

Università degli Studi di Padova

Degree in Physics

QFT

*Notes written by master students on some of the topics
covered during the Quantum Field Theory course*

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January 19, 2020
Temporary Version

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Course Program

(i) 26/02/2018, 14.30 - 16.15.

Aim and scope of the course. Description of the program and main references. Differences between Quantum Mechanics and Quantum Field Theories (QFT). Outlines on the perturbative formulation, operator formalism, path integral formalism. Comments on ϕ^4 and Quantum Electrodynamics (QED). Triviality of ϕ^4 in $D = 4$ and its non-existence for $D > 4$. Non-Borel summability of QED. Wigner's theorem and exact symmetries. Algebra of the observables and their *-automorphisms. The von-Neumann theorem and unitary equivalence of theories with finitely many degrees of freedom. Spontaneous symmetry breaking as a phenomenon in theories with infinitely many degrees of freedom. Formulation of QFT on the lattice. Axiomatic approach. Wightman's axioms and Wightman's reconstruction theorem. Euclidean formulation. Schwinger functions and Osterwalder-Schrader's reconstruction theorem.

The excellent book by F. Strocchi, "Elements of QM of infinite systems", SISSA, World Scientific, 1985, includes the analysis of some key non-perturbative phenomena of QFT's. Spontaneous symmetry breaking, shortly illustrated during the lecture, is reported at pp. 115-120. Other useful references are <http://arxiv.org/pdf/1201.5459.pdf>, <http://arxiv.org/pdf/1502.06540.pdf> and the two excellent books F. Strocchi, "An introduction to non-perturbative foundations of quantum field theory", Oxford, 2013, Haag, "Local quantum physics, fields, particles, algebras", Springer-Verlag, 1996. See also the notes: A brief introduction to different QFT approaches.

(ii) 01/03/2018, 13.30 - 15.15.

Review of the Dirac equation. Gamma's algebra and its representations. Covariance of the Dirac equation. Transformation of the spinors under the Poincaré group, $\psi'(x') = S(\Lambda)\psi(x)$. Representations of the Poincaré group. $S(\Lambda)$ in the case of parity transformations. Charge conjugation.

Itzykson-Zuber, sect. 2-1-2, 2-1-3, 2-4-2. Further references are chapters 2 and

3 of Peskin-Schroeder, “Quantum Field Theory”, and the chapters 11, 12 and 13 of Bjorken-Drell, “Relativistic Quantum Fields”. Notes: Unitary representation of the Poincaré Group - Wigner classification. Behaviour of local fields under the Poincaré group. Relativistic covariance. Finite-dimensional irreducible representations of the Lorentz group.

(iii) 08/03/2018, 13.30 - 15.15.

Clifford Algebra and bilinear spinors. $S(\Lambda)$ in the case of charge conjugation. Discrete symmetries in the case of quantised fields. Parity operator.

Itzykson-Zuber, sect. 3-4-1.

(iv) 12/03/2018, 14.30 - 16.15.

Charge conjugation and time reversal. Transformation properties of the Dirac bilinears under P , C and T transformations. PCT theorem.

Itzykson-Zuber, sect. 3-4-2, 3-4-3, 3-4-4. See also chapter 2 and 3 of Peskin-Schroeder, “Quantum Field Theory”, and the chapters 11, 12 and 13 of Bjorken-Drell, “Relativistic Quantum Fields”.

(v) 15/03/2018, 13.30 - 15.15.

Lehmann, Symanzik and Zimmerman reduction formula. Dirac formulation of the path integral.

*Sect. 5 of M. Srednicki, “Quantum Field Theory”, Cambridge. See also the notes: Källén-Lehmann representation. N.B.: Srednicki uses the metric $g' = -g$, $\text{diag}(g') = (-1, 1, 1, 1)$. The scalar products defined by the two metrics have opposite sign. Ramond, sect. 2.1 e 2.2. Notes: On the Dirac paper where it has been first formulated the path integral. P. A. M. Dirac, “The Lagrangian in quantum mechanics”, *Phys. Z. Sowjetunion* **3** (1933) 64. Reprinted in, *Selected papers on quantum electrodynamics*, J. Schwinger Ed., Dover, 1958. See also, <http://arxiv.org/pdf/quant-ph/0004090v1.pdf>. The standard text for the path integral is Feynman-Hibbs, “Quantum mechanics and path integrals”, McGraw Hill, 1965, and the 2010 Dover edition commented by Styer.*

(vi) 19/03/2018, 14.30 - 16.15.

Dirac paper on the formulation of the path integral. The need of the Hamiltonian formulation: the case $H = p^2v(q)/2$.

Ramond, sect. 2.2.

(vii) 22/03/2018, 13.30 - 15.15.

Functional derivatives. Forced harmonic oscillator. Convergence methods: $i\epsilon$ prescription and Euclidean formulation. Vacuum to vacuum amplitude in the presence of an external force.

Ramond, sect. 2.3. Notes: Functional derivatives.

(viii) 26/03/2018, 14.30 - 16.15.

Path integral for quadratic Lagrangians. Bohm-Aharonov effect. Path integral in the case of scalar theories. Euclidean formulation.

*Felsager, "Geometry, Particles and Fields", Springer, 1998, pp. 51-56, pgg. 81-82.
Ramond, sect. 3.1, 3.2.*

(ix) 29/03/2018, 13.30 - 15.15.

Green's functions in the free case and their representation in momentum space. Feynman diagrams for the free theory. Feynman propagator. Legendre transform and effective action. Schwinger method to extract the potential density from the path integral.

Ramond, sect. 3.2 and 3.3. The rigorous proof that the path integral reproduces the N -point functions, topic not treated in the lectures, can be found in the notes. These follow Peskin-Schroeder, "An introduction to Quantum Field Theory", ABP 1995, pp. 282-284.

(x) 05/04/2018, 13.30 - 15.15.

Effective action in the general case. Schwinger-Dyson equations. Inverse of the Feynman propagator. Linked-cluster theorem and $W[J]$ as generating functional of the connected Green's functions. Invariance of the N -point connected Green's functions under the shift $\phi(x) \rightarrow \phi(x) + f(x)$.

Ramond sect. 3.3. Notes: $\phi_{cl}(x)$ and the Schwinger-Dyson equation. $W[J]$ as generator of the connected Green's functions (based in Kerson Huang, "Quantum Field Theory. From operators to path integrals", 2010. pp. 188-189. Notes: Comment on the connected Green functions.

(xi) 09/04/2018, 14.30 - 16.15.

Saddle point approximation. Determinants from gaussian integrals. Perturbative solution of the Klein-Gordon equation for ϕ^4 in the presence of an external current. Green functions at tree level. Topology of the Feynman diagrams.

Ramond, sect. 3.4 and appendix A.

(xii) 12/04/2018, 13.30 - 15.15.

Relation between the connected two-point function and $\Gamma^{(2)}$. Determinants and heat equation. Effective action at order \hbar . Dependence of the coupling constant on the mass scale (Coleman-Weinberg).

Ramond, sect. 3.4 and 3.5. Notes: $\tilde{\Gamma}_E^{(2)}(p)\tilde{G}_{cE}^{(2)}(p) = 1$. Comments on $\Gamma[\varphi]$ at order \hbar .

(xiii) 19/04/2018, 13.30 - 15.15.

Breaking of dilatation symmetry by quantum effects. Perturbation theory and Feynman rules. Examples of Feynman diagrams: tadpole, setting sun, fish. Normal ordering singularities.

Ramond, sect. 3.6 and 4.1.

(xiv) 23/04/2018, 14.30 - 16.15.

Loop expansion as power expansion in \hbar . Truncated Green functions and LSZ reduction formula. Superficial degree of divergence. Renormalisable, super-renormalisable and non-renormalisable theories. Weinberg theorem. Effective action as generating functional of proper vertices, Jona-Lasinio theorem.

Ramond, sect. 4.2. Notes: Loop expansion as power expansion in \hbar . Truncated green functions and LSZ reduction formula. Effective action as generating functional of proper vertices, Jona-Lasinio theorem. The discussion at pp. 111-112 of the Ramond book is also reported, with more care and clarity, in the Casalbuoni lectures: pp. 92-97 of <http://theory.fi.infn.it/casalbuoni/dott1.pdf>. See also pp. 139-142 of <http://theory.fi.infn.it/casalbuoni/lezioni99.pdf> and sect. 11.5 of Kleinert book, "Particles and Quantum Fields", World Scientific, 2016.

(xv) 26/04/2018, 13.30 - 15.15.

Jona-Lasinio theorem. Regularisation methods. Dimensional regularisation.

Notes: Effective action as generating functional of proper vertices, Jona-Lasinio theorem. Ramond, sect. 4.3.

(xvi) 03/05/2018, 13.30 - 15.15

Dimensional regularisation. Proof of the Feynman parametrisation formula. Euclidean action in 2ω dimension, dimensionless λ_{new} and 't Hooft mass parameter μ . Tadpole and fish diagrams.

Ramond, sect. 4.3. The proof of the Feynman parametrisation formula follows the one in http://kodu.ut.ee/~kkannike/english/science/physics/notes/feynman_param.pdf.

(xvii) 07/05/2018, 14.30 - 16.15.

Calculation of the fish and double scoop diagrams. Notes on the calculation of the setting sun diagram, analysis of the divergences, residue depending on the moment. Renormalisation. On the Feynman rules. Mass term considered as a two-leg vertex, Feynman propagator with mass as diagrammatic series of the massless one with interaction given by the mass term. Counterterms for $\tilde{\Gamma}^{(2)}(p)$ and $\tilde{\Gamma}^{(4)}(p)$. Recursive structure of the renormalisation procedure.

Ramond, sect. 4.4 and 4.5. Notes: On the Feynman rules. $\tilde{\Gamma}^{(2)}(p)$ at one-loop with the counterterm contribution. A useful reference for further information is section 11 of the Kleinert text, "Particles and Quantum Fields", World Scientific, 2016.

(xviii) 10/05/2018, 13.30 - 15.15

Renormalised Lagrangian density. Relation between the bare and renormalised proper vertex functions. Renormalisation group equation. Scale equation for $\tilde{\Gamma}_{\text{ren}}^{(N)}$. Bare parameters in terms of λ , m/μ and ϵ . Renormalisation prescriptions. 't Hooft and Weinberg prescriptions. The β function. Landau pole. Ultraviolet and infrared fixed points of β . Asymptotic freedom and confinement. Scaling of $\tilde{\Gamma}_{\text{ren}}^{(N)}$ and anomalous dimension.

Ramond, sect. 4.5 and 4.6. Notes: Relation between the bare and renormalised proper vertex functions. Scaling of $\tilde{\Gamma}_{\text{ren}}^{(N)}$ and anomalous dimension. The explicit steps concerning the equations 4.6.10 - 4.6.15 of the Ramond book are reported in the equations 31.11 - 31.23 of <http://theory.fi.infn.it/casalbuoni/dott1.pdf>. The english version is reported in Chapter 6 of <http://theory.fi.infn.it/casalbuoni/lezioni99.pdf>.

(xix) 14/05/2018, 14.30 - 16.15.

Calculation of γ_m and γ_d . Vertex functions in the limit of large momenta in the case of a UV fixed point. Prescription dependence of the renormalisation group coefficients. Prescription independence of the existence of a UV fixed point of the

β function. Grassmann algebra. Derivation and integration for anticommuting variables.

Ramond, sect. 4.6 and 4.7. The prescription dependence of the renormalisation group coefficients is also discussed at pp. 131-132 of <http://theory.fi.infn.it/casalbuoni/dott1.pdf>. The english version is reported in Chapter 6 of <http://theory.fi.infn.it/casalbuoni/lezioni99.pdf>. The Grassmann algebra is reported in L.H. Ryder, "Quantum Field Theory", 2nd Edition. 1996, sect. 6.7.

- (xx) 17/05/2018, 13.30 - 15.15 Review of the properties of the self-energy, Green's functions and the vertex proper functions. Fermionic path integral.

L.H. Ryder, "Quantum Field Theory", 2nd Edition. 1996, sect. 6.7.

- (xxi) 21/05/2018, 14.30 - 16.15.

Gauge theories. Path integral for gauge fields. Gauge fixing. Propagator of the gauge field. Feynman and Landau gauges. Non-covariant gauge for QED. Faddeev e Popov method. Feynman rules in the covariant gauge. The case of QED. Ward-Takahashi identities in QED.

L.H. Ryder, sect. 7.1, 7.2 and 7.4.

- (xxii) 24/05/2018, 13.30 - 15.15

Ward-Takahashi identities in QED. BRS transformations. Furry theorem. γ_5 in dimensional regularisation.

L.H. Ryder, "Quantum Field Theory", sec. 7.4, 7.5. A reference for the Furry theorem is the solution of problem 58.2 of Srednicki's book reported his solution manual "Quantum Field Theory: Problem Solutions", available at <https://drive.google.com/file/d/OB0xb4crDvCgTM2x6QkhKREg0WW8/edit>. The Dirac algebra in arbitrary dimension is discussed in the book: Collins, "Renormalisation". A good paper on the subject is the one by S. Weinzierl, Equivariant dimensional regularisation, hep-ph/9903380, available at <https://arxiv.org/pdf/hep-ph/9903380.pdf>.

- (xxiii) 28/05/2018, 14.30 - 16.15.

Feynman rules for QED. Superficial degree of divergence. Divergent Feynman diagrams at one loop. Electron and photon self-energies. Vertex function.

L.H. Ryder, sect. 9.5.

(xxiv) 31/05/2018, 13.30 - 15.15

Counterterms of QED at one loop. Renormalisation at one-loop. Lamb Shift. Anomalous magnetic moment.

L.H. Ryder, "Quantum Field Theory", sect. 9.5 and 9.6. Itzykson-Zuber, sect. 2-2-3.

Main References

In an ideal World any theoretical physicists should read the magnificent and presumably best book of Quantum Field Theory

- [1] Bryce S. DeWitt, *The Global Approach to Quantum Field Theory. Vol. 1, 2*, Int. Ser. Monogr. Phys., 114, 2003.

For a near-perfect understanding of Quantum Field Theory, reading DeWitt's book should be complemented by the following masterpieces

- [2] P. A. M. Dirac, "The Lagrangian in quantum mechanics," Phys. Z. Sowjetunion **3** (1933) 64. Available [here]
- [3] R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All that*, Landmarks in Physics, Princeton University Press, 2000.
- [4] J. Glimm and A.M. Jaffe, *Quantum Physics. A Functional Integral Point of View*, New York, Usa: Springer, 1987, p. 535.
- [5] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Berlin, Germany: Springer (1992) p. 356. (Texts and monographs in physics), 1992.
- [6] F. Strocchi, *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*, Int. Ser. Monogr. Phys., 158, 2013.
- [7] C. Itzykson and J. Zuber, *Quantum Field Theory*, Dover Pubns, 2006.
- [8] S. Weinberg, *The Quantum Theory of Fields. Vol. 1: Foundations*, Cambridge University Press, 2005.
- [9] S. Weinberg, *The Quantum Theory of Fields. Vol. 2: Modern Applications*, Cambridge University Press, 2013.
- [10] L.D. Faddeev and A.A. Slavnov, *Gauge Fields. Introduction to Quantum Theory*, Front. Phys., 50, 1980, p. 232.
- [11] M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press, 1995.

Chapter 1

Overview of the formulations of Quantum Field Theory¹

The aim of these chapter is to provide a short overview on the various approaches to quantum field theory (QFT), whose main task is to compute physical quantities such as the S -matrix and therefore the cross section of the theory.

We will start with the axiomatic approach, based on the Wightman axioms, which is mathematically well-defined and is therefore used for rigorous proofs. Another approach is the perturbative one, which is the most used for studying quantum field theories. This can be formulated in terms of the operator approach or in the framework of the path integral formalism. A non-perturbative approach to QFT concerns the formulation on a lattice, where space-time is discretised. We will also shortly review the formalism based on the Schrödinger representation of quantum fields. The last section concerns a short introduction to the phenomenon of spontaneous symmetry breaking. Here is the list of acronyms used in these notes.

CCR	canonical commutation relations
GF	Green's functions
QCD	quantum chromodynamics
QED	quantum electrodynamics
QFT	quantum field theory
QFTL	quantum field theory on a lattice
QM	quantum mechanics
SSB	spontaneous symmetry breaking

¹ Matteo Sighinolfi

1.1 What QFT is: differences with QM and consequences

In the past courses the student has been introduced to QM and its formalism. In QM, as in classical physics, it is possible to discriminate between a theory with a finite number of degrees of freedom and a theory with an *infinite* number of degrees of freedom. We will denote the first as QM_{fin} and the latter as QM_{∞} . As shown in the lectures of “Theoretical Physics A”, in QM one says that the map

$$T : |\phi\rangle \rightarrow |\phi'\rangle ,$$

where $|\phi\rangle$ and $|\phi'\rangle$ belong to the Hilbert space of states, is an exact (or unbroken) symmetry if it preserves the transition probabilities

$$|\langle\phi|\psi\rangle|^2 = |\langle\phi'|\psi'\rangle|^2 .$$

A theorem by Wigner states that such a transformation must be represented by the transformation

$$|\phi'\rangle = U|\phi\rangle ,$$

where U is a unitary or antiunitary operator. This is true both for QM_{fin} and QM_{∞} , but there is a great difference between the two cases.

QFT is a QM_{∞} theory, and it is possible to show that only for this type of theory there are *inequivalent representations* of CCR not connected by unitary (or antiunitary) transformations. Later on, we will see that this is related to SSB.

The main point is that QFT has infinite degrees of freedom, so it must be treated differently than QM_{fin} . We will consider some of the possible approaches illustrating their successes and problems.

1.2 Axiomatic approach

The axiomatic approach² was developed by Wightman in the 50's, with the will of quantising fields following von Neumann's idea of quantum theory (so the Dirac's formalism, involving bra and ket, is not followed).

