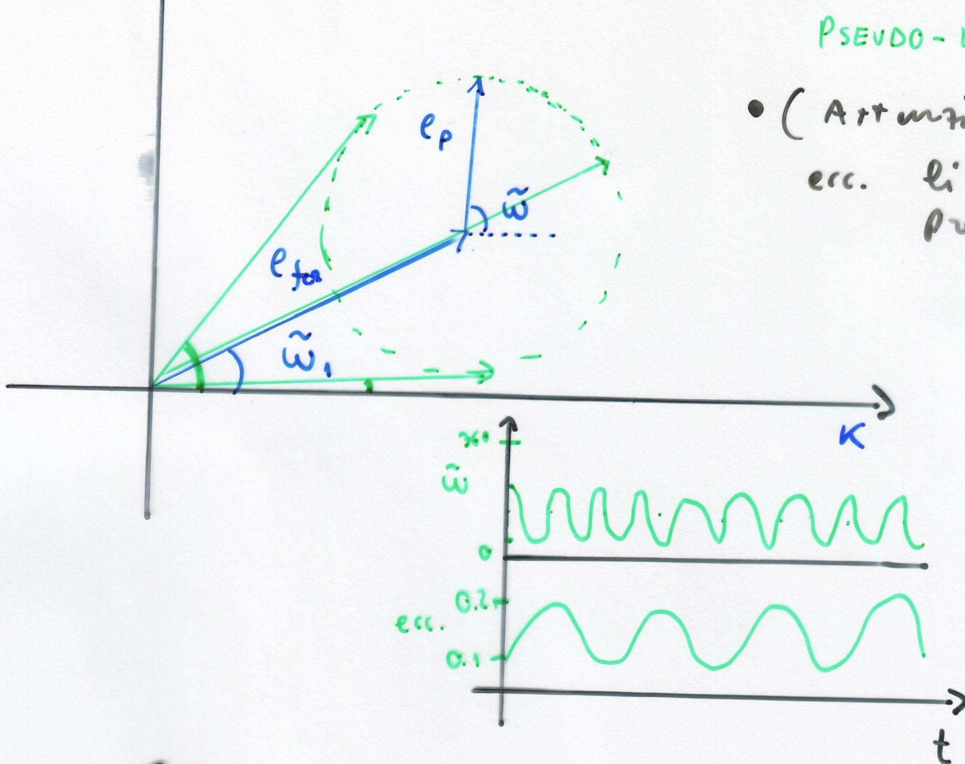
Rappresentazione grafica dell'evoluzione orbitale