Let us first consider a classical relativistic field theory. Here, one considers a field $\phi(t, \mathbf{x})$ whose dynamics are consistent with special relativity. For a free field with mass $m \geq 0$

² The following description of the Wightman axioms and Wightman reconstruction theorem is based on the article http://www.scholarpedia.org/article/Wightman_quantum_field_theory.

this means that $\phi(t, \mathbf{x})$ satisfies the free wave equation

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi(t, \mathbf{x}) - \nabla^2 \phi(t, \mathbf{x}) + m^2 \phi(t, \mathbf{x}) = 0 . \quad (1.1)$$

It is now possible to choose as units of time and space $x^0 = ct$, x^j , $j = 1, 2, 3$. In this way, the Minkowski metric is the familiar mostly negative

$$g \equiv g_{\mu\nu} = \text{diag}(1, -1, -1, -1) .$$

Adopting the standard convention for covariant and contravariant variables, the free wave equation is now written in the Lorentz covariant form

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0 . \quad (1.2)$$

This equation can be obtained from the free action

$$S_0 = \frac{1}{2} \int d^4x (\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) . \quad (1.3)$$

To have an interacting theory one adds a term to S_0 which is usually a polynomial in ϕ with grade higher than two, for example

$$S_I = - \int d^4x \frac{\lambda}{4!} \phi^4 ,$$

implying the classical equation of motion

$$\partial^\mu \partial_\mu \phi + m^2 \phi + \frac{\lambda}{3!} \phi^3 = 0 . \quad (1.4)$$

Until now there is nothing new or tricky in our physics, but by now things starts getting more difficult. If $\phi(t, \mathbf{x})$ is a real field, then Eq.(1.4) has smooth solutions for any smooth bounded initial conditions at some initial time t_0 . The field is determined at every position and time knowing its value and its time derivative at $t = t_0$. At any time, there is a Poisson bracket between the field and its time derivative $\dot{\phi}$

$$\left\{ \phi(t, \mathbf{x}), \dot{\phi}(t, \mathbf{y}) \right\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}) .$$

If one tries to quantise the field ϕ , it is clear that it cannot be a function of \mathbf{x} because of the above Poisson bracket containing $\delta^{(3)}(\mathbf{x} - \mathbf{y})$, which is a distribution. The only possibility for ϕ is to be a distribution in the sense of Schwartz. Looking back at Eq.(1.4) we see that the term ϕ^3 is problematic because non-linear distributions are undefined. Actually, while quantising the theory one unavoidably gets the divergences in the calculations, like the infinities arising in the Dyson-Feynman theory.

A different approach was successfully implemented by Wightman in 1956 for free fields. Wightman found that to give sense to the space-time derivatives of the free field, and also to field polynomials and their derivatives, it is enough to smear the field with an infinitely smooth function of Schwartz class $\mathcal{S}(\mathbb{R}^4)$ in space-time.³ In particular, Wightman showed that the smeared field

$$\phi_l^{(k)}(f) = \int d^4x \phi_l^{(k)}(x) f(x) , \quad (1.5)$$

with $f(x)$ a test function and $\{\phi_l^{(k)}(f)\}$ linear operators in a Hilbert space \mathcal{H} , is a well-defined operator on the Fock space. The main problem with the Wightman axioms, is that all known four-dimensional theories satisfying Wightman's axioms have a trivial scattering matrix. Nevertheless, non-trivial theories satisfying the Wightman axioms exist in lower dimension.

1.2.1 Wightman's axioms

It is now necessary to introduce a set of axioms to work with our QFT, where the fields are the smeared ones in (1.5).

W1 (Relative invariance of the space of states). *It exists a Hilbert space \mathcal{H} that carries a continuous unitary representation $\mathcal{U}(\Lambda, \mathbf{a})$ of the Poincaré spinorial group (universal covering group of the Poincaré proper group).*

W2 (Spectral properties). *The spectrum of p^μ is concentrated exclusively in the superior closed cone*

$$\bar{V}^+ := \{p \in \mathbb{M} \mid p^2 \geq 0, p^0 \geq 0\} \quad m = 0 \quad \text{included} .$$

W3 (Existence and uniqueness of the vacuum). *$\exists!$ a vacuum state $|0\rangle$ (up to a phase $e^{i\alpha}$) for \mathcal{H} that is invariant under $\mathcal{U}(\Lambda, \mathbf{a})$.*

With these three axioms Wightman noticed that for the quantised field ϕ , $\phi(f)$ is unbounded. For an unbounded operator it is necessary to define a domain \mathcal{D}

W4 (Fields' domain of definition). *The components $\phi_l^{(k)}$ of the field $\phi^{(k)}$ are operators with distributional values on the Schwartz's space $\mathcal{S}(\mathbb{M})$, with domains of definition \mathcal{D}*

³ The space $\mathcal{S} := \mathcal{S}(\mathbb{R}^4)$ consists of infinitely differentiable real functions of real variables that goes to zero at infinity faster than any power of the Euclidean distance. For an introduction to distributions see, for example, [12].

common for all the operators and dense in \mathcal{H} . The vacuum lies in \mathcal{D} and $\mathcal{D} \rightarrow \mathcal{D}$ under $\phi_l^{(k)}$ and $\mathcal{U}(\Lambda, \mathbf{a})$.

W5 (Poincaré covariance). The fields transform under $\mathcal{U}(\Lambda, \mathbf{a})$ according to the law

$$\mathcal{U}(\Lambda, \mathbf{a})\phi_l^{(k)}(x)\mathcal{U}^{-1}(\Lambda, \mathbf{a}) = \sum_{l,m} V_{l,m}^{(k)}(\Lambda^{-1})\phi_m^{(k)}(\Lambda x + a) ,$$

with $V_{l,m}^{(k)}(\Lambda^{-1})$ finite representation of $SL(2, \mathbb{C})$.⁴

W6 (Locality and microcausality). Two fields $\phi_l^{(k)}(x)$ and $\phi_m^{(k')}(y)$ commute or anti-commute when there is a space-like separation between two points x, y of \mathbb{M} , i.e.

$$\left[\phi_l^{(k)}(x), \phi_m^{(k')}(y) \right]_{\mp} = 0 \quad \text{for} \quad (x - y)^2 < 0 .$$

W7 (Cyclicity of the vacuum). The set of finite linear combinations of vector of the form

$$\phi_{l_1}^{(k_1)}(f_1) \dots \phi_{l_n}^{(k_n)}(f_n)|0\rangle , \quad n = 1, 2, \dots ,$$

is dense in \mathcal{H} . A vector with this property is called cyclic, so the vacuum is cyclic.

It should be stressed that the axiom **W6** is hard to satisfy. In particular, all known examples are derived from free fields and, if one proceeds in the usual way by looking at the vacuum representation, then get a trivial scattering-matrix.

1.2.2 Wightman's distributions

Finding the fields ϕ that satisfy Wightman's axioms is very difficult, for this reason it is useful to introduce the Wightman *distributions* \mathcal{W}_n . Through these objects, the QFT problem is reduced to finding a set of distributions \mathcal{W}_n satisfying certain properties.

First, we must define what is a Wightman distribution. Consider the vacuum Ψ_0 of a Wightman field ϕ and test functions f_1, \dots, f_n and the multifunctional

$$\langle \Psi_0 | \phi(f_1)\phi(f_2) \dots \phi(f_n) | \Psi_0 \rangle , \tag{1.6}$$

which is a map from the n test functions into complex numbers. In addition, this

⁴ $V_{l,m}^{(k)}(\mathbb{I}_4) = \pm 1$, +1 if $\phi^{(k)}$ is a tensorial field while -1 if $\phi^{(k)}$ is a spinorial field.

mapping is continuous because of the assumption that the field is a distribution, and this is still true for each f_i keeping all the others fixed. Using Schwartz's nuclear theorem it is possible to prove that there is a unique distribution in $4n$ variables, denoted by $\mathcal{W}_n(f)$ and called Wightman's distribution, defined for all test functions $f(x_1, \dots, x_n)$, that coincides with (1.6) when $f(x_1, x_2, \dots, x_n) = f_1(x_1) \cdots f_n(x_n)$. Therefore,

$$\mathcal{W}_n(f_1 \otimes \cdots \otimes f_n) = \langle \Psi_0 | \phi(f_1) \cdots \phi(f_n) | \Psi_0 \rangle .$$

If the field is assumed to be a tempered distribution, i.e. is a continuous linear map $\mathcal{S} \rightarrow \mathbb{C}$, then also \mathcal{W}_n is tempered.

Starting from the axioms **W1-6** it is possible to find the corresponding properties for \mathcal{W}_n . These are quite easy to find and are formalised in a set of theorems not showed here explicitly. A key consequence is that two fields are physically the same if they have the same Wightman distributions, because \mathcal{W}_n determine the field up to unitary transformations. This means that, giving a set of \mathcal{W}_n obeying some properties, then there exists a separable Hilbert space⁵ on which acts a Wightman field ϕ that obeys the axioms **W1-6**.

In conclusion, the problem is no longer to directly find the field ϕ but the \mathcal{W}_n obeying some specific properties.

1.2.3 Reconstruction theorem

Suppose that we were able to find the \mathcal{W}_n introduced before: how are they linked to QFT and to the quantities of interest? Answering this question is the aim of the reconstruction theorem. An approach is to reconstruct the fields directly from \mathcal{W}_n , another one is to perform an analytic continuation to find the so-called Schwinger's function $\mathcal{S}(x_1, \dots, x_n)$, defined in the Euclidean space.

Reconstructing fields directly from a given set of \mathcal{W}_n is like reconstructing the representation of a \mathcal{C}^* -algebra from a state, with the remarkable difference that Wightman's operators are generally unbounded. In addition, Borchers proved that Wightman fields $\phi(f)$ generate a $*$ -algebra over the complex numbers called \mathcal{A} . An element $A \in \mathcal{A}$ and Ψ in D , domain of $\phi(f)$, defines the expectation functional on \mathcal{A}

$$A \rightarrow \langle \Psi | A | \Psi \rangle .$$

This map is linear and positive and has the properties of the state of the algebra, in particular the vacuum expectation values \mathcal{W}_n define a state on \mathcal{A} . It is now possible to define by these elements the Hilbert space \mathcal{H} , the domain D and the field operator $\phi(f)$, i.e. all the elements necessary to our QFT.

⁵ A Hilbert space is separable if contains a countable dense subset, i.e. \exists a sequence $\{x_n\}_{n=1}^{\infty}$ of \mathcal{H} such that every nonempty open subset of \mathcal{H} contains at least one element of the sequence.

The Wightman distributions can be continued analytically to the Euclidean space

$$\{-ix^0, \mathbf{x}\}, \quad x^0 \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^3.$$

The Schwinger's functions are

$$\mathcal{S}_n(\dots, \mathbf{x}_k, x_k^0, \dots) := \mathcal{W}_n(\dots, \mathbf{x}_k, -ix_k^0, \dots),$$

with

$$x_{k+1}^0 - x_k^0 > 0.$$

The properties of Schwinger's function have been studied axiomatically by Osterwalder and Schrader. The axioms of the Osterwalder and Schrader formulation concern the following properties of the Schwinger functions

E1 invariance under Euclidean transformations

$$\mathcal{S}(x_1, x_2, \dots, x_n) = \mathcal{S}(\Lambda x_1 + a, \dots, \Lambda x_n + a), \quad \Lambda \in SO(4),$$

E2 satisfy the so-called reflection positivity property, related to the time reversal in Minkowski space (see Glimm and Jaffe book [4]),

E3 are completely symmetric in their arguments,

E4 satisfy the cluster property. This is related to an asymptotic factorisation of $\mathcal{S}_n(\dots, \mathbf{x}_k, x_k^0, \dots)$ (see Glimm and Jaffe book [4]).

By the Schwinger's functions it is possible to reconstruct the Wightman's functions \mathcal{W}_n and then the corresponding QFT. The advantage of working with Schwinger's functions is that they are defined in Euclidean space, so that they obey to simpler properties and are easier to manipulate than the Wightman functions or field operators.

1.2.4 Applications, successes and fails

The axiomatic approach is successful in describing free fields and is the framework in which most of the properties of the QFT are rigorously proven. In particular, this is done for the PCT theorem, proving invariance under parity transformation, charge conjugation and time reversal of a Wightman theory, and the spin-statistics theorem, proving the connection between the spin of the particle and the statistics it satisfies.

On the other hand, the axiomatic approach has the problem that only a small number of concrete derivations are known. In addition, Wightman theory deals with unbounded

operators. This can be resolved using the algebraic approach that treats only limited operators, but this causes a loss of generality in the theory.

1.3 Perturbative approach

While free theories are surely easier to study, the interacting ones are the most interesting and necessary to describe Nature. Unfortunately, no exactly solvable interacting theory is known in more than two space-time dimensions⁶. An alternative approach is to use a perturbative approach. This approach was derived independently by Tomonaga, Schwinger and Feynman by removing the special role of time in QM, and then applying this viewpoint to recast each term of perturbation expansion as a space-time process. We will now draw a sketch of how this is possible through the method of functional integration and the result one may obtain.

1.3.1 Path integral formulation

Let us consider the probability amplitude of finding a particle at (x, t) knowing that it was at (x_0, t_0)

$$\langle \mathbf{x}, t | \mathbf{x}_0, t_0 \rangle .$$

This is given by the sum of the amplitudes of all possible paths, each path weighted by its quantum mechanical amplitude. This sum over paths is the *path integral* and can be expressed in the form

$$\langle \mathbf{x}, t | \mathbf{x}_0, t_0 \rangle = \int \mathcal{D}x(t) \exp \left(i \int_{t_0}^t dt \mathcal{L}[\dot{x}, x, t] \right) = \int \mathcal{D}x(t) \exp (iS[x(t)]) , \quad (1.7)$$

where \mathcal{L} is the classical Lagrangian, S is the classical action and $\mathcal{D}x(t)$ denotes the functional integration over all possible paths. By means of the amplitudes $\langle \mathbf{x}, t | \mathbf{x}_0, t_0 \rangle$ it is possible to calculate all the quantities of physical interest (observables). However, also in the case of QM path integral one may get analytic solutions for few systems only, e.g. the free particle, harmonic and forced oscillator. In the general case the only way to compute $\langle \mathbf{x}, t | \mathbf{x}_0, t_0 \rangle$ is to use the perturbative approach. We have seen that the path integral is useful in QM, but how can we derive its QFT version?

In the path integral representation of QM, one integrates over the phase space $(\mathbf{x}_i, \mathbf{p}_i)$. In a second quantised system, the field $\phi(\mathbf{x})$ is an operator, so we should expect that in QFT the path integral is constructed integrating in a phase space of functions $(\phi(\mathbf{x}), \pi(\mathbf{x}))$, $\pi(\mathbf{x})$ being the appropriate momentum. Defining $\langle \phi, t | \phi_0, t_0 \rangle$ as the prob-

⁶ For example, the Ising model is exactly solved in two dimensions but not in three.

ability amplitude for a field in the configuration $\phi_0(\mathbf{x})$ at t_0 to evolve to $\phi(\mathbf{x})$ at t , after some mathematical preliminaries we find⁷

$$\langle \phi, t | \phi_0, t_0 \rangle = \int \mathcal{D}\phi \exp \left(i \int_{t_0}^t dt \int d^{D-1}x \mathcal{L}[\dot{\phi}, \phi, t] \right) = \int \mathcal{D}\phi \exp (iS[\phi(t)]) , \quad (1.8)$$

where \mathcal{L} is the classical Lagrangian density and S is the classical action functional. To probe the dynamics, one may add an arbitrary external source J for ϕ . In this way one gets a path integral representation of the generating functional of the vacuum expectation values of time-ordered products of the ϕ 's. Such expectation values, also called Green's functions, can be evaluated through a perturbative expansion. As we will see, a key result, known as Lehmann, Symanzik and Zimmerman reduction formula, shows that the GF are the building blocks to obtain the S -matrix and therefore the cross sections.

Let us consider $Z[J] := \langle \Omega | \Omega \rangle_J$, denoting the vacuum to vacuum amplitude in the presence of the external source J . It turns out that

$$Z[J] = N \int \mathcal{D}\phi \exp \left[i(S + \int d^D x J\phi) \right] , \quad (1.9)$$

where N is a constant usually ill-defined.

Set

$$Z[J] = \exp(iW[J]) .$$

It turns out that $W[J]$ is the generating functional of connected GF

$$\begin{aligned} G_c^{(N)}(x_1, \dots, x_n) &= \frac{1}{i^{N-1}} \frac{\delta^N W[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0} \\ &= \langle \Omega | T\phi(x_1) \dots \phi(x_N) | \Omega \rangle_c . \end{aligned}$$

A problem with (1.9) is that the integrand is an oscillatory one, so that the path integral is not well-defined. A possibility is to define Z in the Euclidean space (calling it Z_E) and then computing GF in the Euclidean space. After this, we recover the GF in the Minkowski space by analytic continuation.

Before proceeding, it is worth stressing the basic fact that in calculating the S -matrix the relevant quantity is the product of the residues of the GF involved in the process. Since such a product is invariant under diffeomorphisms of the fields, it follows that the scattering matrix S is invariant under such transformations.

⁷ This is the case if the Hamiltonian is quadratic in the momentum $\pi(\mathbf{x})$.

1.3.2 Renormalisation

An important aspect of QFT is that they have a useful representation in momentum space, in which Feynman diagrams become a systematic powerful tool to compute the cross sections of a QFT. In this beautiful procedure there is anyway a problem, because Feynman amplitudes are quite often *divergent* quantities and therefore the GF of our QFT. For such a reason, it is necessary to build a procedure, called renormalisation, whose role is to remove such divergences maintaining the structure of the theory. This can appear quite magic, but it works!

Renormalisation, that involves the redefinition of fields and coupling constants, works fine only for certain theories, called renormalisable. We will not sketch renormalisation here, but we just note that it is possible to build several renormalisation procedures. In particular, a first step in renormalising a theory is to introduce a regularisation of the relevant integrals so that they are finite, e.g. by dimensional regularisation or by introducing a cut-off in the domain of integration.

1.3.3 An example: ϕ_4^4 theory

A simple example of theory with physical relevance is the so-called ϕ_4^4 theory.⁸ This theory is not solved exactly but one can evaluate the GF perturbatively. Let us start by considering the generating functional in Euclidean space

$$Z_E[J] = e^{-W_E[J]} = N \int \mathcal{D}\phi \exp \left[- \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 - J\phi \right) \right],$$

and then computes the Euclidean GF

$$G_E^{(N)}(x_1, \dots, x_N) = \frac{\delta^N Z_E[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0}.$$

Expanding the interaction term in power of λ one gets the perturbative series. In doing this it is necessary to build a renormalisation scheme to obtain finite GF in the Euclidean space and therefore at the end one expresses GF in Minkowski space by analytic continuation and then compute the cross sections.

1.3.4 Applications, successes and fails

The perturbative approach is largely used in QFT, because through it we can make a great number of physical predictions, e.g. in QED and other successful theories. In particular, in QED such an approach leads to predictions of incredible accuracy like the so-called “ α running” that explains how the fine structure constant α depends on the

⁸ The ϕ_d^n theory denotes the d -dimensional scalar field theory with potential density ϕ^n .

energy scale.