$$e_{for} = \frac{M}{A - g_r}$$

$e_p < e_{for}$

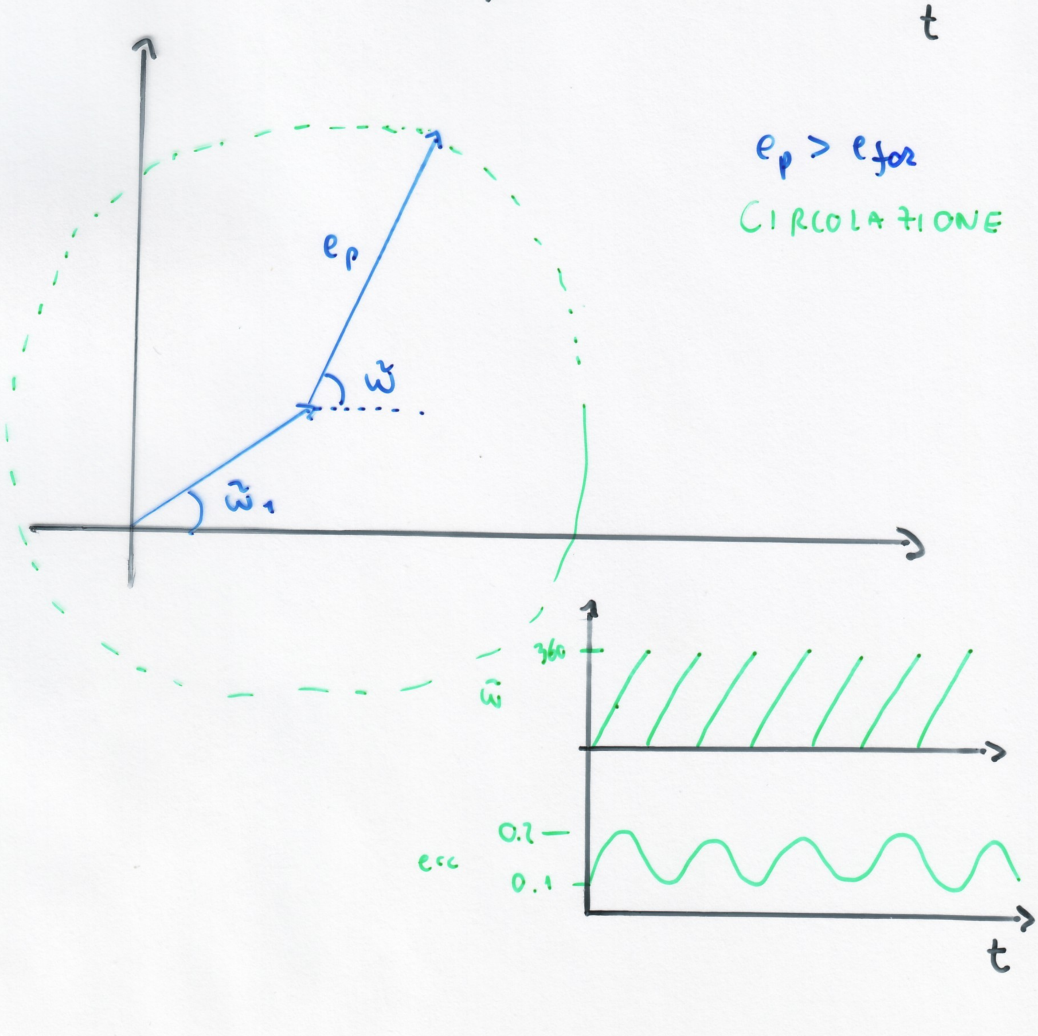
PSEUDO-LIBRAZIONE

- (Attenzione $e_f = e_p$ ecc. libera o propria)



$e_p > e_{for}$

CIRCOLAZIONE



Soluzione generale: N pianeti + corpo minore (m=0).

$$h = e_f \sin(At+\beta) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{A_j \cdot M_{jk}}{A-g_k} \sin\tilde{\omega}_k$$

$$k = e_f \cos(At+\beta) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{A_j M_{jk}}{A-g_k} \cos\tilde{\omega}_k$$

$$p = i_f \sin(Bt+\gamma) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{B_j N_{jk}}{B-f_k} \sin\Omega$$

$$q = i_f \cos(Bt+\gamma) - \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{B_j N_{jk}}{B-f_k} \cos\Omega$$

e_f, i_f, β, γ Dipendono dalle condizioni iniziali

Ex: 3 corpi di cui 1 a massa nulla

(ex: Giove + Asteroide)

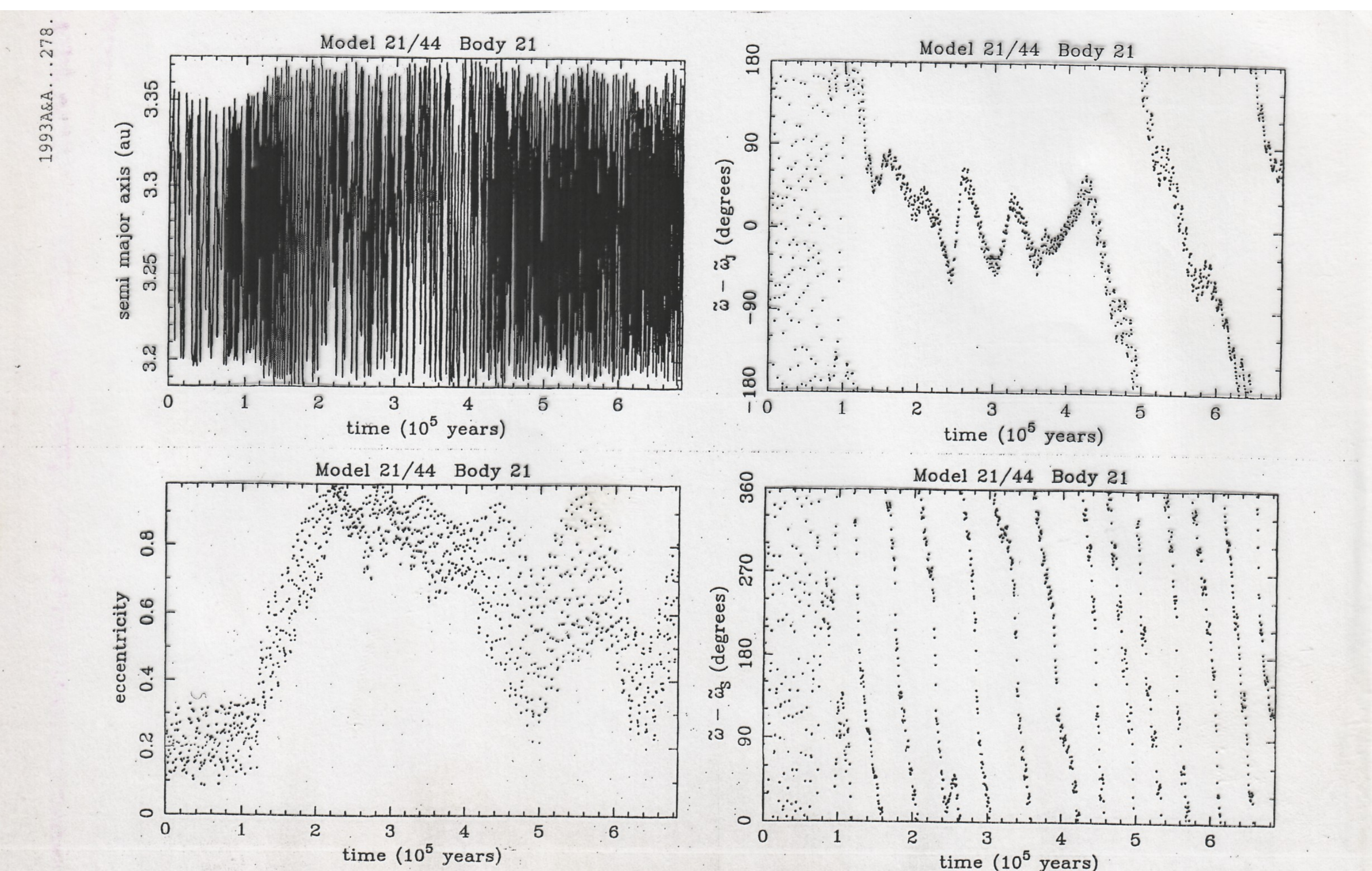
$$h = e\sin\tilde{\omega} = e_f \sin(At+\beta) + \frac{M}{A-g_1}\sin(g_1 t+\varphi)$$

$$e^2 = e_f^2 + \left(\frac{M}{A-g_1}\right)^2 + 2\frac{AM}{A-g_1}\cos\left((A-g_1)t+(\beta-\varphi)\right)$$

•) Se Giove non perturbato $g_1=0$ $\quad h = A\sin(At+\beta) + \frac{M}{A}\sin\varphi$

Si capisce l'origine del caos nelle Kirkwood gaps: risonanze secolari interagiscono con quelle in moto medio causando diffusione nello spazio delle fasi. Ad esempio, all'interno della risonanza in moto medio 2:1 nella fascia asteroidale sono presenti anche le risonanze $A=g_5$ e $A=g_6$

Le figure mostrano l'integrazoine numerica dell'orbita di un asteroide nella risonanza 2:1. Anche l'argomento critico della risonanza secolare $A=g_5$ libra causando una crescita caotica dell'eccentricità.



La sovrapposizione tra risonanze secolari e in moto medio avviene anche per i Toriani (risonanza 1:1).

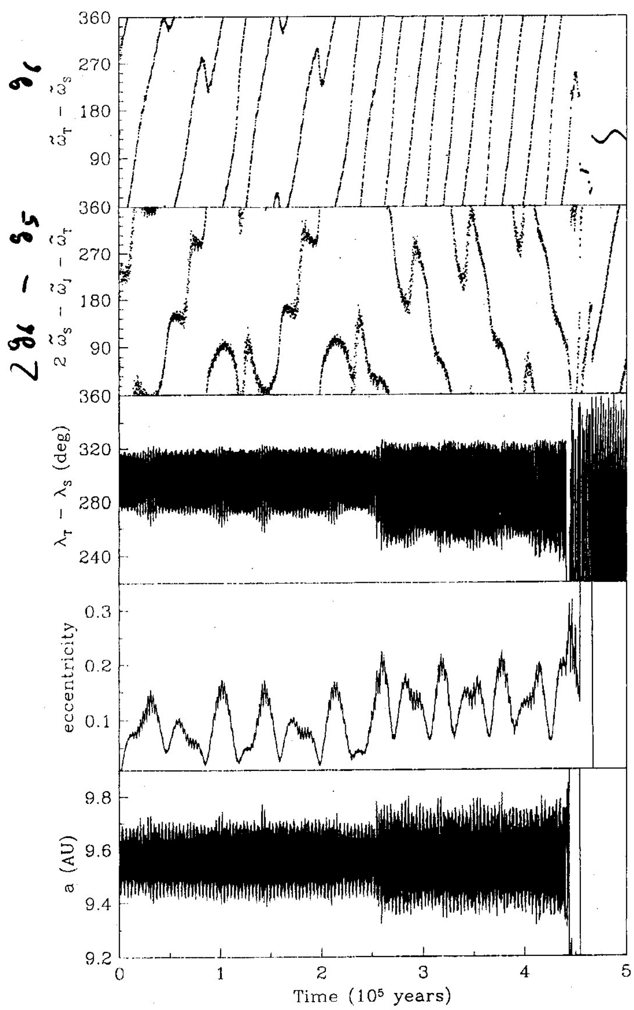


FIG. 3. Orbital evolution of a L5 Saturn Trojan. The crossing of the mixed secular resonance with critical argument $2\tilde{\omega}_S - \tilde{\omega}_J - \tilde{\omega}_T$ is shown by the change of the direction of circulation after about 2.5×10^5 year. Ejection occurs near 4.5×10^5 year. Indices T, J, S refer, respectively, to Trojan, Jupiter, Saturn.

amplitude of both the semimajor axis and the critical argument of the Trojan resonance $\lambda_T - \lambda_S$, and in eccentricity. This jump occurs at the same time as the inversion in the circulation direction. After the crossing, instability builds up and the eccentricity grows until the body is ejected out of the resonance into a chaotic orbit, eventually, after a close encounter with Saturn. The inversion of circulation direction during the resonance crossing is very similar to that shown in Fig. 1b for the ν_{16} secular resonance and Jupiter Trojans. However, for the mixed resonance we cannot plot the h, k variables (same as p, q variables, but for resonances involving the perihelia) to show in more detail the resonance crossing. We lack, in fact, the definition of the h, k variables for the mixed resonance, since we need a perturbative scheme to derive the correct action variable related to the critical argument. However, the resonance crossing is evident in Fig. 3 and is well supported by the related changes in the orbital elements. Regarding the short-period modulations superimposed onto the circulation trends of the secular arguments in Fig. 3, they are related to the changes in the precession rate of $\tilde{\omega}_T$. These are caused by the oscillations in the orbital eccentricity. For most of our low-amplitude libration orbits we found that this kind of behavior leads to escape from both L4 and L5 orbits on a time scale of the order of 10^5 years. Since it was suggested by De la Barre *et al.* (1996) that the ν_6 secular resonance can also play an important role in destabilizing Saturn Trojans, we also plot in Fig. 3 the corresponding critical argument. However, in this case, it does not show any particular behavior that may be related to the instability of the orbit.