The main problem of this approach is that it is an approximation. In principle, it is possible to be as precise as one wants simply by computing the perturbation series to higher order of the coupling constant, i.e. λ or α , but a priori this procedure is not sure as it may look. In building the renormalisation scheme one must redefine the coupling constant using the so-called counterterms. An example of this is given by QED: the series in the coupling constant $\alpha \propto e^2$

$$S(e^2) = a_0 + a_1 e^2 + a_2 e^4 + \dots ,$$

is not convergent, because if we assume a finite radius of convergence, $S(e^2)$ must be analytic at $e^2 = 0$. This means that $S(-e^2)$, i.e. $e \rightarrow ie$, is analytic. A theory with imaginary charge possesses an instable vacuum, leading to a production of space separated electron positron pairs, i.e. electrons attract each other. Since this is a contradiction of known physics, the only possibility is that the series above is not convergent.

The fact that the perturbative expansion is mathematically ill-defined is mainly do to the interchange of the summation of the series expansion with the functional integral. Such an interchange is admitted only when, according to Levi's theorem, the conditions of monotone convergence are fulfilled. Actually, according to Levi's theorem, if $(f_k(x))_{k=1}^{\infty}$ is a sequence of Lebesgue integrable functions, almost everywhere non-negative in a set A , and such that $\sum_{k=1}^{\infty} \int_A dx f_k(x)$ converges, then $\sum_{k=1}^{\infty} f_k(x)$ converges to a Lebesgue integrable function f almost everywhere on A and

$$\sum_{k=1}^{\infty} \int_A dx f_k(x) = \int_A dx \sum_{k=1}^{\infty} f_k(x) .$$

This does not in general hold in QFT expansions.

QED is also not Borel summable, and for this reason some physicist think it is not a consistent theory, due also to the existence of the Landau pole. In fact, QED fails at very high energy while it gives extremely good predictions at low energy. This could not be an essential problem, because at the scale of energy where QED loses because of Landau pole is greater than the Planck energy ($\sim 10^{19}$ GeV), so we do not even know if our description of reality with electrons and positrons is still valid at such energy scale.

In conclusion, the perturbative approach to QFT provides, in some range of energies, excellent numerical predictions, but one sometimes sacrifices mathematical rigorousness in the construction of the theory.

1.3.5 The origin of the divergences in quantum field perturbation theory

We already mentioned that divergences are a consequence of the fact that the Poisson brackets for fields contain a Dirac's δ , implying that ϕ is a distribution. The trouble is that non-linear terms, such as ϕ^n , correspond to power of distributions at the same point, which are not defined. This means that some field theory presents singularities, even at the classical level. Due to loop integrations, the problem is much harder in QFT.⁹

The problem already arises when one replaces test functions by distributions. To see this, we consider the Wightman 2-point distribution in the case of the free particle

$$\langle 0|\phi(f)\phi(g)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \tilde{f}^*(p)(2\pi)\theta(p^0)\delta(p^2 - m^2)\tilde{g}(p) , \quad (1.10)$$

where $\tilde{f}(p)$ and $\tilde{g}(p)$ are the Fourier transform of the test functions $f(x)$ and $g(x)$, respectively. In the standard formulation, the test functions are replaced by the δ -distribution. In other words, one makes the identification

$$f(z) = \delta^{(4)}(z - x) , \quad g(z) = \delta^{(4)}(z - y) ,$$

corresponding to

$$\tilde{f}(p) = e^{ipx} , \quad \tilde{g}(p) = e^{ipy} ,$$

so that (1.10) becomes

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{d^4p}{(2\pi)^3} \theta(p^0)\delta(p^2 - m^2)e^{-ip(x-y)} \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{-ip(x-y)} , \end{aligned}$$

$\omega_{\mathbf{p}} := \sqrt{\mathbf{p}^2 + m^2}$, which has a pole at $y = x$. We then see that a non-singular quantum field cannot be a well-defined operator for any sharp spacetime point.

The problem is even harder when the theory, such as ϕ_4^4 , has a self-interaction at any time. As a matter of fact perturbation theory erroneously treats the quantum fields evolving as the free ones between point-like interaction events. From the physical point of view, the role of renormalisation is to iteratively change the parameters of the theory, that then will depend on the physical scale. In other words, perturbation theory is a way to mimic the interacting theory by a free one, with the parameters becoming scale

⁹ As we will see, the classical theory of ϕ_4^4 , which is the tree-level contribution of its quantised version, involves products of the Feynman propagator $\Delta_F(y - x)$. This is in fact divergent at $y = x$.

dependent.

1.3.6 Resurgence

A recent development of interest is the phenomenon of resurgence in QM,¹⁰ which could be of great interest if extended to QFT. This approach shows that in one dimensional QM systems with a bound-state potential V that admits an alternative potential $\hat{V}(x; \lambda, \lambda_0)$ any observable can be exactly computed by a single perturbative series. This is the so-called exact perturbation theory (EPT) and $\hat{V}(x; \lambda, \lambda_0)$ admits always a Borel resumable perturbation theory in λ and coincides with V when $\lambda = \lambda_0$. This approach is an alternative way for computing the instanton contributions due to deformation of the contour of integration of the path integral if one wants to restore Borel summability.

The resurgence phenomenon has been studied for QM, but a possible extension to QED could be of great interest, because EPT works well at strong coupling constants where QFT in the perturbative approach is not always well-defined. In addition, it is in principle possible to extend the results obtained for QM to non-Borel resumable QFT, like gauge theories in 4 dimensions as QED.

1.4 QFT on a lattice

QFT can be formulated on a lattice instead that on a continuous space-time. Such an approximation allows the application of analytical and numerical techniques that are very useful for studying quarks and gluons in strong interactions. In defining a lattice QFT it is extremely important to have a well-defined continuum limit, i.e. as the lattice parameter a goes to zero the continuous QFT must be restored.

By now, we will refer to QFT on a lattice as QFTL for simplicity.

Even in QFTL we are interested in defining a path integral because we know well how to compute cross sections by those integrals. However, we are not anymore in a continuum space-time, but on an ipercubical lattice

$$\Lambda = a\mathbb{Z}^4 = \left\{ x \left| \frac{x_\mu}{a} \in \mathbb{Z} \right. \right\}, \quad (1.11)$$

on which is defined the scalar field $\phi(x)$. Even on the lattice it is possible to define a derivative, but one must distinguish between forward and backward derivatives. Defining the scalar product in analogy with the continuum case

$$(g, h) = \sum_x a^4 g(x)h(x),$$

¹⁰ See, for example, [13][14][15].

the forward and backward derivatives are defined as

$$\begin{aligned}\Delta_\mu^f g(x) &= \frac{1}{a} (g(x + a\hat{\mu}) - g(x)) , && \text{forward} , \\ \Delta_\mu^b g(x) &= \frac{1}{a} (g(x) - g(x - a\hat{\mu})) , && \text{backward} ,\end{aligned}$$

with $\hat{\mu}$ the unit vector and $(\Delta_\mu^f g, h) = -(g, \Delta_\mu^b h)$. This leads to the following definition of the lattice d'Alembertian operator

$$\square = -\Delta_\mu^b \Delta_\mu^f .$$

It is then possible to define the lattice action in the general case as

$$S[\phi, a] = S_0[\phi, a] + S_I[\phi, a] = \frac{1}{2} (\phi, (\square + m^2)\phi) + S_I[\phi, a] , \quad (1.12)$$

which enters in the generating functional of GF

$$Z[J, a] = \frac{1}{Z[0, a]} \int \prod_x d\phi(x) \exp(-S[\phi, a] + (J, \phi)) . \quad (1.13)$$

In the free case, i.e. $S[\phi, a] = S_0[\phi, a]$, it is easy to show that (1.13) restores correctly the limit in the continuum when $a \rightarrow 0$. The problem is to find this limit in the interacting case to obtain well-defined GF.

It is worth stressing that in defining the two-point correlation function one may use the so-called *transfer matrix* \mathbf{T} that plays the role of an evolution operator. \mathbf{T} is a bounded, symmetric and positive operator, which are essential properties for having a self-adjoint Hamiltonian. If it is not possible to have an explicit representation of the transfer matrix, then one must have time reflection positivity on the lattice.¹¹ If this is the case, the Hamiltonian can be defined, and a Hilbert space formalism exists.

1.4.1 Renormalisation in the continuum limit and renormalisation group

The lattice regularisation provides a cut-off even for the momenta. For this reason, loop integration in QFTL are finite¹² and no renormalisation is needed. However, in the continuum limit one must send lattice spacing to zero and therefore there is not cut-off in the range of the momentum and renormalisation is again needed. Renormalisation introduces renormalised fields, coupling etc. that are treated to blow away

¹¹ There are two possible types of reflection positivity: site-reflection positivity and link-reflection positivity.

¹² With lattice spacing the momentum lies in the first Brillouin zone, so it is different from both zero and infinity.

divergences. At this point, one may see an analogy between QFTL and statistical mechanics and some concepts of the latter, such as the one of transfer matrix, can be applied successfully to QFTL. One can also use the correlation length ξ , which governs the exponential decay of the correlation functions, and therefore to the propagator, and has the behaviour

$$\xi = \frac{1}{ma} .$$

In taking the continuum limit, with a suitable choice of the renormalisation parameter a that goes to zero while m stays finite and ξ diverges. This is related to the existence of the so-called critical point, that is a point in the phase space corresponding to a phase transition.

A useful method to study the theory is the *renormalisation group*. The main point is to move from infinite-dimensional space of actions to the finite-dimensional subspace parameterised by those quantities like coupling constant, mass etc. renormalised in our theory. The idea is to see how renormalised quantities change when there is a changing in the lattice parameter, especially when the continuum limit is driven. For doing this it is very important to study the fixed points in the subspace defined above.

1.4.2 An example: ϕ_4^4 theory

Even in QFTL one can investigate ϕ_4^4 theory and expects to find the same results derived before. What is important to notice is that the cut-off provided by the space-time lattice used in QFTL is not particularly convenient for perturbative calculations. The main purpose of the lattice is to provide a regularisation which allows the application of various non-perturbative methods. However, sometimes it is necessary to perform perturbative calculations with a lattice cut-off, in particular if quantities calculated by non-perturbative methods are related to quantities calculated perturbatively. Furthermore, some quantities of numerical interest, such as finite volume effects, can be calculated in lattice perturbation theory.

Even in QFTL one can find the Feynman rules to evaluate GF, but there are some differences with the continuum case. Especially, in performing loop integrations only the momenta in the first Brillouin zone are involved. In this way, one may compute GF on the lattice for the perturbative expansion of ϕ_4^4 theory, but such quantities diverge in taking the continuum limit. When this happens, one applies a renormalisation scheme and removes the divergences.

1.4.3 Applications, successes and fails

An interesting application of QFTL is the one with QCD. As the reader probably knows, QCD has the property of being an asymptotically free theory, i.e. the coupling constant increases with the distance. For this reason, the description of the long distance strong

colour force requires a non-perturbative approach, and this can be done in QFTL. In particular, lattice QCD gives a prediction on the mass of the quarks.

Another interesting application is the construction of simulation algorithms: as easily understandable it is not possible to calculate continuum quantities as a field numerically. The only possibility is to discretise space-time, and this means that we must build a QFTL for numerical applications.

Finally, QFTL is used in solid state physics and condensed matter physics, where it is not rare to work with systems with a particular symmetry or with a lattice.

As pointed before, QFTL is essentially a non-perturbative approach that works thanks to the discretisation of space-time. Anyway, even in this case most of the theory needs, for explicit calculations, to use perturbative techniques.

1.5 Schrödinger representation formalism

In QFT is worth of mention the Schrödinger representation, a natural extension of non-relativistic QM used for atomic physics.

The idea is to proceed analogously to what we did in QM but using a mathematics consistent with the fact that we are working with fields, so we will expect to work with functional differential equations instead of differential equations as in QM.

Let us consider the case of the free scalar field theory with action (1.3). One can construct the conjugate field momentum π and the Hamiltonian H as

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(x))} = \dot{\phi}(x) ,$$

$$H = \frac{1}{2} \int d^3x (\pi^2 + |\nabla \phi|^2 + m^2 \phi^2) .$$

As in QM we defined the CCR for position and momentum, here we do the same for the field operator ϕ and its conjugated π

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) , \quad (1.14)$$

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0 . \quad (1.15)$$

It is now possible to switch to a coordinate Schrödinger representation and work with a basis for the Fock space where the field operator ϕ is diagonal. If $|\phi\rangle$ is an eigenstate of ϕ with eigenvalue ϕ then the coordinate representation of the state $|\Psi\rangle$ is the wave functional

$$\Psi[\phi] = \langle \phi | \Psi \rangle .$$

It is also possible to give a functional differential representation of the equal time

commutator (1.14) using

$$\pi(\mathbf{x}) = -i \frac{\delta}{\delta\phi(\mathbf{x})} ,$$

so that

$$\left[\frac{\delta}{\delta\phi(\mathbf{x})}, \phi(\mathbf{y}) \right] = \delta^{(3)}(\mathbf{x} - \mathbf{y}) .$$

The differential representation of the momentum field operator turns the Hamiltonian operator in a functional differential operator

$$H_0 = \frac{1}{2} \int d^3x \left(-\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + |\nabla\phi|^2 + m^2\phi^2 \right) ,$$

and the Schrödinger equation in a differential functional equation

$$i \frac{\partial}{\partial t} \Psi[\phi, t] = \frac{1}{2} \int d^3x \left(-\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + |\nabla\phi|^2 + m^2\phi^2 \right) \Psi[\phi, t] . \quad (1.16)$$

Even if we are dealing with the simplest case of free field theory, this equation can be solved easily only for the ground state because we can use the property that the wave functional of the ground state is always positive and has no nodes. Although, through a little bit of calculation it is possible to find Ψ even for excited states.

1.5.1 Results

Resolving (1.16) when is time independent is possible to find the energy of the ground state E_0 and also the energy of the excited state E_i .

It is also possible to show that the energy eigenstate $\Psi_1[\phi]$ with energy ω_{k_1} is also a momentum eigenstate with momentum \mathbf{k}_1 . This can be used to describe a state with one particle with four-momentum k_1 and mass m . This leads to the Schrödinger representation of creation and destruction operators, respectively a^\dagger and a

$$\begin{aligned} a(\mathbf{k}) &= \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} \left(\omega_k \phi(\mathbf{x}) + \frac{\delta}{\delta\phi(\mathbf{x})} \right) , \\ a^\dagger(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \left(\omega_k \phi(\mathbf{x}) - \frac{\delta}{\delta\phi(\mathbf{x})} \right) . \end{aligned}$$

It is also possible to compute the propagator. If the initial state is a particle located at \mathbf{x} at time t and the final state is the one with the particle located at \mathbf{x}' at t' , then the initial wave functional is $\phi(\mathbf{x})\Psi_0[\phi, t]$ and the final one is $\phi(\mathbf{x}')\Psi_0[\phi, t']$. One may check that the propagator is

$$\langle 0 | \phi(x') \phi(x) | 0 \rangle \theta(t' - t) = \int \mathcal{D}\phi \phi(\mathbf{x}') \phi(\mathbf{x}) \Psi_0^*[\phi, t'] \Psi_0[\phi, t] ,$$

where $|0\rangle$ denotes the vacuum state and the theta function is necessary because $t' > t$. It turns out that¹³

$$\langle 0|\phi(x')\phi(x)|0\rangle\theta(t' - t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} e^{ik(x-x')}\theta(t' - t) ,$$

where $\omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}$.

1.5.2 Interacting fields

One of the main goals of interacting QFT is to compute the cross section for scattering processes. In the perturbative operator formalism this means computing the S -matrix elements in terms of initial and final states and field operators and we have seen that such quantities are related to GF computed in perturbation theory.

In the Schrödinger representation the dynamics instead resides in the states, not in the operators, so we do not compute GF. Since S -matrix elements are defined as an overlap between initial and final states, what one needs to compute is the initial and final interacting states. Such states are computed perturbatively.

It should be mentioned that the formalism extends to photon and spinor fields. It is also possible to give a Feynman diagram interpretation of this procedure, but the most important thing is to underline that this way of computing the S -matrix is completely equivalent, and gives the same results, as the other formulations. Let us consider the case of the ϕ_4^4 theory.

1.5.3 An example: ϕ_4^4 theory

The Hamiltonian of the ϕ_4^4 theory reads

$$H = H_0 + H_{int} = H_0 + \int d^3x \left(\frac{1}{2}\delta m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \right) ,$$

where $\delta m^2\phi^2/2$ is the mass correction term to the free one, $m^2\phi^2/2$, in H_0 . It is possible to obtain the vacuum state and the energy spectrum of such an interacting theory by using the Rayleigh-Schrödinger perturbation theory. This is developed by first writing $H = H_0 + \alpha H_{int}$, where the dimensionless parameter α ranges between 0 and 1, and then performing a series expansion both in the wave functional Ψ and in the energy

¹³ See, for example, page 208 of B. Hatfield, “*Quantum Field Theory of Point Particles and Strings*”, Perseus Books, 1992.

eigenvalues E

$$\begin{aligned}\Psi_N[\phi] &= \Psi_N^{(0)}[\phi] + \alpha\Psi_N^{(1)}[\phi] + \alpha^2\Psi_N^{(2)}[\phi] + \dots, \\ E_N &= E_N^{(0)} + \alpha E_N^{(1)} + \alpha^2 E_N^{(2)} + \dots,\end{aligned}$$

and then placing this expansion in $H\Psi_N[\phi] = E_N\Psi_N[\phi]$.

In analogy with $Z[J]$ it is possible to define a functional $\mathcal{G}[J]$ which is the generator of the momenta of $\Psi_0^{*(0)}\Psi_0^{(0)}$ (where $H_0\Psi_0^{(0)} = E_0^{(0)}\Psi_0^{(0)}$), namely

$$\langle \Psi_0^{(0)} | \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_n) | \Psi_0^{(0)} \rangle = \frac{\delta^n \mathcal{G}[J]}{\delta J(\mathbf{x}_1) \dots \delta J(\mathbf{x}_n)} \Big|_{J=0}. \quad (1.17)$$

By means of $\mathcal{G}[J]$ it is possible to compute, order-by-order, the energy excitations and the corrections to the wave functional.

1.5.4 Applications, successes and fails

The Schrödinger representation approach is, even now, less favorite than others approaches to QFT. This fact has some historical reasons, but it is also due to the fact that the Schrödinger representation is not explicitly Lorentz invariant and its renormalisability was proven only in 1980 by Symanzik. Since Lorentz invariance and renormalisability play a central role in QFT, the Schrödinger representation approach was initially less considered.

However, this approach is very versatile, the reason is that it is focused on the time evolution of the state of the system. The problems with such a formulation are essentially the same of the others perturbative approaches, i.e. it is very difficult to find analytical solutions for interacting systems and renormalisation is needed.

1.6 Spontaneous symmetry breaking

The concept of symmetry, and therefore of symmetry breaking, is of central importance in physics. It is necessary to make a distinction between a theory with a finite number of degrees of freedom, that we call QM_{fin} , and a theory with infinite number of degrees of freedom, QM_∞ .