The passage through the $2\tilde{\omega}_S - \tilde{\omega}_J - \tilde{\omega}_T$ mixed secular resonance may occur after a longer time scales. Figure 4 illustrates

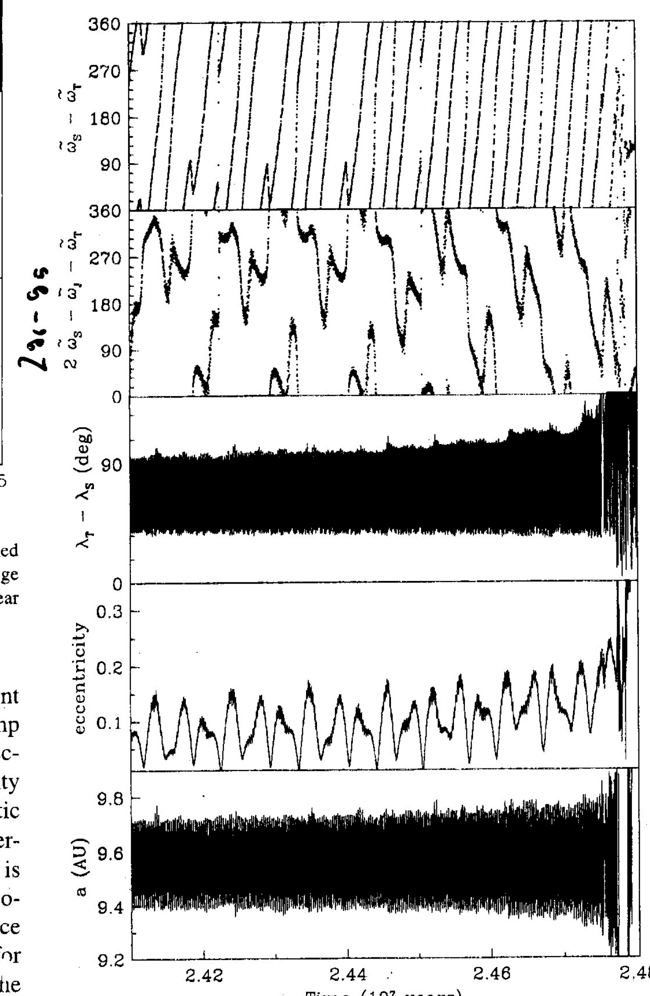


FIG. 4. Instability of a Saturn Trojan on a longer time scale (2.4×10^7 year) than in Fig. 2.

Per i Troiani, le risonanze secolari possono 'pompare' l'inclinazione a valori elevati.