For the first type of theory is valid the following theorem by von Neumann

von Neumann unicity theorem. *An algebraic symmetry $(q, p) \rightarrow (q', p')$ with $[q'_i, p'_j] =$*

$i\hbar\delta_{ij}$ in \mathbb{R}^n , with n finite, is inducted by a unitary operator U

$$q'_i = Uq_iU^\dagger, \quad p'_i = Up_iU^\dagger.$$

This implies that in QM_{fin} every symmetry in the equation of motion is an exact symmetry. For this reason, it is not possible to have SSB in QM_{fin} .

The scenario changes when we deal with QM_∞ , that is essentially QFT. With this type of theory the von Neumann unicity theorem is no longer valid, therefore there are inequivalent representations of the CCR, i.e. not connected by unitary or antiunitary operators. It follows that a symmetry of the equations of motion does not necessarily imply an exact symmetry. So, there could be a correspondence between a symmetry in the equations of motion and a transformation law that does not preserve transition amplitudes, unlike QM_{fin} .

A key theorem in studying SSB is the one by Goldstone, stating that

Goldstone theorem. *Consider a generic continuous symmetry which is spontaneously broken, i.e. currents are conserved but the ground state is not invariant under the action of corresponding charges. Then new massless particles, called Goldstone bosons, appear: in particular, there is a Goldstone boson for every broken generator of the symmetry group.*

Goldstone theorem is fundamental for the classical description of the Higgs mechanism, whose effect is the prediction of Higgs boson.

In the Higgs mechanism, there is a spontaneously broken global symmetry within a theory (the electroweak theory) that has a local gauge invariance. It is important to cite the *Elitzur theorem*, that states

Elitzur theorem. *An Abelian gauge theory formulated on the lattice cannot be spontaneously broken.*

1.6.1 Physical examples

SSB is of basic importance in physics, because it plays a central role in a variety of physical processes. Here we give a sketch of some of the main processes where SSB is involved:

- ▷ Ferromagnetism.
- ▷ Superfluidity.
- ▷ Superconductivity.

- ▷ Higgs mechanism.
- ▷ Convection cells in fluids.

Chapter 2

Lie Groups, Algebras and Representations

2.1 Lie groups¹

In this section we give a brief overview on the basic notions of Lie groups and Lie algebra, topics that will be used in the following.

Definition 1. Given X a topological space,² $U \subset X$ an open subset and

$$\phi : U \xrightarrow{\sim} \phi(U) = V \subset \mathbb{R}^n ,$$

with ϕ an homeomorfism. The pair (U, ϕ) is called *chart*.

Consider the charts (U_i, ϕ_i) and (U_j, ϕ_j) with $U_i \cap U_j \neq \emptyset$. We define the *transition map*

$$\eta_{ij} := \phi_i \circ \phi_j^{-1} ,$$

satisfying

$$\eta_{ii} = \text{id} , \quad \eta_{ji} = \eta_{ij}^{-1} ,$$

and, in $U_i \cap U_j \cap U_k$,

$$\eta_{ij} \circ \eta_{jk} = \eta_{ik} .$$

¹ Elia de Sabbata and Pietro Oreglia

² Recall that a topological space is a set with a structure of open subsets (i.e. a topology), usually considered separable.

A topological space X with a family of charts $\{(U_i, \phi_i)\}_{i \in I}$, called *atlas*, such that

$$X = \bigcup_{i \in I} U_i ,$$

is a *topological manifold*. If all the transition maps are C^∞ we have a *differentiable manifold*.

Let us give the definition of group.

Definition 2. A *group* is a set G with a multiplication law “ \circ ”, i.e. a map

$$\begin{aligned} \circ : G \times G &\longrightarrow G , \\ (a, b) &\longmapsto a \circ b , \end{aligned}$$

which respects the following properties

(i) it is associative, i.e. $\forall a, b, c \in G$

$$(a \circ b) \circ c = a \circ (b \circ c) ,$$

(ii) there is the identity, i.e. an element $e \in G$ such that

$$e \circ a = a \circ e = a ,$$

(iii) there exists an inverse for each element, i.e. $\forall a \in G, \exists a^{-1} \in G$ such that

$$a \circ a^{-1} = a^{-1} \circ a = e .$$

Definition 3 (Lie group). A *Lie group* is a differentiable manifold G with a group structure, i.e. there is a map

$$\circ : G \times G \longrightarrow G ,$$

which respects the group axioms.

The Lie groups are finite dimensional groups³ and can be

(i) compact, if the topological space is compact,

(ii) non-compact, otherwise,

which are important properties in the context of representation theory.

³ There are groups resembling Lie groups, except for being infinite-dimensional.

Another topic concerns the structure of a Lie group near the identity. In particular, we will consider the connected part containing the identity, that is the elements of the Lie group such which are described by only one coordinate system $\{a^1, \dots, a^n\}$ with $\dim G = n$. The generic element will be denoted by $g(a)$ and the product

$$g(a)g(b) = g(f(a, b)) ,$$

with $f(a, b)$ a set of n functions of the coordinates of the two elements. It is also convenient to assume that the origin of the coordinates is the identity, i.e $g(0) = e$, hence

$$f^k(a, 0) = a_k , \quad f^k(0, b) = b_k .$$

We denote by $i^k(a)$, $k = 1, \dots, n$ the functions linking an element to its inverse,

$$g^{-1}(a) = g(i(a)) .$$

From now on, we assume that G is a group of matrices, in such a way that one can introduce the concept of derivative with the usual definition

$$\frac{\partial g(a)}{\partial a} = \lim_{\delta a \rightarrow 0} \frac{g(a + \delta a) - g(a)}{\delta a} .$$

This cannot be done in general because there is no definition of the sum in our construction, however this is not restrictive for the cases of physical interest.

Let us notice an important identity: deriving with respect to a^j the identity

$$f^i(f(a, b), i(b)) = a^i ,$$

corresponding to $(a \circ b)b^{-1} = a$, we get

$$\frac{\partial f^i}{\partial p^k}(f(a, b), i(b)) \frac{\partial f^k}{\partial a^j}(a, b) = \delta_j^i , \quad (2.1)$$

with $\partial f^i / \partial p^k$ denoting the partial derivative with respect to the first argument of f^i . Such an identity implies that $\partial f^k(a, b) / \partial a^j$ is an invertible matrix. The partial derivative of $g(a)$ with respect to the coordinates a^i , evaluated at the origin, defines the *generators* of G

$$T_i = \left. \frac{\partial g(a)}{\partial a^i} \right|_{a=0} , \quad i = 1, \dots, n . \quad (2.2)$$

The T_i 's themselves determine the partial derivatives of $g(a)$ at each point. This follows by first deriving the identity

$$g(a) = g(a)g^{-1}(b)g(b) = g(f(a, i(b)))g(b) ,$$

with respect to a^i

$$\frac{\partial g(a)}{\partial a^i} = \left(\frac{\partial g(f(a, i(b)))}{\partial f^k} \frac{\partial f^k(a, i(b))}{\partial a^i} \right) g(b) ,$$

and then setting $b = a$, so that, by $f(a, i(a)) = 0$, we get

$$\frac{\partial g(a)}{\partial a^i} = (T_k A_i^k(a)) g(a) ,$$

with

$$A_i^k(a) := \left. \frac{\partial f^k(a, i(b))}{\partial a^i} \right|_{b=a} .$$

Note that the identity (2.1) implies that $A_i^k(a)$ is invertible.

Let us now consider a path $a(t)$ on the group, such that $a(0) = 0$. We have

$$\begin{aligned} \dot{g}(a(t)) &= \dot{a}^i(t) \frac{\partial g(a)}{\partial a^i} \\ &= \left(\dot{a}^i(t) T_k A_i^k(a(t)) \right) g(a(t)) . \end{aligned}$$

We ask then if there exist paths such that

$$\frac{\dot{g}(a(t))}{g(a(t))} = v^k T_k , \quad (2.3)$$

with v a constant vector. In this case, one would have

$$g(a(t)) = \exp(v^k T_k t) . \quad (2.4)$$

The condition (2.3) is equivalent to

$$A_i^k(a) \dot{a}^i(t) = v^k ,$$

or, being $A_i^k(a)$ invertible,

$$\dot{a}^i(t) = (A^{-1}(a))^i_k v^k . \quad (2.5)$$

It can be proved that there exists a neighborhood U of $v = 0$ in the space of vectors v , and a correspondent neighborhood V of $a = 0$ in the coordinates space, such that

- (i) if $v \in U$ the equation (2.5) can be solved for $t \in (0, 1)$, and it determines the value of the $a^k(t)$ for $t = 1$ as function of v . Therefore, if $v \in U$, then (2.4) describes an element of the group,
- (ii) if $a \in V$ the v^k can be considered as functions of a . Hence every element of the group can be expressed with $a \in V$ in the form given by (2.4).

This means that (2.4) expresses an element of the group G for every v . One can then show the following theorem.

Theorem 2.1.1. *The generators T_i of the group G are linearly independent.*

2.2 Lie algebras⁴

This last theorem allows us to see the generators T_i as a basis for a vector space, i.e. the *Lie algebra*.

Definition 4 (Algebra). An *algebra* \mathfrak{A} over the field F (usually \mathbb{R} or \mathbb{C}) is a vector space over F equipped with a bilinear product, i.e. a binary operation

$$\circ : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A} ,$$

such that

$$\begin{aligned} (x + y) \circ z &= x \circ z + y \circ z , & x \circ (y + z) &= x \circ y + x \circ z , \\ (ax) \circ (by) &= (ab)x \circ y , \end{aligned}$$

$\forall x, y, z \in \mathfrak{A}$ and $\forall a, b \in F$.

Definition 5 (Lie algebra). A *Lie algebra* \mathfrak{g} is an algebra whose bilinear operation is the so-called *Lie bracket*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} ,$$

such that

$$[x, x] = 0 , \quad \forall x \in \mathfrak{g} ,$$

and satisfying the *Jacobi identity*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 , \quad \forall x, y, z \in \mathfrak{g} .$$

As a consequence of bilinearity and of the first defining property of the Lie bracket we have

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x] ,$$

that is

$$[x, y] = -[y, x] ,$$

i.e. the Lie bracket is antisymmetric. An example of a Lie algebra is the set of $N \times N$

⁴ Elia de Sabbata and Pietro Oreglia

matrices, identifying

$$[v, w] = v \cdot w - w \cdot v .$$

To prove the generators of the Lie group defined in (2.2) are the vectors of a basis of a Lie algebra, let us start from the expression (given yet) of the product of two element in G

$$g(a)g(b) = g(f(a, b)) ,$$

and derive it with respect to a^k and b^k

$$\begin{aligned} \frac{\partial g(a)}{\partial a^k} g(b) &= \frac{\partial g(f(a, b))}{\partial f^m} \frac{\partial f^m(a, b)}{\partial a^k} , \\ g(a) \frac{\partial g(b)}{\partial b^k} &= \frac{\partial g(f(a, b))}{\partial f^m} \frac{\partial f^m(a, b)}{\partial b^k} . \end{aligned}$$

Hence, deriving them respectively with respect to b^l and a^l and subtracting, we obtain

$$\frac{\partial g(a)}{\partial a^k} \frac{\partial g(b)}{\partial b^l} - \frac{\partial g(a)}{\partial a^l} \frac{\partial g(b)}{\partial b^k} = \frac{\partial g(f(a, b))}{\partial f^m} \left(\frac{\partial^2 f^m(a, b)}{\partial a^k \partial b^l} - \frac{\partial^2 f^m(a, b)}{\partial b^k \partial a^l} \right) .$$

Taking now $a = b = f(a, b) = 0$ we have

$$[T_k, T_l] = f_{kl}^m T_m , \tag{2.6}$$

where $[T_k, T_l]$ is the commutator of the two operators, and the f_{kl}^m 's are real constant, antisymmetric in k and l

$$f_{kl}^m = -f_{lk}^m ,$$

defined by

$$f_{kl}^m := \left(\frac{\partial^2 f^m(a, b)}{\partial a^k \partial b^l} - \frac{\partial^2 f^m(a, b)}{\partial b^k \partial a^l} \right) \Big|_{a=b=0} .$$

We then proved that the T_k 's form a basis for a Lie algebra associated to the Lie group G . The f_{kl}^m 's are said *structure constants* of \mathfrak{g} . Being $[\cdot, \cdot]$ a commutator, the Jacobi identity is automatically satisfied, and, in terms of the structure constants, reads

$$f_{im}^n f_{jk}^m + f_{jm}^n f_{ki}^m + f_{km}^n f_{ij}^m = 0 .$$

We can interpret (2.4) as defining the *exponential maps* locally as

$$\exp : \mathfrak{g} \longrightarrow G ,$$

in such a way that the elements $g(v)$ of the group can be expressed in terms of the generators of the algebra T_i . However, such a relation is ambiguous, in the sense that in general the same Lie algebra can generate different groups. When it happens the

group whose manifold is simply connected⁵ is called *universal covering group*. Given these basic concepts we can now give some useful definitions.

Definition 6 (Subalgebra). A subset $\mathfrak{h} \subseteq \mathfrak{g}$ of a Lie algebra \mathfrak{g} is called *subalgebra* if itself is a Lie algebra, i.e.

$$[h, h'] \in \mathfrak{h} , \quad \forall h, h' \in \mathfrak{h} .$$

Any Lie algebra \mathfrak{g} has two subalgebras, namely \mathfrak{g} itself and the subspace $\{0\}$, which are called *trivial subalgebras*, while the others are called *proper subalgebras*.

Definition 7. An *invariant subalgebra* \mathfrak{h} , also called *ideal*, is a subalgebra of \mathfrak{g} such that

$$[h, g] \in \mathfrak{h} , \quad \forall h \in \mathfrak{h} , \forall g \in \mathfrak{g} .$$

Definition 8. A *simple* Lie algebra is a non-Abelian Lie algebra which contains no proper invariant subalgebras. A *semisimple* Lie algebra is a direct sum of simple Lie algebras.

The previous definition of semisimple Lie algebra is equivalent to saying that it is a non-Abelian Lie algebra with no Abelian proper invariant subalgebras.

2.3 Representations⁶

Let us consider the definitions of group representation and Lie algebra representation.

Definition 9 (Group representation). A *representation* R of a group G is a group homomorphism of G into V^V , the group of all the functions from an arbitrary set V to itself, with the composition map as the group product. In other words, R is a map

$$R : G \longrightarrow V^V ,$$

such that $\forall x, y \in G$ we have

$$R(x) \circ R(y) = R(x \circ y) ,$$

where “ \circ ” on the left denotes the composition map in V^V , whereas on the right it denotes the product in G . The set V is sometimes called *representation space* of R . If

⁵ Simply connected means that any closed curve can be shrunk to a point with a continuous transformation.

⁶ Elia de Sabbata and Pietro Oreglia

G is a Lie group we further require R to be a differentiable map⁷.

Definition 10 (Lie algebra representation). A *representation* R of a Lie algebra \mathfrak{g} is a map

$$R : \mathfrak{g} \longrightarrow V^V ,$$

such that $\forall x, y \in \mathfrak{g}$ we have

$$R(x) \circ R(y) - R(y) \circ R(x) = R([x, y]) ,$$

where V is an arbitrary set, with V^V we denote the set of all the functions from V to itself, and with “ \circ ” the composition map. The set V is sometimes called *representation space* of R .

As an example, we consider the group of translations of the n -dimensional Euclidean space $T = \{\tau_w \mid \tau_w : \mathbb{E}^n \rightarrow \mathbb{E}^n, v \mapsto \tau_w(v) = v + w\}$, we choose on \mathbb{E}^n a basis $\{e_i\}_{i=1, \dots, n}$ and we denote with w^i the unique n -tuple such that $\sum w^i e_i = w \in \mathbb{E}^n$. If we define the family of functions

$$\begin{aligned} \Phi_w : \mathbb{R}^n &\longrightarrow \mathbb{R}^n , \\ u^i &\longmapsto \Phi_w(u^i) = u^i + w^i , \end{aligned}$$

with $w \in \mathbb{E}^n$, then the map $R : w \mapsto \Phi_w(\cdot)$ is a representation of the group T with representation space \mathbb{R}^n . Notice that in this case the maps $\Phi_w(\cdot)$ are not linear. Another interesting example is the action of the gauge group on the gauge potential density in electrodynamics or in Yang-Mills theories. In particular, in electrodynamics a gauge transformation acts on the four-potential density $A_\mu(x)$ by adding the gradient of a smooth function $\phi(x)$, i.e

$$A_\mu \longmapsto A_\mu + \partial_\mu \phi .$$

In the definition of Lie algebra representation we have done no assumptions on the set V or the properties of the functions $R(x)$, however of particular interest is the case in which V is a vector space and $R(x)$ are linear maps $\forall x \in \mathfrak{g}$. To give a more convenient formulation we have to introduce the *general linear algebra* of V labelled as $\mathfrak{gl}(V)$, i.e the space of all linear maps from V to V , which, if V is a finite-dimensional space of dimension n , can be seen as the space of all $n \times n$ matrices, usually denoted $\mathfrak{gl}(n)$ (thus $\mathfrak{gl}(n)$ is a Lie algebra of dimension n^2). Note that if the map

$$R : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) ,$$

is an homomorphism of Lie algebras, then it defines a *linear representation* of \mathfrak{g} . If

⁷ Note that in this case, we need to define a differentiable manifold structure on V^V .

the homomorphism is also injective, R is called a *faithful* representation. There always exists a (not faithful) representation for any Lie algebra, namely the one which maps each element of \mathfrak{g} on the zero vector, called trivial representation or *singlet* representation. There is another representation that exists for all Lie algebra. This is the *adjoint representation*, defined as

$$\begin{aligned} R_{ad} : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) , \\ x &\longmapsto \text{ad}_x , \end{aligned}$$

with

$$\text{ad}_x(y) := [x, y] .$$

It turns out that the adjoint representation of a simple Lie algebra is faithful, while for any Abelian Lie algebra it is not faithful. Furthermore, the dimension of the adjoint matrix representation is equal to the dimension of the algebra. In terms of the generators, the entries of the matrices $R_{ad}(T_a)$ read

$$(R_{ad}(T_a))_c^b = f_{ac}^b .$$

Another type of representations very common in physics are the *unitary representation*, in which the representation matrices are unitary. It can be shown

Theorem 2.3.1. *All the finite-dimensional representations of a finite group, or of a compact Lie group, are linearly equivalent to unitary representations.*

By *linearly equivalent* we mean that the representations are linked by a similarity transformation of the type

$$B(g) = SA(g)S^{-1} ,$$

with A and B two representations of the group element g , and S a non-singular matrix.

Definition 11. A representation is said *irreducible* if there are no non-trivial invariant subset. Otherwise it is said *reducible*.