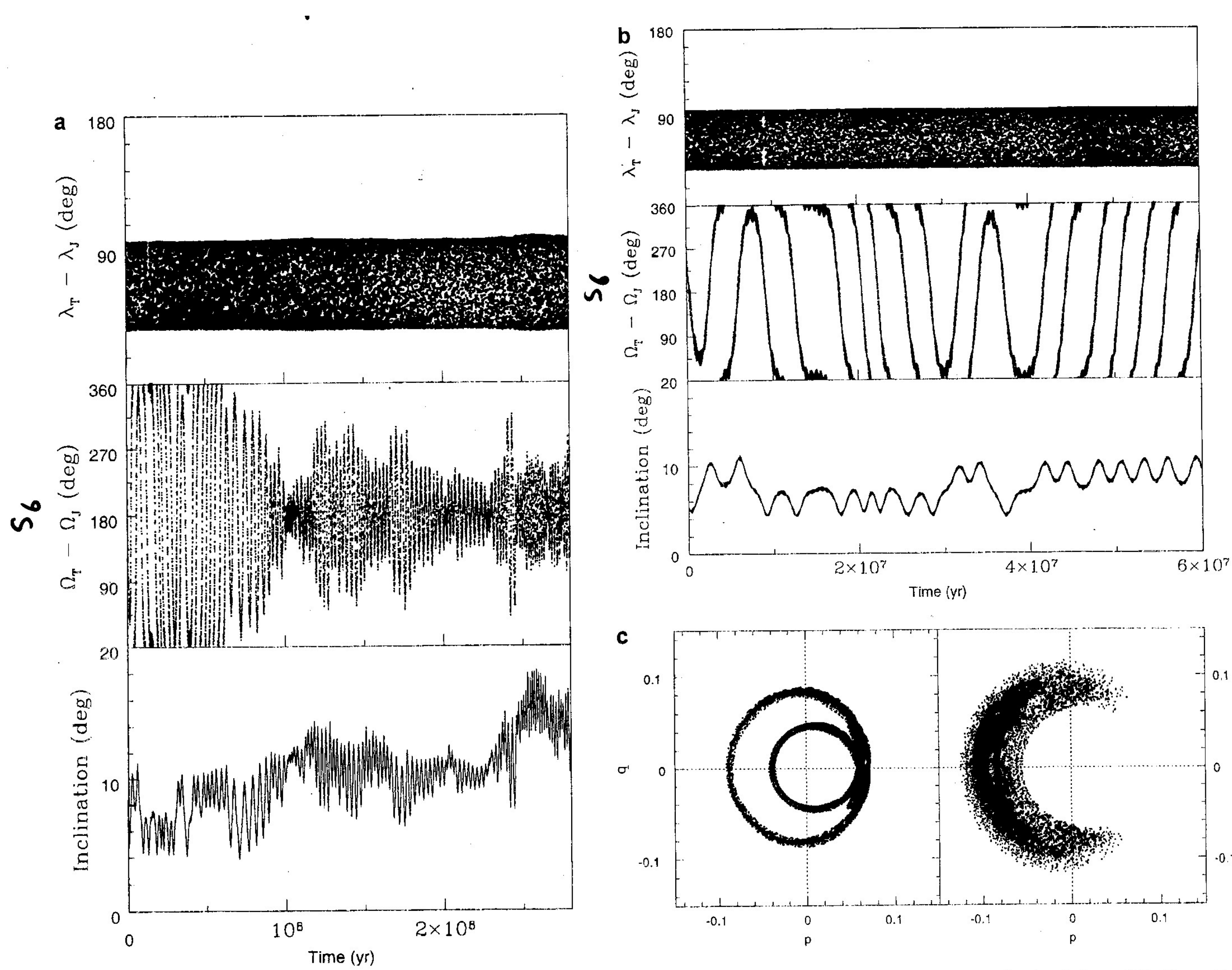


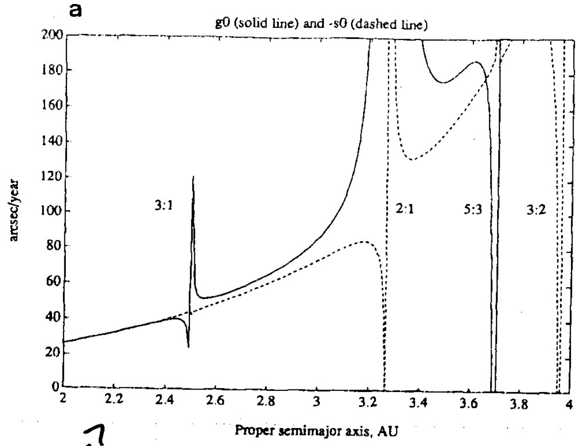
FIG. 1. (a) Evolution of the inclination of a Jupiter L4 Trojan orbit with an initial inclination of 4.5° and libration amplitude of about 59°. The critical argument of the ν_{16} resonance shows inversions of the circulation direction in the first 0.5×10^8 year of evolution, typical of resonance crossing; then it enters the resonance and librates for the remaining 2×10^8 year. (b) Focus on the first 10^8 year of orbital evolution of the case shown in Fig. 1a. The changes from positive to negative circulation and vice versa are clearly visible and are related to the ν_{16} resonance crossings. A jump in inclination occurs at every reversion. (c) The variables p and q are plotted for the case in Fig. 1a. On the left the data points cover the time interval 2×10^7–5×10^7 year; the resonance crossing is marked by the change from negative to positive circulation. On the right the variables p and q are shown during the time interval 1×10^8–2×10^8 year. The classical banana shape characterizes the libration of the ν_{16} critical argument.

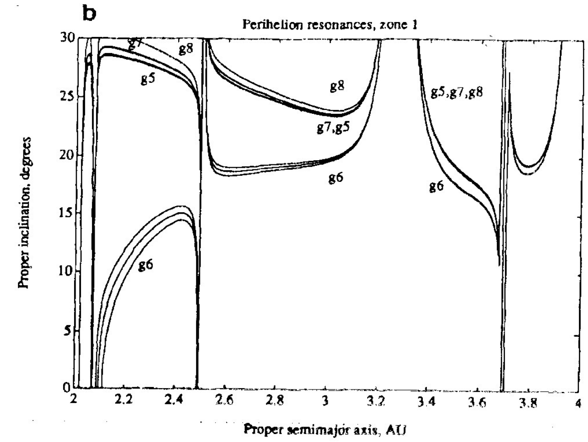
first 10^8 years of evolution (Fig. 1b). Any time a reversion occurs, the body crosses the secular resonance and the inclination jumps. A similar behavior, but for the eccentricity, is known when a mean motion resonance is crossed (e.g., Peale 1986, Marzari *et al.* 1997). Finally, the body enters the resonance and the critical argument starts to librate around 180° with a period of about 1.7×10^6 years. The resonance crossing and the libration are made more clear in Fig. 1c, where the values of the variables p and q are plotted from $t = 2 \times 10^7$ year to $t = 5 \times 10^7$ year, and from $t = 1 \times 10^8$ year to $t = 2 \times 10^8$ year. The inclination grows from 4.5° to a maximum of 19.5° in 2.8×10^8 year. The libration amplitude of the Trojan critical argument $\lambda_p - \lambda_j$ grows from ~58.9° to 64° during this evolution.

The second case (Fig. 2) has an initial libration amplitude of 58° and an initial inclination of 4°. Its behavior is more chaotic, with frequent changes between circulation and libration. The eccentricity remains low ($\lesssim$0.05) while the inclination grows up to about 30°. Shortly after such a high inclination is reached, the libration amplitude increases, the orbit becomes unstable, and it escapes from the Trojan swarm.

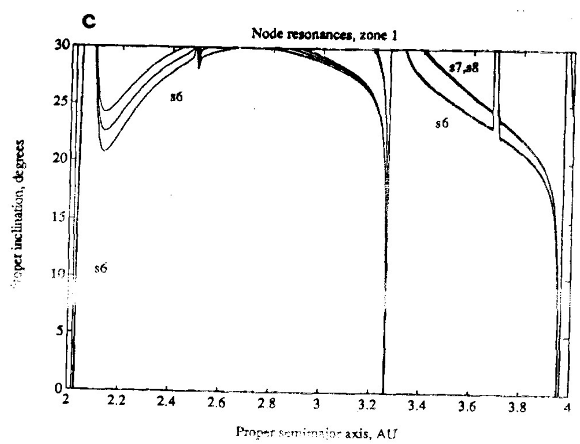
Due to the presence of the ν_{16} resonance at low inclinations, the scenario for the origin of highly inclined Trojans at L4 and L5 may have been the following. Planetesimals with low inclinations are trapped as Trojans by the mass growth of Jupiter in the early Solar System with a broad distribution of libration amplitudes (see Fig. 10 in Marzari and Scholl, 1998b). Trojans which

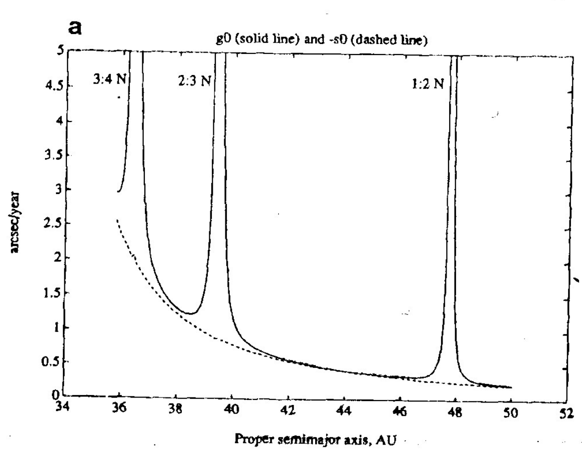
Mappa delle risonanze secolari nel Sistema Solare

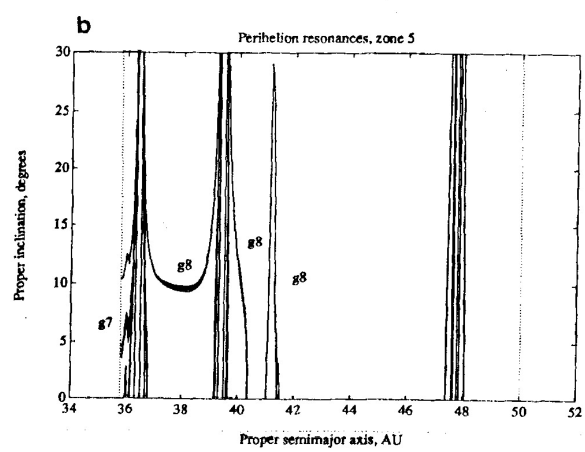


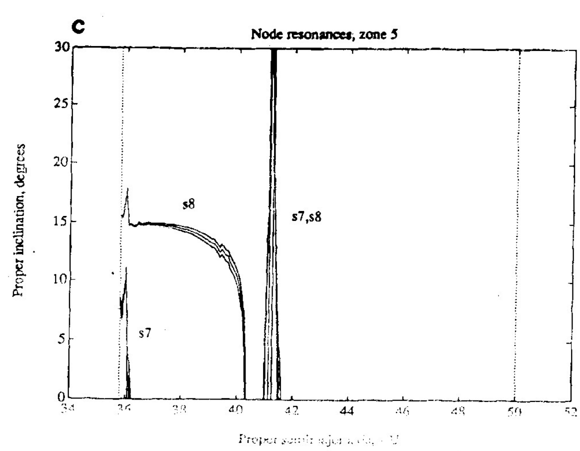


frequenze libere
(o proprie)