Irreducible representations play a key role, also thanks to the following theorem.

Theorem 2.3.2 (Schur's Lemma). *If a matrix commutes with all representations matrices of an irreducible representation, it must be a multiple of the unit matrix.*

We conclude with following definition.

Definition 12. A representation over \mathbb{C} is *fully reducible* if and only if there is a basis of the underlying vector space V such that all the representation matrices $R(x)$ are

simultaneously of a block-diagonal form

$$R(x) = \begin{pmatrix} R_1(x) & 0 & \dots & 0 \\ 0 & R_2(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_n(x) \end{pmatrix},$$

where R_i are squared matrices of appropriate dimension that describe irreducible representations.

Similarly, it can be seen that, in the case of reducible representations, but not fully reducible, there is a basis of V such that all the representation matrices have the block form

$$R(x) = \begin{pmatrix} R_1(x) & Q_1(x) \\ 0 & Q_2(x) \end{pmatrix}.$$

2.4 Unitary representation of the Poincaré group: Wigner classification⁸

The study of the representation theory of the Poincaré group could start from Wigner's idea for the classification of elementary particles. In non-relativistic Quantum Mechanics, elementary particles are identified as the spaces of irreducible representations of the algebra generated by the set of observables $\{\hat{\mathbf{x}}, \hat{\mathbf{p}}, \hat{\mathbf{S}}\}$. Nevertheless, when Quantum Mechanics and Special Relativity are put together, this classification becomes meaningless since the position \hat{x}^μ can no longer be an observable.

Wigner's idea was that elementary particles might have been classified as spaces of *irreducible representations* of space-time symmetry group, *i.e. proper Poincaré group* $\mathcal{P}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \mathcal{L}_+^\uparrow$, where “ \rtimes ” stands for the *semi-direct product* of groups. \mathcal{L}_+ (*special Lorentz group*) and \mathcal{L}^\uparrow (*orthochronous Lorentz group*) stand for the subgroups of $\mathcal{L} \simeq O(1,3)$ whose elements respectively satisfy the conditions

$$\det \Lambda = 1, \quad \Lambda^0_0 \geq 0.$$

\mathcal{L}_+^\uparrow is the so-called *proper Lorentz group*.

The spaces of irreps of \mathcal{P}_+^\uparrow describe physical states, which are rays in a Hilbert space, that is sets of non-zero vectors differing by a complex scalar factor or, if one is considering normalised states, by a phase factor. Therefore, one must consider *projective representations* of the Poincaré group acting on ray spaces, or equally, via *Bargmann's the-*

⁸ Stefano De Angelis

orem, the unitary representations of its *universal covering group* $\tilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^{1,3} \rtimes \mathrm{SL}(2, \mathbb{C})$.

Unitary representations will be characterised by mean of Wigner's trick. Later an alternative and more conventional way to find group representations will be shown. This brings to the same results in a simpler, even though less intuitive, way. Wigner's trick splits into four main steps.

First of all, characterise unitary irreps of $\mathbb{R}^{1,3}$, which have to be one dimensional, since the group is Abelian. Consider the four-momentum operator \hat{P}^μ and its (generalised) eigenstates $|p\rangle$

$$\hat{P}^\mu |p\rangle = p^\mu |p\rangle , \quad (2.7)$$

where $p^\mu \in \sigma(\hat{P}^\mu) = \{p^\mu \in (\mathbb{R}^{1,3})^*\}$.⁹ Take $a^\mu \in \mathbb{R}^{1,3}$ and define

$$\hat{U}(a^\mu) = e^{ia^\mu \cdot \hat{P}_\mu} . \quad (2.8)$$

This gives the one-dimensional unitary irreps which one is looking for. Indeed,

$$\hat{U}(a^\mu) |p\rangle = e^{ia^\mu \cdot p_\mu} |p\rangle , \quad (2.9)$$

i.e. $\hat{U}(a^\mu)$ simply multiplies basis vectors $\{|p\rangle\}_{p \in (\mathbb{R}^{1,3})^*}$ by a number $\exp(ia^\mu p_\mu)$. Since $|p\rangle$ is not normalisable, it cannot be a ray of the Hilbert space. Therefore consider $|\psi\rangle \in \mathcal{H}$ and let $\hat{U}(a)$ act on $|\psi\rangle$

$$\hat{U}(a) : \psi(p) = \langle p | \psi \rangle \rightarrow \langle p | \hat{U}(a) | \psi \rangle = e^{ia \cdot p} \langle p | \psi \rangle = e^{ia \cdot p} \psi(p) . \quad (2.10)$$

Once $p^\mu \in (\mathbb{R}^{1,3})^*$ is fixed, $\psi(p) = \langle p | \psi \rangle \in \mathbb{C}$ and $\dim(U) = 1$. Then the p^μ 's characterise the unitary representations of $\mathbb{R}^{1,3}$.

Secondly, \mathcal{H} must be the space of unitary irrep of the group $\mathbb{R}^{1,3} \rtimes \mathrm{SL}(2, \mathbb{C})$ (not $\mathbb{R}^{1,3}$ itself), then the structure of the set of p^μ 's in an irrep of this group must be studied. Consider the space $(\mathbb{R}^{1,3})^*$, thus it is necessary to find how this space is divided by curves which are invariant under the action of $\mathrm{SL}(2, \mathbb{C})$, because these have to be related to \mathcal{H} (for hypothesis this is the space of an irreducible representation). Let us define

$$\sigma^\mu := (\sigma^0, \sigma^i) = (\sigma_0, -\sigma_i) ,$$

with σ_0 the 2×2 identity matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

⁹ $(\mathbb{R}^{1,3})^*$ denotes the space of four-momenta, eigenvalues of \hat{P}_μ , in order to distinguish this from the space of translations $\mathbb{R}^{1,3} \ni a^\mu$.

the Pauli matrices. Once p^μ is fixed, associate it with a 2×2 matrix

$$\sigma_\mu p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \quad (2.11)$$

and, by $\text{Tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}$ or, equivalently, $\text{Tr}(\sigma_\mu \sigma^\nu) = 2g_{\mu\nu}$,

$$p^\mu = \frac{1}{2} \text{Tr}(p_\nu \sigma^\nu \sigma_\mu). \quad (2.12)$$

Now consider $A \in \text{SL}(2, \mathbb{C})$,

$$A \sigma_\mu p^\mu A^\dagger = \Lambda^\mu{}_\nu(A) \sigma_\mu p^\nu, \quad (2.13)$$

where

$$\Lambda^\mu{}_\nu(A) = \frac{1}{2} \text{Tr}(\sigma^\mu A \sigma_\nu A^\dagger). \quad (2.14)$$

It follows from the previous relation that every transformation $\Lambda^\mu{}_\nu$ can be specified by two matrices $\pm A$, *i.e.* $\Lambda^\mu{}_\nu(A) = \Lambda^\mu{}_\nu(-A)$ (indeed it can be shown that $\mathcal{L}_+^\dagger \simeq \text{SL}(2, \mathbb{C})/\mathbb{Z}_2$). After few calculations one finds

$$\det(A \sigma_\mu p^\mu A^\dagger) = \det(\sigma_\mu p^\mu) = p^\mu p_\mu = m^2 \in \mathbb{R}. \quad (2.15)$$

Therefore, every orbit of $\text{SL}(2, \mathbb{C})$ in $(\mathbb{R}^{1,3})^*$ is characterised by m^2 . Three main cases can be distinguished:

- ▷ $m^2 < 0$: these are hyperboloids of one sheet;
- ▷ $m^2 = 0$: this is the conical surface;
- ▷ $m^2 > 0$: these are hyperboloids of two sheets.

Since p^μ is the four-momentum of the particle, m is its mass. Thus, it is possible to restrict to physical cases only, that are the orbits on which $m^2 \geq 0$ and $p^0 \geq 0$. In other words, one considers only the positive light cone. Thus, physical orbits are

- ▷ $m^2 > 0$ and $p^0 > 0$: one sheet of the hyperboloids of the third case, they stand for massive particles;
- ▷ $m^2 = 0$ and $p^0 > 0$: this is the light cone (except for the origin) and it stands for massless particles;
- ▷ $m^2 = 0$ and $p^0 = 0$: the origin is itself an orbit, which is invariant under the action of all $\text{SL}(2, \mathbb{C})$ and it stands for the *vacuum state*.

There is now some difficulty since it is not possible to define the Hilbert space as

$$\mathcal{H} = \bigoplus_{p \in \text{orbit}} \mathcal{H}_p , \quad (2.16)$$

because it is quite evident that \mathcal{H} is not separable. This kind of problem is the same observed when one introduces the Hilbert space $L^2(\mathbb{R})$. Indeed, the space of one variable functions is not separable, but, by introducing the *Lebesgue measure*, contributions of single points are avoided and the space becomes separable. In this sense the sum over p^μ is substituted by a *direct integral*

$$\bigoplus_{p \in \text{orbit}} \rightarrow \int^\oplus . \quad (2.17)$$

Then it must be defined a measure supported on the orbit and $\text{SL}(2, \mathbb{C})$ invariant. It can be proved that there exists only one metric satisfying these conditions, that is

$$d\mu(p) = d^4p \delta(p^2 - m^2) \theta(p^0) , \quad (2.18)$$

where $\theta(p^0)$ is Heaviside's theta. \mathcal{H} is the space of a unitary irrep of $\tilde{\mathcal{P}}_+^\uparrow$, thus

$$\mathcal{H} = \int^\oplus d^4p \delta(p^2 - m^2) \theta(p^0) \mathcal{H}_p . \quad (2.19)$$

The scalar product in \mathcal{H} is generated by the scalar product in \mathcal{H}_p , which is a complex Hilbert space of $d(p)$ dimensions: if $\psi, \phi \in \mathcal{H}$

$$(\psi, \phi)_\mathcal{H} \equiv \int d^4p \delta(p^2 - m^2) \theta(p^0) (\psi_p, \phi_p)_{\mathcal{H}_p} . \quad (2.20)$$

Thus, $\phi \in \mathcal{H}$ iff

$$(\phi, \phi)_\mathcal{H} := \int d^4p \delta(p^2 - m^2) \theta(p^0) (\phi_p, \phi_p)_{\mathcal{H}_p} < \infty . \quad (2.21)$$

The Hilbert space has been completely defined and, to complete the classification, one should find $d(p)$. Let $A \in \text{SL}(2, \mathbb{C})$ act on a representative p^μ of an orbit

$$\hat{P}^\mu U_\mathcal{H}(A)|p, \alpha\rangle = U_\mathcal{H}(A)U_\mathcal{H}(A)^{-1}\hat{P}^\mu U_\mathcal{H}(A)|p, \alpha\rangle = \Lambda^\mu{}_\nu(A)p^\nu U_\mathcal{H}(A)|p, \alpha\rangle ,$$

where α stands for the components in \mathcal{H}_p and $U_\mathcal{H}(A)^{-1}\hat{P}^\mu U_\mathcal{H}(A) = \Lambda^\mu{}_\nu(A)\hat{P}^\mu$. Thus, $U_\mathcal{H}(A)|p, \alpha\rangle$ is a (generalised) eigenstate of \hat{P}^μ . Since $U_\mathcal{H}$ is a unitary representation

$$\langle p, \alpha | U_\mathcal{H}^\dagger(A) = \langle p, \alpha | U_\mathcal{H}(A^{-1}) , \quad (2.22)$$

which tells that the action of $U_{\mathcal{H}}$ on \mathcal{H}_p is

$$U_{\mathcal{H}}(A)\mathcal{H}_p = \mathcal{H}_{\Lambda(A^{-1})p} = \mathcal{H}_{\Lambda^{-1}(A)p} . \quad (2.23)$$

Then, in order to be invariant under $\text{SL}(2, \mathbb{C})$, $d(p)$ remains constant for one single orbit. But now it must be explained how the α components are mixed. The answer is provided by using the following lemma. Once k^μ , representative of an orbit ($p^\mu p_\mu = m^2 \geq 0$), is fixed, the action of $U_{\mathcal{H}}(A)$ on $|p, \alpha\rangle$ with $\alpha = 1, \dots, d(p)$ can be decomposed in the product of two transformations: a boost that maps k in Λp (it is denoted $U_{\mathcal{H}}(B_{\Lambda p})$ and $U_{\mathcal{H}}(B_{\Lambda p})|k, \alpha\rangle := |\Lambda p, \alpha\rangle$) and an element of the isotropy group of k , $\text{Iso}(k)$, that is the subgroup of $\text{SL}(2, \mathbb{C})$ which does not change k^μ

$$U_{\mathcal{H}}(A) = U_{\mathcal{H}}(B_{\Lambda p})U_{\mathcal{H}}(B_{\Lambda p}^{-1}AB_p) , \quad (2.24)$$

where

$$U_{\mathcal{H}}(B_p)|k, \alpha\rangle := |p, \alpha\rangle . \quad (2.25)$$

It is straightforward to prove that $U_{\mathcal{H}}(B_{\Lambda p}^{-1}AB_p := \tilde{A}_p)$ is a representation of $\text{Iso}(k)$:

$$U_{\mathcal{H}}(B_{\Lambda p}^{-1}AB_p)|k, \alpha\rangle = U_{\mathcal{H}}(B_{\Lambda p}^{-1})U_{\mathcal{H}}(A)|p, \alpha\rangle = U_{\mathcal{H}}(B_{\Lambda p}^{-1})|\Lambda p, \beta\rangle = |k, \beta\rangle .$$

Therefore,

$$U_{\mathcal{H}}(\tilde{A}_p)|k, \alpha\rangle := D_{\beta\alpha}(\tilde{A}_p)|k, \beta\rangle \in \mathcal{H}_p , \quad (2.26)$$

where $D_{\beta\alpha}(\tilde{A}_p)$ is an irrep of $\text{Iso}(k)$.

Finally, the fourth and also the last step is to classify the irreps of $\text{Iso}(k)$ in the three different physical cases.

- ▷ $p^\mu p_\mu = m^2 > 0$: $k^\mu = (m, 0, 0, 0)$ can be chosen, that is a rest particle in a reference frame, thus

$$k^\mu \sigma_\mu = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} . \quad (2.27)$$

$A \in \text{SL}(2, \mathbb{C})$ such that

$$A (k^\mu \sigma_\mu) A^\dagger = k^\mu \sigma_\mu , \quad (2.28)$$

is every $A \in \text{SU}(2, \mathbb{C})$. Irreps of $\text{SU}(2, \mathbb{C})$ are labeled by $j \in \mathbb{N}/2$, that is the *spin*. Therefore, it can be argued that massive particles are completely characterised by their mass and spin.

- ▷ $p^\mu p_\mu = 0$: $k^\mu = (1, 0, 0, 1)$ can be a representative, thus

$$k^\mu \sigma_\mu = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} . \quad (2.29)$$

After some calculations it is found that the more general element of $\text{Iso}(k)$ is written as

$$A = \begin{pmatrix} e^{i\theta} & x + iy \\ 0 & e^{-i\theta} \end{pmatrix}, \quad (2.30)$$

where $x, y \in \mathbb{R}$. It is straightforward to show that there are two natural subgroups

$$R(\theta) := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad T(x, y) := \begin{pmatrix} 1 & x + iy \\ 0 & 1 \end{pmatrix}. \quad (2.31)$$

A brief calculation brings

$$R(\theta)T(x, y)R(\theta)^{-1} = \begin{pmatrix} 1 & (x \cos 2\theta - y \sin 2\theta) + i(y \cos 2\theta + x \sin 2\theta) \\ 0 & 1 \end{pmatrix}.$$

Therefore, $\text{Iso}(k) := \mathbb{R}^2 \rtimes \widetilde{\text{SO}}(2)$, where $\widetilde{\text{SO}}(2)$ is the double (non-universal) covering of $\text{SO}(2)$. Indeed, from the above expression, one sees that $R(\theta) \in \widetilde{\text{SO}}(2)$ is a rotation of angle 2θ in the \mathbb{R}^2 plane, such that $\theta = 2\pi$ corresponds to two complete rotations. Let consider only the trivial representation of \mathbb{R}^2 , because one is interested in finite-dimensional representations of $\text{Iso}(k)$ (particles with continuous spin have not been observed in nature). Irreps of $\text{SO}(2)$ are labeled by $n \in \mathbb{Z}$ ($n : \theta \rightarrow e^{in\theta}$), then the ones of $\widetilde{\text{SO}}(2)$ are labelled by $\epsilon \in \mathbb{Z}/2$. Then massless elementary particles are characterised by their *helicity* ϵ , which is the projection of the spin on the direction of \mathbf{p} . Indeed, the direction of motion of massless particles cannot be reversed by a proper Lorentz transformation, while for massive ones $\mathbf{S} \cdot \mathbf{p}$ depends on the reference frame.

- ▷ The last and simpler case that must be considered is the *vacuum state*: the measure of the direct integration in this case is $d\mu(p) = d^4p \delta^{(4)}(p^\mu)$ and then $\mathcal{H} \equiv \mathcal{H}_0$. $p^\mu = 0$ is invariant under the action of all $\text{SL}(2, \mathbb{C})$, then the vacuum state is a one dimensional Hilbert space.

2.5 Behaviour of local fields under the Poincaré group: relativistic covariance¹⁰

Let us start this section by recalling some notational aspects. Consider the four-vector $V^\mu := (V_0, \mathbf{V})$ and note that $V_\mu := (V_0, -V_1, -V_2, -V_3) = g_{\mu\nu}V^\nu$ implies

$$\mathbf{V} := (V_x, V_y, V_z) = (V^1, V^2, V^3) = -(V_1, V_2, V_3).$$

¹⁰Stefano De Angelis

In particular, for the three-gradient, we have

$$\nabla := (\nabla_x, \nabla_y, \nabla_z) = \left(\frac{\partial}{\partial x^k} = \partial_k \right)_{k=1,2,3} = - \left(\frac{\partial}{\partial x_k} = \partial^k \right)_{k=1,2,3} . \quad (2.32)$$

Also note that the four-momentum operator reads

$$p^\mu = i\partial^\mu := (i\partial^0, -i\nabla) . \quad (2.33)$$

Take a point in Minkowski space-time and let $x^\mu := (x_0, \vec{x})$ with respect to a reference frame I . The coordinates x'^μ , in a different reference frame I' , can be expressed in terms of a Poincaré transformation

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu , \quad (2.34)$$

where

$$g_{\rho\sigma} = g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma . \quad (2.35)$$

Let us recall that a contravariant vector is the one transforming as dx^μ , that is

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu ,$$

whereas a covariant vector transforms as the four-gradient

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} .$$

Since constant translations have no effect on covariant and contravariant vectors, it follows that x^μ transforms as a contravariant vector only under Lorentz transformations.

Let us derive the expression of the ten generators of the Poincaré transformations. We first consider an infinitesimal Lorentz transformation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu , \quad (2.36)$$

where $\delta^\mu{}_\nu$ is the Kronecker delta. Evaluation of (2.35) yields to

$$0 = g_{\nu\rho} \epsilon^\rho{}_\mu + g_{\mu\rho} \epsilon^\rho{}_\nu , \quad (2.37)$$

which becomes

$$0 = \epsilon_{\nu\mu} + \epsilon_{\mu\nu} , \quad (2.38)$$

that is $\epsilon_{\mu\nu}$ is an antisymmetric tensor, with six independent entries. An infinitesimal

variation due to a Lorentz transformation can be written as

$$\delta x^\mu = \epsilon^{\mu\rho} x_\rho := \frac{i}{2} \epsilon^{\rho\sigma} L_{\rho\sigma} x^\mu , \quad (2.39)$$

where the $L_{\mu\nu}$'s are Hermitian operators

$$L_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu) . \quad (2.40)$$

It is easy to verify that the $L_{\mu\nu}$'s satisfy Lie algebra of $\text{SO}(1,3)$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i g_{\nu\rho} L_{\mu\sigma} - i g_{\mu\rho} L_{\nu\sigma} - i g_{\nu\sigma} L_{\mu\rho} + i g_{\mu\sigma} L_{\nu\rho} . \quad (2.41)$$

The most general representation of the generators of $\text{SO}(1,3)$ that obeys the commutation relations (2.41) is given by

$$J_{\mu\nu} := L_{\mu\nu} + S_{\mu\nu} , \quad (2.42)$$

where the $S_{\mu\nu}$'s satisfy the same Lie algebra as the $L_{\mu\nu}$'s and commute with them.

As it can be seen from (2.34), Poincaré transformations include also uniform translations in space and time

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu , \quad (2.43)$$

where a^μ is an arbitrary constant four-vector. The translations do not commute with the Lorentz transformations: indeed two successive Poincaré transformations give

$$x^\mu \rightarrow \Lambda_1^\mu{}_\nu x^\nu + a_1^\mu \rightarrow \Lambda_2^\mu{}_\rho \Lambda_1^\rho{}_\nu x^\nu + \Lambda_2^\mu{}_\rho a_1^\rho + a_2^\mu , \quad (2.44)$$

i.e. the translation parameters a_1^μ get rotated as a four-vector do. In this sense it can be said that \mathcal{P}_+^\uparrow is the semi-direct product of \mathcal{L}_+^\uparrow and $\mathbb{R}^{1,3}$.

In order to obtain the algebra of the generators, observe that the change in x^μ under an infinitesimal translation is

$$\delta x^\mu = \epsilon^\mu := i \epsilon^\rho P_\rho x^\mu , \quad (2.45)$$

such that the P_μ 's are the Hermitian operators

$$P_\mu = -i \partial_\mu . \quad (2.46)$$

They satisfy the commutation relations

$$[P_\mu, P_\nu] = 0 , \quad (2.47)$$

and

$$[J_{\mu\nu}, P_\rho] = -ig_{\mu\rho}P_\nu + ig_{\nu\rho}P_\mu , \quad (2.48)$$

that is P_μ transforms like a four-vector. The commutation relations (2.41), (2.47) and (2.48) define the Lie algebra of the Poincaré group.

Now irreps of \mathcal{P}_+^\uparrow can be classified in an alternative way than before simply doing some considerations on its algebra. It is quite obvious that the “length” $P_\mu P^\mu$ of the four-vector P^μ is invariant under Poincaré transformations, thus it can be seen as a Casimir operator. Since the Lie algebra of \mathcal{P}_+^\uparrow has rank 2, one must construct another Casimir operator. The length of any four-vector which commutes with the P^μ 's will be a good one: the *Pauli-Lubanski four-vector* does it and it is defined by

$$W^\mu := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu S_{\rho\sigma} . \quad (2.49)$$

Then irreps of \mathcal{P}_+^\uparrow (or, to be more precise, of its covering group) are characterised according the values of Casimir operators and three cases can be distinguished.

- ▷ The eigenvalues of $P^2 = m^2$ are real positive numbers. $W^2 = -m^2\mathbf{S}^2 = -m^2s(s+1)$, where $s \in \mathbb{N}/2$. These representations are labeled by the mass m and the spin s . States within them are distinguished by the third component of the spin $s_3 = -s, -s+1, \dots, s-1, s$ and the continuous eigenvalues of P_i .
- ▷ The eigenvalue of P^2 is zero, corresponding to a particle of zero rest mass. W^2 is also zero. From (2.49) it is easy to verify $P^\mu W_\mu = 0$, *i.e.* P^μ and W^μ are proportional. The constant of proportionality, called helicity, well labels representations and it is equal to s , where $s \in \mathbb{Z}/2$. States with same helicity are distinguished by the three values of their momenta along x , y and z directions, \mathbf{P} .
- ▷ Finally the case previously excluded: $P^2 = 0$ and $W^2 = \alpha^2\mathbb{I}$, where α is a real number. In this case the corresponding infinite-dimensional representation is characterised by continuous spin. Most likely, this type of representation does not correspond to any real particles.

Consider an arbitrary field as a function of space-time point in a reference frame I , $f_a(x^\mu)$ with $a = 1, \dots, n$. If one moves to another inertial frame I' , the field will be written as $f'_a(x'^\mu)$, because the functional transformation will be in general frame-dependent. Write the change in the function for an infinitesimal transformation as

$$\delta f_a(x) := f'_a(x') - f_a(x) . \quad (2.50)$$

Let observe that

$$f'_a(x + \delta x) - f_a(x) = f'_a(x) - f_a(x) + \delta x^\mu \partial_\mu f'_a(x) + O(\delta x) . \quad (2.51)$$

To $\mathcal{O}(\delta x)$, $\partial_\mu f'_a$ is replaced by $\partial_\mu f_a$

$$\delta f_a(x) = \delta_0 f_a(x) + \delta x^\mu \partial_\mu f_a(x) + O(\delta x) , \quad (2.52)$$

where the *functional change* at the same x has been introduced

$$\delta_0 f_a(x) = f'_a(x) - f_a(x) . \quad (2.53)$$

The second term on the right side in (2.52) is called *transport term*. One can formally see (2.53) as an operator equation

$$\delta = \delta_0 + \delta x^\mu \partial_\mu . \quad (2.54)$$

To refer to representations of Poincaré group means to consider how the functional structure of $f_a(x)$ changes under its transformations. Thus, if one wants to study generators of Poincaré group, $\delta_0 f_a$ must be considered, not δf_a .

To know how a given field transforms helps to define different Lagrangian densities which are Poincaré invariant and from which motion equations are extracted. Therefore, the general aspects of dynamics can be investigated from the study of Poincaré group representations and their properties. Equivalently, it is also possible to start from motion equations of fields and, vice versa by requiring relativistic covariance, to determine transformation properties of them.

Under a translation in space-time, there is no change in a local field, that is

$$\delta f_a = 0 , \quad (2.55)$$

or

$$\delta_0 f_a(x) = -\epsilon^\mu \partial_\mu f_a(x) = -i\epsilon^\mu P_\mu f_a(x) . \quad (2.56)$$

Under Lorentz transformation the situation is more complicated

$$f_a(x) \rightarrow f'_a(x') = D_{ab}(\Lambda) f_b(x) , \quad (2.57)$$

where $D(\Lambda)$ is a $n \times n$ matrix, *i.e.* a finite-dimensional representation of Lorentz group. As explained before, functional change $f'_a(x) - f_a(x)$ is needed:

$$f'_a(x) = D_{ab}(\Lambda) f_b(\Lambda^{-1}x) , \quad (2.58)$$

whose infinitesimal form is

$$f'_a(x) = \left(\mathbb{I}_{ab} - \frac{i}{2} \epsilon^{\mu\nu} (S_{\mu\nu})_{ab} \right) f_b(x^\mu - \epsilon^\mu_\nu x^\nu) . \quad (2.59)$$

Therefore, the functional change is

$$\delta_0 f_a(x) = -\frac{i}{2}\epsilon^{\mu\nu} (S_{\mu\nu})_{ab} f_b(x) - \frac{i}{2}\epsilon^{\mu\nu} L_{\mu\nu} f_a(x) , \quad (2.60)$$

where the $L_{\mu\nu}$'s have been defined in (2.40). Then the generators of Lorentz transformations for fields are

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} , \quad (2.61)$$

where the matrices $S_{\mu\nu}$'s are a finite-dimensional representation of the Lie algebra of \mathcal{L}_+^\uparrow . In the next section they are going to be classified and their properties discussed.

Now consider how quantised field operators $\hat{f}_a(x)$ transform. Physical observables are given in matrix element form, that is

$$\langle \Phi_\alpha | \hat{f}_a(x) | \Phi_\beta \rangle . \quad (2.62)$$

These matrix elements are the analogue of the amplitude $f_a(x)$. An observer in a different reference frame sees the amplitude

$$\langle \Phi'_\alpha | \hat{f}_a(\Lambda x + a) | \Phi'_\beta \rangle , \quad (2.63)$$

where $|\Phi'_\alpha\rangle$ and $|\Phi'_\beta\rangle$ stand for the states as seen in the second reference frame. For what concerns the field operator, notice that transformation is only for its argument, consistently with Heisenberg picture. The amplitude (2.63) is the quantum analogue of $f'_a(x')$, thus, as seen before,

$$\langle \Phi'_\alpha | \hat{f}_a(\Lambda x + a) | \Phi'_\beta \rangle = D_{ab}(\Lambda) \langle \Phi_\alpha | \hat{f}_b(x) | \Phi_\beta \rangle . \quad (2.64)$$

According to Wigner theorem, transformations of states corresponding to exact symmetries are represented by unitary or antiunitary operators

$$|\Phi'_\alpha\rangle = U(\Lambda, a) |\Phi_\alpha\rangle . \quad (2.65)$$

We then have

$$U^{-1}(\Lambda, a) \hat{f}_a(x) U(\Lambda, a) = D_{ab}(\Lambda) \hat{f}_b(\Lambda^{-1}x - a) . \quad (2.66)$$

Note that sometimes in the literature it is used the inverse of such a relation, obtained by the transformation $\Lambda, a \rightarrow \Lambda^{-1}, -a$, that is

$$U(\Lambda, a) \hat{f}_a(x) U^{-1}(\Lambda, a) = D_{ab}(\Lambda^{-1}) \hat{f}_b(\Lambda x + a) . \quad (2.67)$$

2.6 Finite-dimensional irreducible representations of the Lorentz group¹¹

Physical motivations for the study of finite-dimensional representations of \mathcal{L}_+^\uparrow , or to be more precise of its covering group $\mathrm{SL}(2, \mathbb{C})$, has just been discussed, thus the classification of its irreps are going to be presented.

Let us start with some mathematical considerations. Finite-dimensional representations of $\mathrm{SL}(2, \mathbb{C})$ are in one-to-one correspondence with representations of its algebra $\mathfrak{sl}(2, \mathbb{C})$ (since the group is connected and simply connected), which are one-to-one with those of $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}}$. It can be shown that $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$, whose finite-dimensional representations are in one-to-one correspondence with those of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and thus with those of $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$:

$$\begin{aligned} \mathrm{SL}(2, \mathbb{C}) &\leftrightarrow \mathfrak{sl}(2, \mathbb{C}) \leftrightarrow \mathfrak{sl}(2, \mathbb{C})_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \\ &\leftrightarrow \mathfrak{su}(2) \oplus \mathfrak{su}(2) \leftrightarrow \mathrm{SU}(2) \otimes \mathrm{SU}(2) . \end{aligned}$$

$\mathrm{SU}(2) \otimes \mathrm{SU}(2)$ is a compact group, thus its representations are finite-dimensional and (equivalent to) unitary (ones). Therefore, representations of its algebra have to be Hermitian finite-dimensional matrices because of the exponential. Let us now construct, by opportune (complex) combinations of generators of the Lorentz group, two sets of generators obeying the $\mathrm{SU}(2)$ algebra.

We start with the algebra of finite-dimensional generators of Lorentz group (2.41)

$$[S_{\mu\nu}, S_{\rho\sigma}] = ig_{\nu\rho}S_{\mu\sigma} - ig_{\mu\rho}S_{\nu\sigma} - ig_{\nu\sigma}S_{\mu\rho} + ig_{\mu\sigma}S_{\nu\rho} , \quad (2.68)$$

where, of course, $S_{\mu\nu} = -S_{\nu\mu}$, and the indices of $S_{\mu\nu}$ are usual raised or lowered by contraction with $g_{\mu\nu}$ or $g^{\mu\nu}$. To see how to construct desired matrices, first divide the six components of $S_{\mu\nu}$ into two three-vectors, *i.e.* *angular momentum* matrices

$$J_1 = S_{23} , \quad J_2 = S_{31} , \quad J_3 = S_{12} , \quad (2.69)$$

and *boost* ones

$$K_1 = S_{10} , \quad K_2 = S_{20} , \quad K_3 = S_{30} . \quad (2.70)$$

¹¹ Stefano De Angelis

Algebra (2.68) reads

$$[J_i, J_j] = i\epsilon_{ijk}J_k , \quad (2.71)$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k , \quad (2.72)$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k , \quad (2.73)$$

where i, j, k run over the values 1, 2, 3 and ϵ_{ijk} is the total antisymmetric quantity with $\epsilon_{123} := +1$. The algebra (2.71) just says that \mathbf{J} matrices generate a representation of the rotational subgroup of the Lorentz group (which determines the spin of the representation), and (2.72) just represents the fact that \mathbf{K} is a three-vector. The minus sign in the right-hand side of (2.73) arises from the fact that $g_{ii} = -1$ and it plays a crucial role in what follows. Indeed, replacing the matrices \mathbf{J} and \mathbf{K} by the two decoupled spin-like three-vectors

$$\mathbf{A} := \frac{1}{2}(\mathbf{J} + i\mathbf{K}) , \quad (2.74)$$

$$\mathbf{B} := \frac{1}{2}(\mathbf{J} - i\mathbf{K}) , \quad (2.75)$$

we see that commutation relations (2.71)-(2.73) are equivalent to

$$[A_i, A_j] = i\epsilon_{ijk}A_k , \quad (2.76)$$

$$[B_i, B_j] = i\epsilon_{ijk}B_k , \quad (2.77)$$

$$[A_i, B_j] = 0 . \quad (2.78)$$

One finds matrices satisfying commutation relations (2.76)-(2.78) in the same way that one finds matrices representing the spins of a pair of uncoupled particles as direct sum. That is, let label the rows and columns of these matrices with a pair of integers and/or half-integers a, b , running over the values

$$a = -A, -A + 1, \dots, +A ,$$

$$b = -B, -B + 1, \dots, +B ,$$

and take

$$(\mathbf{A})_{a'b', ab} = \delta_{bb'}\mathbf{J}_{a'a}^{(A)} , \quad (2.79)$$

$$(\mathbf{B})_{a'b', ab} = \delta_{aa'}\mathbf{J}_{b'b}^{(B)} , \quad (2.80)$$

where $\mathbf{J}^{(A)}$ and $\mathbf{J}^{(B)}$ are the standard spin matrices for spins A and B

$$\left(J_3^{(A)}\right)_{a'a} = a\delta_{a'a} , \quad (2.81)$$

$$\left(J_1^{(A)} \pm iJ_2^{(A)}\right)_{a'a} = \delta_{a',a\pm 1} \sqrt{(A \mp a)(A \pm a + 1)} , \quad (2.82)$$

and likewise for $\mathbf{J}^{(B)}$. The representation is labeled by the values of the positive integers and/or half-integers A and B . Therefore, the (A, B) representation has dimensionality $(2A + 1)(2B + 1)$.

As said before, \mathbf{A} and \mathbf{B} have to be Hermitian, and therefore \mathbf{J} is Hermitian (as expected for the spin, which is an observable) but \mathbf{K} is *anti-Hermitian*.¹² This is because of the i in equation (2.76) and (2.77), which is required by the minus sign in (2.73) in order to obtain such \mathbf{A} and \mathbf{B} satisfying the right algebra. Thus, the finite-dimensional representations of Lorentz group are not unitary. This is a general result of group theory: simple non-compact Lie groups do not have any finite-dimensional non-trivial unitary irreducible representation. There is no problem working with non-unitary representations, because the objects one is now concerning with are fields, not states, and do not need to have a Lorentz-invariant positive norm.

In contrast, the rotation group is represented unitarily, with its generators represented by Hermitian matrices

$$\mathbf{J} = \mathbf{A} + \mathbf{B} . \quad (2.83)$$

This sum of generators \mathbf{A} and \mathbf{B} can be seen as the direct sum of the matrix-vectors. Therefore, the corresponding representation of the group can be seen as the direct product of two $SU(2)$ representations, $D^{(j)}$. By the Clebsch-Gordon decomposition one gets

$$(A, B) = D^{(A)} \otimes D^{(B)} = \bigoplus_{j=|A-B|}^{A+B} D^{(j)} . \quad (2.84)$$

The field which transforms according to the (A, B) representation of the Lorentz group has components that rotate like objects of spin j , with

$$j = A + B, A + B - 1, \dots, |A - B| . \quad (2.85)$$

Furthermore, note that the product of two representations $(j_1, 0)$ and $(j_2, 0)$ is reducible

¹²In the text of P. Ramond, “*Field Theory: A Modern Primer*”, on page 8 after the formula (1.2.34) there is an error: the K_i 's are incorrectly carried as Hermitian generators. Indeed, when Ramond constructs the one-to-one correspondence with $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators, this requirement makes N_i and N_i^\dagger to be not Hermitian, thus their eigenvalues are not $n(n + 1)$ and $m(m + 1)$. Furthermore in this way he would get finite-dimensional unitary irreps, even though it is forbidden by mathematical theorem. However it can be obtained infinite-dimensional unitary irreps: *e.g.* (2.40) is Hermitian. Note in formula (1.4.20) on page 16 that \mathbf{K} is anti-Hermitian.

and can be decomposed into the sum

$$(j_1, 0) \otimes (j_2, 0) = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} (j, 0) . \quad (2.86)$$

Representations of the proper Lorentz group have been considered, thus, if one wants to include space inversion, there must be a matrix β which behaves such that

$$\beta \mathbf{J} \beta^{-1} = +\mathbf{J} , \quad \beta \mathbf{K} \beta^{-1} = -\mathbf{K} , \quad (2.87)$$

or, in terms of matrices (2.74) and (2.75)

$$\beta \mathbf{A} \beta^{-1} = \mathbf{B} , \quad \beta \mathbf{B} \beta^{-1} = \mathbf{A} . \quad (2.88)$$

Thus, an irreducible (A, B) representation of the proper Lorentz group does not provide a representation including space inversion unless $A = B$. It will be shown that (A, A) representations are the scalar, the vector and the symmetric traceless tensors. For $A \neq B$, the irreducible representation of Lorentz group including space inversion are the direct sums $(A, B) \oplus (B, A)$, of dimensionality $2(2A + 1)(2B + 1)$.