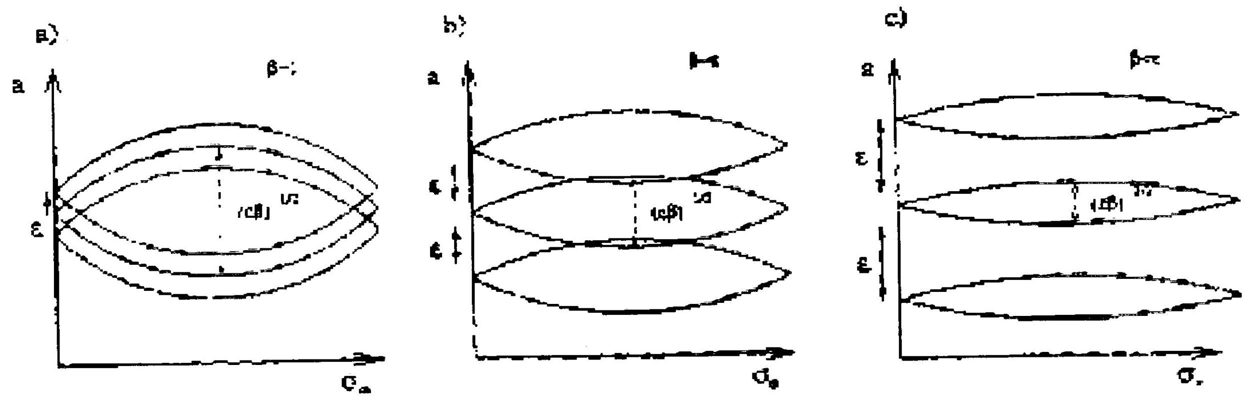


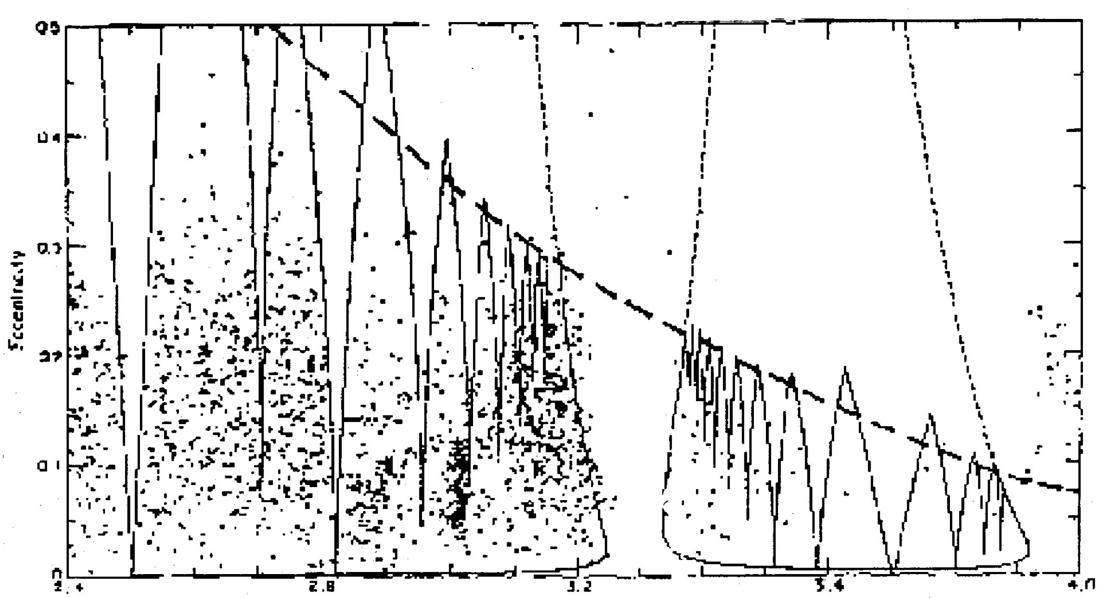






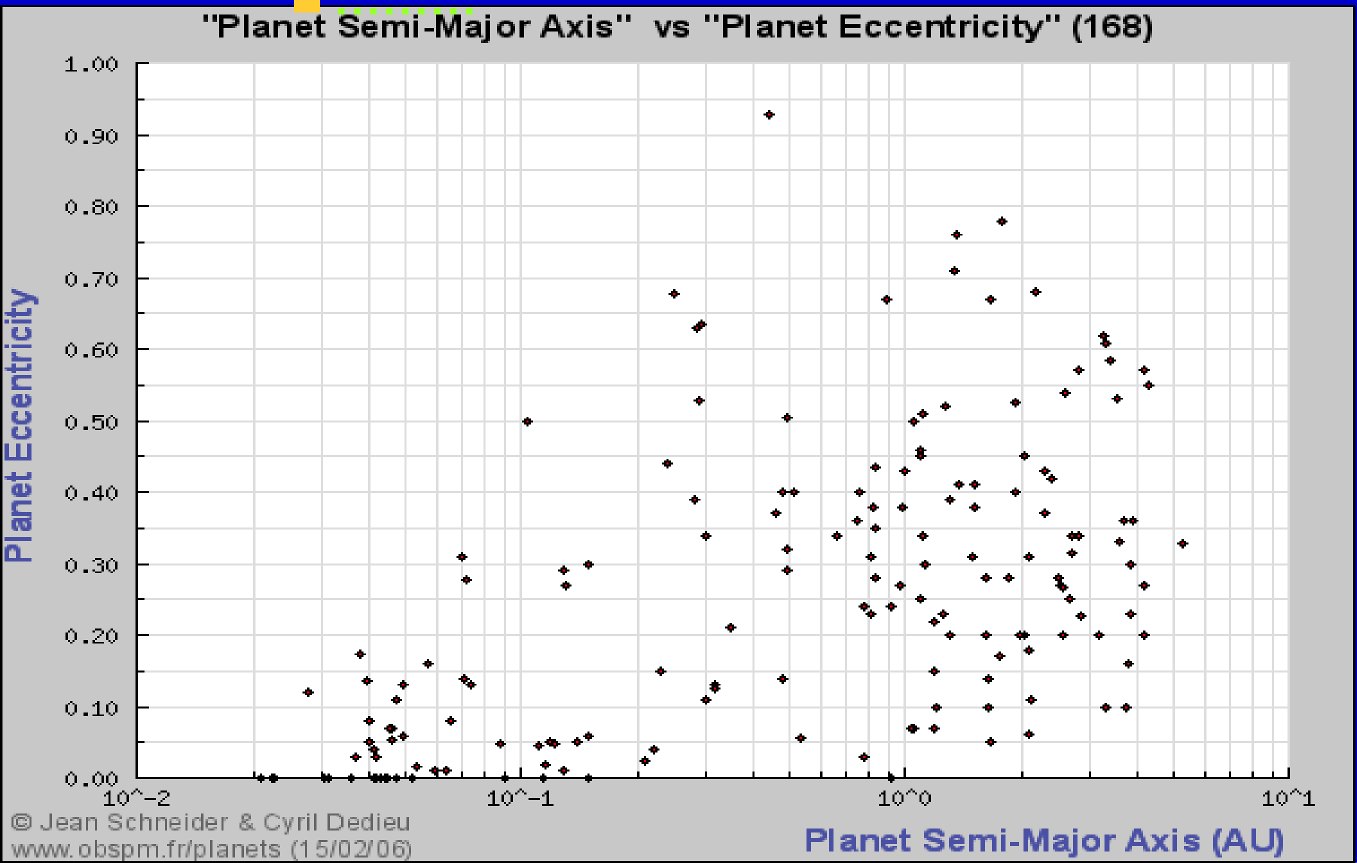
Il principio di sovrapposizione delle risonanze funziona anche nella parte esterna della fascia asteroidale. Al di là della risonanza 2:1, al crescere dell'eccentricità le numerose risonanze in moto medio adiacenti si allargano portando a moto caotico causato dalla loro sovrapposizione.