At this stage some (A, B) 's are going to be identified with the perhaps more familiar scalars, vectors, spinors and tensors.

Let start from the simplest one, the $(0, 0)$ representation. This corresponds to transformation

$$S_{\mu\nu} = 0 , \quad (2.89)$$

that is, if $\phi(x)$ is the field transforming according to this representation,

$$\delta_0 \phi = -\frac{i}{2} \epsilon^{\rho\sigma} J_{\rho\sigma} \phi(x) = -\frac{i}{2} \epsilon^{\rho\sigma} L_{\rho\sigma} \phi(x) . \quad (2.90)$$

Therefore, under Lorentz transformations,

$$\phi'(x') = \phi(x) , \quad (2.91)$$

which is a *scalar field* (it has the same value when measured in different inertial frames).

Consider now the most important representations of proper Lorentz group, that are the Weyl spinors $(1/2, 0)$ and $(0, 1/2)$. These are realised by two-component complex spinors. Let call conventionally $\psi_L(x)$, left-handed spinor, and $\psi_R(x)$, right-handed spinor, respectively. If one imposes that these fields satisfy Dirac-like equations, it simply turns out that they describe massless particles with helicity $\pm 1/2$. They are so important because, as it will be seen later, one is able to generate any other represen-

tation by opportunely multiplying them together. Write

$$\psi_L(x) \rightarrow \psi'_L(x') = \Lambda_L \psi_L(x) , \quad (2.92)$$

$$\psi_R(x) \rightarrow \psi'_R(x') = \Lambda_R \psi_R(x) , \quad (2.93)$$

where $\Lambda_{L,R}$ are 2×2 matrices with complex entries. When the transformation is a rotation, the form of $\Lambda_{L,R}$ is quite obvious from the spinor representation of $SU(2)$:

$$\Lambda_{L,R} = e^{i(\boldsymbol{\sigma} \cdot \boldsymbol{\omega})/2} \text{ (rotation) } , \quad (2.94)$$

where the ω_i 's are rotation parameters and the σ_i 's are the Hermitian 2×2 Pauli spin matrices. In other words, the rotation generators J^i are $\sigma^i/2$. Boosts cannot be represented unitarily. The representation

$$\mathbf{K} = -\frac{i}{2} \boldsymbol{\sigma} , \quad (2.95)$$

satisfies all the required commutation relations. Therefore, write

$$\Lambda_L = e^{i\boldsymbol{\sigma} \cdot (\boldsymbol{\omega} - i\boldsymbol{\nu})/2} , \quad (2.96)$$

where the ν^i 's are boost parameters. Since $(1/2, 0)$ and $(0, 1/2)$ representations are related by parity, construct Λ_R from Λ_L by changing the sign of boost parameters

$$\Lambda_R = e^{i\boldsymbol{\sigma} \cdot (\boldsymbol{\omega} + i\boldsymbol{\nu})/2} . \quad (2.97)$$

Note that Λ_L and Λ_R are related by

$$\Lambda_R = (\Lambda_L^{-1})^\dagger . \quad (2.98)$$

Let switch subject for few lines in order to present a frequently used notation. One already knows from representation theory of groups that, for a given group G , a n -dimensional representation $r(G)$ is a set of $n \times n$ matrices acting on a vector space. Consider a vector v_a , with $a = 1, \dots, n$, this transforms under the action of any matrix $r(g)_{ab}$ like

$$v_a \rightarrow v'_a = r(g)_{ab} v_b . \quad (2.99)$$

Once this representation $r(G)$ is given, one is able to construct other three ones: *complex conjugated* $r(G)^*$, *inverse transposed* $(r(G)^{-1})^T$ and *inverse Hermitian* $(r(G)^{-1})^\dagger$. One is interested in the last one, as suggested by (2.98). Conventionally one may write $v_{\dot{a}}$ a vector of the space on which $(r(G)^{-1})^\dagger$ acts and its transformation

$$v_{\dot{a}} \rightarrow v'_{\dot{a}} = r(g)_{\dot{a}b} v_b . \quad (2.100)$$

Also note that, if $r(G)$ is unitary, then there is no need for dotted indices, since the two representations are trivially equivalent. But this is not the case for representations of Lorentz group. Therefore, when spinor indices appear explicitly, they are written undotted (dotted) for $(1/2, 0)$ ($(0, 1/2)$)

$$\Lambda_L := \Lambda_{\alpha\beta} , \quad (2.101)$$

$$\Lambda_R := \Lambda_{\dot{\alpha}\dot{\beta}} . \quad (2.102)$$

If parity is concerned, one must consider the Dirac spinor representation $(1/2, 0) \oplus (0, 1/2)$. The simplest way to realise it is

$$\Psi := \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_a \\ \psi_{\dot{a}} \end{pmatrix} , \quad (2.103)$$

on which the operation of parity is well-defined

$$P : \Psi \rightarrow \Psi^P = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \Psi := \gamma_0 \Psi . \quad (2.104)$$

One projects only the left and the right spinors by means of the projection operators

$$\frac{1}{2}(\mathbb{I}_4 \pm \gamma_5) , \quad (2.105)$$

where

$$\gamma_5 := \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} . \quad (2.106)$$

Transformation properties are trivially

$$\Psi(x) \rightarrow S(\Lambda)\Psi(x) = \begin{pmatrix} \Lambda_L \psi_L(x) \\ \Lambda_R \psi_R(x) \end{pmatrix} = \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \Psi(x) . \quad (2.107)$$

An alternative method to obtain this representation is based on *Clifford's algebra*, i.e. γ -matrix algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{I}_4 . \quad (2.108)$$

Define

$$S_{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu] . \quad (2.109)$$

The $S_{\mu\nu}$'s satisfy the correct algebra (2.68). Furthermore it is easy to show that

$$[\gamma^\mu, S^{\nu\rho}] = i(g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu) , \quad (2.110)$$

or rather γ^μ behaves like a four-vector. Consider $A \in \text{SL}(2, \mathbb{C})$ and denote with $S(A)$ its Dirac representation, latter property suggests that by exponentiation

$$S(A)^{-1} \gamma^\mu S(A) = \Lambda^\mu{}_\nu(A) \gamma^\nu . \quad (2.111)$$

Let define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma , \quad (2.112)$$

and verify

$$\{\gamma^5, \gamma^\mu\} = 0 , \quad (\gamma^5)^2 = \mathbb{I}_4 . \quad (2.113)$$

Then one introduces two projectors like before

$$P_\pm = \frac{1}{2}(\mathbb{I}_4 \pm \gamma^5) , \quad (2.114)$$

such that subspaces obtained by projection are invariant under the action of $S(A)$

$$S(A)\gamma^5 S(A)^{-1} = \det(S(A)) \gamma^5 = \gamma^5 . \quad (2.115)$$

Fields are four-component spinors $\Psi(x)$ obeying Dirac equation

$$(i\gamma^\mu \partial_\mu + m)\Psi(x) = 0 , \quad (2.116)$$

which is covariant for the properties found above.

Let us consider the representation $(1/2, 1/2) = (1/2, 0) \otimes (0, 1/2)$. The decomposition (2.84) shows that the field in the representation $(1/2, 1/2)$ has components with $j = 1$, which is a three-vector, and $j = 0$, corresponding to a scalar under rotations. Thus, it describes a particle with spin 1. These fields can be represented by four-vectors

$$A^\mu(x) = (A^0, \mathbf{A}) , \quad (2.117)$$

where A^0 and \mathbf{A} are, with respect to rotations, the scalar and vector components, respectively. It can be shown that

$$(S_{\rho\sigma})_\mu{}^\nu = i(g_{\rho\mu}g^\nu{}_\sigma - g_{\sigma\mu}g^\nu{}_\rho) . \quad (2.118)$$

Let us consider a similar case: $(1/2, 0) \otimes (1/2, 0) = (0, 0) \oplus (1, 0)$. The scalar representation is given by the antisymmetric product. The representation $(1, 0)$ can be represented by an antisymmetric, self-dual second rank tensor, *i.e.* a tensor $F_{\mu\nu}$ which

obeys

$$\begin{aligned} F_{\mu\nu} &= -F_{\nu\mu} , \\ F_{\mu\nu} &= \frac{i}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma} . \end{aligned}$$

Indeed, the elements of $F_{\mu\nu}$ can be written as functions of the components of a three-vector $\mathbf{F} = (F_1, F_2, F_3)$

$$F_{\mu\nu} = \begin{pmatrix} 0 & F_1 & F_2 & F_3 \\ -F_1 & 0 & -iF_3 & iF_2 \\ -F_2 & iF_3 & 0 & -iF_1 \\ -F_3 & -iF_2 & iF_1 & 0 \end{pmatrix} . \quad (2.119)$$

Then the $(0, 1)$ representation would correspond to a tensor that is antisymmetric and anti-self-dual

$$F_{\mu\nu} = -\frac{i}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma} . \quad (2.120)$$

A matrix form, equal up to some signs to (2.119), corresponds to anti-self-dual tensors. For example, Maxwell's field strength tensor $F_{\mu\nu}$ transforms under the Lorentz group as $(0, 1) \oplus (1, 0)$. However, it is only in four dimensions that an antisymmetric two-index tensor can be divided into such self-dual and anti-self-dual parts.

A general tensor of rank N transforms as the direct product of N four vector representations $(1/2, 1/2)$. It can be decomposed into irreducible terms (A, B) with $A = N/2, N/2 - 1, \dots$ and $B = N/2, N/2 - 1, \dots$

General (A, A) fields contain terms with only integer spins $2A, 2A - 1, \dots, 0$ and they can be represented as traceless symmetric tensor of rank $2A$. Note in fact that the number of independent components of a symmetric traceless tensors of rank $2A$ in four dimensions is

$$(2A + 1)^2 , \quad (2.121)$$

as expected for (A, A) fields.

It has been shown how to realise spin-0, 1/2 and 1 fields. Now let build two different spin-3/2 fields. The first procedure is to take the product of three $(1/2, 0)$

$$(1/2, 0) \otimes (1/2, 0) \otimes (1/2, 0) = (3/2, 0) \oplus (1/2, 0) \oplus (1/2, 0) . \quad (2.122)$$

The spin-3/2 corresponds to the completely symmetric part of the product. Thus, a spin-3/2 field can be represented by a field totally symmetric in the interchange of its three L-like spinor indices. Its transformation properties are obtained by a suitable generalisation of the action on one L-like index. To include parity, one must combine

left and right contributions

$$(3/2, 0) \oplus (0, 3/2) . \quad (2.123)$$

A more convenient representation of spin-3/2 field is obtained through the product of a vector and a spinor

$$(1/2, 1/2) \otimes [(1/2, 0) \oplus (0, 1/2)] = (1, 1/2) \oplus (0, 1/2) \oplus (1/2, 1) \oplus (1/2, 0) . \quad (2.124)$$

The corresponding field quantity has four-vector and spinor indices

$$\Psi^\mu = \begin{pmatrix} \psi_L^\mu \\ \psi_R^\mu \end{pmatrix} , \quad (2.125)$$

which is the *Rarita-Schwinger* field when one projects out the extra $(1/2, 0) \oplus (0, 1/2)$ components imposing Lorentz invariant condition

$$\gamma^\mu \Psi_\mu = 0 . \quad (2.126)$$

According to the (2.84), such a field transforms under ordinary rotations as a direct sum of two $j = 3/2$ and two $j = 1/2$ components. The doubling is eliminated by imposing the Dirac equation $(\gamma^\nu \partial_\nu + m)\Psi^\mu = 0$ and the remaining $j = 1/2$ component is eliminated by requiring that

$$\partial_\mu \Psi^\mu = 0 . \quad (2.127)$$

With these conditions the field describes a single particle of spin $j = 3/2$.

The last considered example is that of spin-2 fields. Again there are many possible ways to describe a spin-2 field: $(2, 0)$, $(0, 2)$, $(1, 1)$. Let choose the latter for our brief discussion. It appears in the product

$$(1/2, 1/2) \otimes (1/2, 1/2) = [(0, 0) \oplus (1, 1)]_s \oplus [(0, 1) \oplus (1, 0)]_a , \quad (2.128)$$

where s and a denote the symmetric and antisymmetric parts. Thus, spin-2 field can be described by a second rank symmetric tensor $h_{\mu\nu}(x)$. The scalar component corresponds to its trace which can be subtracted by the traceless condition

$$g^{\mu\nu} h_{\mu\nu}(x) = 0 . \quad (2.129)$$

Finally let discuss one of the possible ways to realise fields transforming according to (A, B) . The following construction has been shown in previous examples for particular cases and it is going to be generalised. Define symmetric product \odot of representations

as the symmetric part of a tensor product \otimes . Thus, one can write

$$(A, B) = \underbrace{(1/2, 0) \odot \cdots \odot (1/2, 0)}_{A \text{ times}} \otimes \underbrace{(0, 1/2) \odot \cdots \odot (0, 1/2)}_{B \text{ times}} . \quad (2.130)$$

Therefore, this type of fields has $2A$ undotted indices and $2B$ dotted indices

$$\psi_{\alpha_1, \dots, \alpha_{2A}; \dot{\alpha}_1, \dots, \dot{\alpha}_{2B}}(x) , \quad (2.131)$$

and remains unchanged as a result of mutual permutations of indices both within the family $\alpha_1, \dots, \alpha_{2A}$ and within the family $\dot{\alpha}_1, \dots, \dot{\alpha}_{2B}$. It transforms like

$$\psi'_{\alpha_1, \dots, \alpha_{2A}; \dot{\alpha}_1, \dots, \dot{\alpha}_{2B}}(x') = \Lambda_{\alpha_1 \beta_1} \cdots \Lambda_{\alpha_{2A} \beta_{2A}} \Lambda_{\dot{\alpha}_1 \dot{\beta}_1} \cdots \Lambda_{\dot{\alpha}_{2B} \dot{\beta}_{2B}} \psi_{\beta_1, \dots, \beta_{2A}; \dot{\beta}_1, \dots, \dot{\beta}_{2B}}(x) .$$

The same result reached in an alternative way can be found in L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory*, in §57. An alternative way to construct quantum field operators which transform like (A, B) , starting from the expression of operators in free theory and requiring causal conditions, can be found in chapter 5 of S. Weinberg, *The Quantum Theory of Fields, Volume I*. Weinberg shows the close relation between commutation relations of the field and spin associated to the representation (A, B) . He also treats the most general form of PCT theorem and the differences between massless and massive particles.

Other good references are the chapter 1 of Ramond's book and R. Slansky, *Group theory for unified model building*: both texts show how to realise Lorentz invariants starting from the fields presented before, in particular they pay much attention to the construction of real scalar invariants, because the Lagrangian density of a chosen theory is so.

Chapter 3

Review of the Dirac Equation¹

Dirac sought a linear first order field differential equation to hopefully circumvent the lack of a positive definite conserved charge and the presence of negative energy solutions, both arising from Klein-Gordon equation. In 1928 he proposed

$$i \frac{\partial}{\partial t} \psi = (-i \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m) \psi =: H_D \psi .$$

The only further requirements Dirac imposed on this equation were the self-adjointness of Hamiltonian H_D and the consistency of the solutions with the relativistic expression for energy $\omega_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$. Using the correspondence principle and identifying the Hermitian operator $-i \boldsymbol{\nabla}$ with the momentum \mathbf{p} , one can see that α^k and β should be Hermitian objects with nontrivial commutation relations

$$\{\alpha^j, \alpha^k\} = 2\delta^{jk} \mathbb{I}_4 , \quad \{\beta, \alpha^j\} = 0 , \quad \beta^2 = \mathbb{I}_4 ,$$

due to

$$(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)^2 = [(\boldsymbol{\alpha} \cdot \mathbf{p})^2 + (\boldsymbol{\alpha} \beta + \beta \boldsymbol{\alpha}) \cdot \mathbf{p} m + \beta^2 m^2] \stackrel{!}{=} (\mathbf{p}^2 + m^2) \mathbb{I}_4 .$$

The simplest possible representation of the previous algebra in terms of complex matrices is four dimensional, thus Dirac assumed ψ to be a four component complex field. As a final comment, notice that, since the consistency with the relativistic expression for the energy is the only requirement of the Klein-Gordon equation, Dirac equation is expected to imply Klein-Gordon.

The matrices α^k, β introduced with the Dirac equation, and the equation itself, can be

¹ Marco Rigobello

expressed in terms of four other matrices

$$\gamma^0 = \beta, \quad \gamma^k = \beta \alpha^k.$$

These gamma matrices are generators of the *Clifford algebra* of Minkowski spacetime, indeed the defining conditions for α^k and β are equivalent to²

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{I}_4. \quad (3.1)$$

Other Clifford algebra matrices of great relevance in physics are

$$\gamma^5 = \gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

Often used in computations involving gamma matrices is the *slashed notation*

$$\not{a} := \gamma^\mu a_\mu.$$

Conditions (3.1) do not fix completely the explicit form of the gamma matrices. If a set of matrices $\{\gamma\}$ satisfies these conditions, the same happens for $\{\gamma'\} = U\{\gamma\}U^{-1}$. Probably, the most common gamma matrices representations are

$$\begin{aligned} \text{Dirac :} \quad \gamma^0 &= \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \\ \text{Weyl/Chiral :} \quad \gamma^0 &= \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}. \end{aligned}$$

Also to mention is Majorana representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix},$$

defined in such a way as to make Dirac equation real.

Some useful γ properties independent of the representation are

$$(\gamma^0)^2 = \mathbb{I}_4, \quad (\gamma^k)^2 = -\mathbb{I}_4, \quad (\gamma^5)^2 = \mathbb{I}_4,$$

scalar products

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}, \quad \text{and, in particular,} \quad \not{a} \not{b} = ab\mathbb{I}_4 - i\sigma^{\mu\nu} a_\mu b_\nu,$$

² Sometimes in the following the $n \times n$ identity matrix \mathbb{I}_n will be omitted.