La teoria di Lagrange-Laplace funziona per piccoli e ed i, mentre in molti sistemi extrasolari i pianeti hanno elevate eccentricità!

- **HD74156** $e_1=0.636$ $e_2=0.583$
- **HD202206** $e_1=0.435$ $e_2=0.267$
- **HD12661** $e_1=0.350$ $e_2=0.20$
- **HD128311** $e_1=0.25$ $e_2=0.17$
- **Ups And** $e_1=0.012$ $e_2=0.27$



Risonanza apsidale (g1 ~ g2)

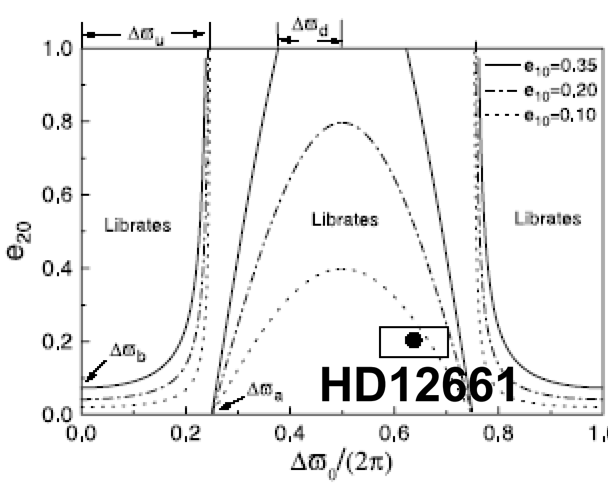
$$e_1 e_2 \cos\Delta\varpi = (e_{11}e_{21} + e_{12}e_{22}) + (e_{11}e_{22} + e_{12}e_{21})\cos(\psi_1 - \psi_2)\,.$$

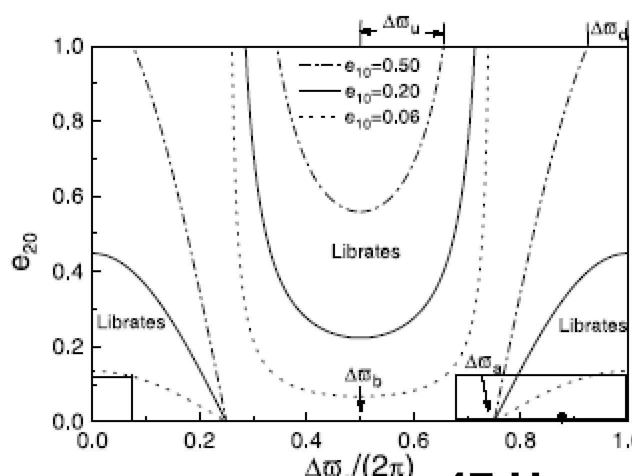
$$\psi_2 - \psi_1 = (g_2 t + \beta_2) - (g_1 t + \beta_1),$$

$$S = \left|\frac{e_{11}e_{22} + e_{12}e_{21}}{e_{11}e_{21} + e_{12}e_{22}}\right|\,.$$

Se S > 1 risonanza apsidale. Librazione di Δω attorno a 0 o 180 gradi.

Perché la risonanza apsidale è importante? Orbite allineate con alta eccentricità e in risonanza apsidale sembrano più stabili di quelle non risonanti (calcolo del coeff. lyapunov).





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