γ^5 anticommutes with all the gamma matrices and commutes with $\sigma^{\mu\nu}$

$$\{\gamma_5, \gamma^\mu\} = 0, \quad [\gamma_5, \sigma^{\mu\nu}] = 0,$$

traces identities

$$\text{Tr}(\text{odd number of } \gamma^{0,1,2,3} \text{ matrices}) = 0, \quad \text{Tr}(\sigma^{\mu\nu}) = 0, \quad \dots,$$

and contraction identities

$$\gamma^\mu \gamma_\mu = 4\mathbb{I}_4, \quad \gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu, \quad \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho}, \quad \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu.$$

The following properties hold in all the representations here introduced

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^k)^\dagger = -\gamma^k, \quad (\gamma^5)^\dagger = \gamma^5,$$

and γ^0 can be used to get Hermitian conjugates

$$\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger, \quad \gamma^0 \gamma_5 \gamma^0 = -(\gamma_5)^\dagger, \quad \gamma^0 \sigma^{\mu\nu} \gamma^0 = (\sigma^{\mu\nu})^\dagger.$$

Note that these properties on Hermitian conjugates, in general, are not independent of the representation.³ However, it can be proved that the Clifford algebra of Minkowski spacetime admits a unique four-dimensional representation up to equivalence. Therefore, one can always assume that these properties hold, for it suffices to change the basis in the representation space.

Using the notions developed above it is possible to rewrite Dirac equation in the form

$$(i\cancel{\not{D}} - m) \psi = 0. \quad (3.2)$$

It is now immediate to verify two requirements outlined at the beginning of this section: the existence of a positive definite conserved charge and the relation with Klein-Gordon.

- (i) Making also use of the equation for the *Dirac conjugate spinor* $\bar{\psi} := \psi^\dagger \gamma^0$, i.e.

³ To see this, suppose that for a particular representation r it holds $r(\gamma_0) = r(\gamma_0)^\dagger$. Assume that the same property holds for all the equivalent representations $\tilde{r}(\gamma^0) = U r(\gamma^0) U^{-1}$. Then it follows $U r(\gamma^0) U^{-1} = (U^\dagger)^{-1} r(\gamma^0) U^\dagger$ for every invertible matrix U , that is $U^\dagger U r(\gamma^0) = r(\gamma^0) U^\dagger U$ for every invertible matrix U . Since every invertible Hermitian matrix can be written in the form $A = U^\dagger U$ (for example with $U = \sqrt{A} = \sqrt{A}^\dagger$), it follows that $r(\gamma^0)$ commutes with all the invertible Hermitian matrices, which is equivalent to the fact that it commutes with all the Hermitian matrices (simply use diagonalisation or Jordan canonical form to write a generic matrix as the sum of two invertible matrices). Since every matrix can be written in the form $C = A + iB$ with A and B Hermitian (for example $C = \frac{C+C^\dagger}{2} + i\frac{C-C^\dagger}{2i}$), it follows that $r(\gamma^0)$ commutes with every matrix, which is a contradiction since we must have $r(\gamma^0)r(\gamma^i) = -r(\gamma^i)r(\gamma^0)$.

$$\bar{\psi} \left(i\overleftarrow{\not{\partial}} + m \right) = 0 ,$$

the sought conserved current is obtained

$$j^\mu = \bar{\psi} \gamma^\mu \psi , \quad \partial_\mu j^\mu = 0 , \quad \rho := j^0 = \psi^\dagger \psi > 0 .$$

- (ii) That the Dirac equation requires each spinor component to satisfy Klein-Gordon equation is just a consequence of $(\square + m^2) = -(i\not{\partial} + m)(i\not{\partial} - m)$.

Negative energy solutions are however still present at this level, as will soon be clear. This problem is solved by the quantisation procedure and the consequent redefinition of the energy and reinterpretation of (old) negative energy solutions as antiparticles.

3.1 Lorentz covariance of the Dirac equation

The relativistic principle states that laws of physics have the same form in all inertial frames of reference. That is, they are invariant in form under Poincaré transformations $x' = \Lambda x + a$. Let us apply this prescription to the Dirac equation. In general

$$\psi'(x') = T_{(\Lambda,a)} \psi(x) ,$$

where T is the action of Poincaré group on the spinorial space. Denoting the Dirac linear operator by $D(x) = i\not{\partial}_x - m$, the relativistic principle reads

$$D(x')\psi'(x') = 0 \quad \Leftrightarrow \quad D(x)\psi(x) = 0 .$$

Assume that also $T_{(\Lambda,a)}$ acts linearly on spinors. Moreover, under translations $x \mapsto x' = x + a$, the Dirac operator is invariant $D(x') = D(x)$, therefore, in this case $\psi'(x') = \psi(x)$. These observations can be summarised by saying that $T_{(\Lambda,a)}$ is a linear map on spinors depending only on Λ . It defines a representation S of the Lorentz group by setting $S(\Lambda) = T_{(\Lambda,a)}$. Thus,

$$\psi'(x') = S(\Lambda)\psi(x) . \tag{3.3}$$

The introduced notation allows to restate the relativistic principle for the Dirac equation arguing that, if ψ is a solution of Dirac equation, than also $S(\Lambda)(\psi \circ \Lambda^{-1})$ is a solution, that is⁴

$$(i\not{\partial} - m) S(\Lambda)\psi(\Lambda^{-1}x) = 0 \quad \Leftrightarrow \quad (i\not{\partial} - m) \psi(x) = 0 .$$

⁴ Simply rename x' to x in the Lorentz transformed equation.

Replacing $\psi'(x')$ in $D(x')\psi'(x') = 0$ by $S(\Lambda)\psi(x)$, we get⁵

$$S(\Lambda) (iS^{-1}(\Lambda)\gamma^\mu S(\Lambda)\Lambda_\mu{}^\nu \partial_\nu - m) \psi(x) = 0 ,$$

that, compared with $D(x)\psi(x) = 0$, gives the following condition on $S(\Lambda)$

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu . \quad (3.4)$$

Explicit realisation of Lorentz transformations for spinors

To find an explicit expression of $S(\Lambda)$ let us parametrise its infinitesimal form in term of the Lie algebra generators. Recall that the defining representation of an orthogonal group is generated by skew-symmetric linear applications

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \dots , \quad \text{with } \omega_{\rho\sigma} = \omega_{[\rho\sigma]} .$$

In particular, for the Lorentz group, via 6 linearly independent generators

$$(\mathcal{J}^{\rho\sigma})^\mu{}_\nu = i(g^{\mu\rho}\delta^\sigma{}_\nu - g^{\mu\sigma}\delta^\rho{}_\nu) , \quad \mathcal{J}^{\sigma\rho} = -\mathcal{J}^{\rho\sigma} ,$$

it is possible to write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu + \dots .$$

In analogy with this, introduce the following parametrisation of the spinorial case⁶

$$S(\Lambda)^\alpha{}_\beta = \delta^\alpha{}_\beta - \frac{i}{2}\omega_{\rho\sigma}(\Sigma^{\rho\sigma})^\alpha{}_\beta + \dots .$$

Notice that again, even if the generators are labeled by two Lorentz indices, only 6 of them are independent. Also, as for the defining representation, we only need the antisymmetric part of $\omega_{\rho\sigma}$ to contribute. This means $\Sigma^{\sigma\rho} = -\Sigma^{\rho\sigma}$.

Expanding (3.4) at first order, one gets

$$[\Sigma^{\rho\sigma}, \gamma^\mu] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu = i(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho) ,$$

which, as a possible solution, has

$$\Sigma^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\sigma^{\rho\sigma} .$$

⁵ Clearly $S^{-1}(\Lambda)$ stands for the matrix inverse of $S(\Lambda)$.

⁶ α and β are spinor indices.

Finite transformations are recovered by exponentiation

$$S(\Lambda) = \exp\left(-\frac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right). \quad (3.5)$$

The spinorial representation of the Lorentz group satisfies

- ▷ $S^\dagger = S$ for $S = \exp\left(-\frac{i}{2}\omega_{0k}\sigma^{0k}\right)$ (boosts),
- ▷ $S^\dagger = S^{-1}$ for $S = \exp\left(-\frac{i}{4}\omega_{jk}\sigma^{jk}\right)$ (spatial rotations).

To prove this, first of all observe that, as anticipated,

$$(\sigma^{\mu\nu})^\dagger = -\frac{i}{2}[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = \frac{i}{2}[\gamma^0\gamma^\mu\gamma^0, \gamma^0\gamma^\nu\gamma^0] = \frac{i}{2}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 = \gamma^0\sigma^{\mu\nu}\gamma^0.$$

Using the Clifford algebra (3.1), this specialises to

- ▷ $(\sigma^{0k})^\dagger = -\sigma^{0k}$ (one $\gamma^0, \gamma^{1,2,3}$ exchange),
- ▷ $(\sigma^{jk})^\dagger = \sigma^{jk}$ (two $\gamma^0, \gamma^{1,2,3}$ exchanges).

Finally, by $(e^A)^\dagger = e^{(A^\dagger)}$, the claim is proved.

Lorentz transformation of Dirac conjugate spinor

Using only the definition $\bar{\psi} = \psi^\dagger\gamma^0$ and $\gamma^{0^2} = \mathbb{I}_4$,

$$\bar{\psi}'(x') = (\psi')^\dagger(x')\gamma^0 = \bar{\psi}(x)\gamma^0 S^\dagger\gamma^0.$$

Introducing (3.5) and using $Af(B)A^{-1} = f(ABA^{-1})$ yields

$$\gamma^0 S^\dagger\gamma^0 = \gamma^0 \exp\left(\frac{i}{4}\omega_{\rho\sigma}(\sigma^{\rho\sigma})^\dagger\right)\gamma^0 = \exp\left(\frac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right) = S^{-1}(\Lambda),$$

so that

$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}(\Lambda), \quad (3.6)$$

as one would expect. This relation is used to derive the transformation properties of important objects called fermionic bilinears, which are introduced in §4.2.5.

Some properties of the spinorial representation⁷

Poincaré group representations are classified according to the eigenvalues of two *Casimir operators*. In a generic representation, let P_μ be the four generators of infinitesimal translations, and $J_{\mu\nu}$ be the generators of Lorentz transformations. The two Casimirs are P^2 and W^2 , where

$$W_\mu := -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma, \quad (3.7)$$

⁷ The topics summarised here are discussed in more detail in the chapter on representations.

is the *Pauli-Lubanski pseudovector*. The representation is then classified by a *mass* M^2 , the eigenvalue of P^2 , and a *spin* s , related to W^2 via

$$W^2 = -M^2 s(s+1) . \quad (3.8)$$

Let us specialise to the representation acting on spinor fields.⁸

It is clear that $P_\mu = -i\partial_\mu \mathbb{I}_4$, indeed at first order

$$(\mathbb{I}_4 - ia^\mu P_\mu)\psi(x) = (\mathbb{I}_4 + a^\mu \partial_\mu)\psi(x) = \psi(x+a) .$$

Both the Klein-Gordon and Dirac equations imply that in the space of states on which the representation acts $M^2 = P^2 = m^2$.

In order to find the expression of $J^{\rho\sigma}$, we first note that at the first order we must have

$$\psi'(x) = \left(\mathbb{I}_4 - \frac{i}{2} \omega_{\rho\sigma} J^{\rho\sigma} \right) \psi(x) .$$

On the other hand, the expansion of (3.3) reads

$$\begin{aligned} \psi'(x) &= \left(\mathbb{I}_4 - \frac{i}{4} \omega_{\rho\sigma} \sigma^{\rho\sigma} \right) \psi(x^\mu - \omega^\mu{}_\nu x^\nu) = \left(1 - \frac{i}{4} \omega_{\rho\sigma} \sigma^{\rho\sigma} - x^\nu \omega^\mu{}_\nu \partial_\mu \right) \psi(x) \\ &= \left[\mathbb{I}_4 - \frac{i}{4} \omega_{\rho\sigma} \sigma^{\rho\sigma} + \frac{1}{2} \omega_{\rho\sigma} (x^\rho \partial^\sigma - x^\sigma \partial^\rho) \right] \psi(x) , \end{aligned}$$

where the antisymmetry of $\omega_{\rho\sigma}$ has been used. Comparing the previous two expansions, we get

$$J_{\rho\sigma} = \frac{1}{2} \sigma_{\rho\sigma} + i(x_\rho \partial_\sigma - x_\sigma \partial_\rho) , \quad (3.9)$$

where the $\sigma_{\rho\sigma}$ term is related to internal (“spin”) angular momentum, whereas the other contribution concerns the external (“orbital”) angular momentum. This nomenclature is clearer by noticing that only the first term contributes to W^2 , hence to the spin of the representation. Indeed, due to the antisymmetry of the Levi-Civita symbol, and after some algebra, we have

$$W_\mu = -\frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \sigma^{\nu\rho} P^\sigma , \quad W^2 = -\frac{1}{2} \left(\frac{1}{2} + 1 \right) M^2 .$$

One concludes that spinor fields have spin 1/2 (trivially scalar fields have spin 0.)

A last mention concerns the reducibility of the representation $S(\Lambda)$. This property is

⁸ Keep in mind that the physical states are spinor valued functions of spacetime, rather than spinors. Notice that the former belong to an infinite dimensional vector space, therefore the corresponding representation of the Poincaré group is infinite dimensional.

manifest if the chiral representation of the γ matrices is used. In this representation

$$\sigma^{0k} = \frac{i}{2}[\gamma^0, \gamma^k] = \begin{pmatrix} -i\sigma^k & 0 \\ 0 & i\sigma^k \end{pmatrix}, \quad \sigma^{jk} = \frac{i}{2}[\gamma^j, \gamma^k] = \varepsilon^{jkl} \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}.$$

Therefore, the two components χ, η of the bispinor $\psi = (\chi, \eta)$, transform independently under Lorentz transformations. Equivalently,

$$\psi \in \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right).$$

The representation becomes irreducible if parity is included.

3.2 Discrete transformations of the Dirac field

In the derivation of the spinors transformations $S(\Lambda)$, we proceeded first identifying infinitesimal transformations, and then exponentiating to get the corresponding finite transformations. In doing so we limited ourselves to the study of the component of the Lorentz group connected to the identity, the restricted Lorentz group. The description of the full Lorentz group is recovered if the parity P and time reversal T discrete transformations are considered. The representative of these two, which are not of the type (3.5), are discussed in the following, together with charge conjugation C .

Parity

The parity transformation acts on spacetime by

$$\mathcal{P} : x^\mu = (x^0, \mathbf{x}) \mapsto \tilde{x}^\mu = (x^0, -\mathbf{x}),$$

that is, in the vectorial representation of the Lorentz group, \mathcal{P} is given by

$$\Lambda_{\mathcal{P}} = \text{diag}(1, -1, -1, -1).$$

To find its spinorial equivalent $S_{\mathcal{P}} := S(\Lambda_{\mathcal{P}})$, rewrite (3.4) as

$$S_{\mathcal{P}}^{-1} (\gamma^0, \boldsymbol{\gamma}) S_{\mathcal{P}} = (\gamma^0, -\boldsymbol{\gamma}) = \gamma^0 \boldsymbol{\gamma} \gamma^0.$$

where in the last step the defining anticommutator of the Clifford algebra has been used. Then, it is straightforward to set

$$S_{\mathcal{P}} = \eta_{\mathcal{P}} \gamma^0,$$

where $\eta_{\mathcal{P}} = e^{i\varphi_{\mathcal{P}}}$ is just an arbitrary phase. The action of parity on spinors is therefore

$$\mathcal{P} : \psi \longmapsto \psi^{\mathcal{P}} = \eta_{\mathcal{P}} \gamma^0 (\psi \circ \Lambda_{\mathcal{P}}^{-1}) . \quad (3.10)$$

Solutions with positive and negative energy have opposite parity, corresponding to opposite γ^0 eigenvalues. This is manifest in the Dirac representation of the γ algebra

$$\psi^{\mathcal{P}}(t, \mathbf{x}) = \eta_{\mathcal{P}} \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} \Big|_{\Lambda_{\mathcal{P}}^{-1}(t, \mathbf{x})} = \eta_{\mathcal{P}} \begin{pmatrix} \chi \\ -\eta \end{pmatrix} \Big|_{(t, -\mathbf{x})} .$$

Charge conjugation

Dirac equation should have a symmetry associated to the exchange

$$\text{particle} \longleftrightarrow \text{antiparticle} , \quad \psi \longleftrightarrow \psi^{\mathcal{C}} ,$$

where $\psi^{\mathcal{C}}$ corresponds to the particle of opposite charge of ψ . We expect the charge conjugation transformation to be local and physically involutory (such that its square amounts at most to the multiplication by an unobservable phase).

To determine the expression of the charge conjugation constructively, couple the Dirac field to an external electromagnetic field. The field equation is obtained substituting the ordinary derivative with the covariant derivative D . This provides,

$$(i\mathcal{D} - m) \psi = 0 , \quad D_{\mu} = \partial_{\mu} + ieA_{\mu} , \quad (3.11)$$

or, more explicitly,

$$[\gamma^{\mu}(i\partial_{\mu} - eA_{\mu}) - m] \psi = 0 . \quad (3.12)$$

The substitution

$$A_{\mu} \rightarrow A_{\mu}^{\mathcal{C}} , \quad \psi \rightarrow \psi^{\mathcal{C}} ,$$

yields the charge conjugated equation of (3.12)

$$[\gamma^{\mu}(i\partial_{\mu} - eA_{\mu}^{\mathcal{C}}) - m] \psi^{\mathcal{C}} = 0 ,$$

Formally, the previous replacements should be equivalent to

$$e \rightarrow -e , \quad \psi \rightarrow \psi^{\mathcal{C}} .$$

In order for this to happen, it is natural to set

$$A_{\mu}^{\mathcal{C}} := -A_{\mu} ,$$

finally obtaining

$$[\gamma^\mu(i\partial_\mu + eA_\mu) - m]\psi^c = 0 . \quad (3.13)$$

From the comparison of (3.13) and (3.12) the expression of ψ^c can be deduced. First of all observe that, in the charge conjugated equation, the terms inside round brackets have the same sign. To emulate this behaviour one could, as a first step, take the complex conjugate⁹ of (3.12). To do so take the adjoint and then the transpose, obtaining

$$[-(\gamma^\mu)^T(i\partial_\mu + eA_\mu) - m]\bar{\psi}^T = 0 .$$

Given the similarity between this equation and (3.13), it seems convenient to parametrise the action of charge conjugation on spinors via

$$\psi^c := \eta_c \mathbf{C} \bar{\psi}^T , \quad (3.14)$$

where η_c is just an unobservable phase. The previous is only a definition, we still have to find a defining condition of C . To this aim, notice that if charge conjugation is a symmetry of the Dirac equation then (3.13) must hold if, and only if, (3.12) is verified. This is equivalent to requiring

$$\mathbf{C}^{-1} [\gamma^\mu(i\partial_\mu + eA_\mu) - m] \mathbf{C} (\mathbf{C}^{-1}\psi^c) = 0 \quad \Leftrightarrow \quad [-(\gamma^\mu)^T(i\partial_\mu + eA_\mu) - m] \bar{\psi}^T = 0 ,$$

that gives immediately the desired condition, defining the action of charge conjugation

$$\mathcal{C} : \psi \longmapsto \psi^c = \eta_c \mathbf{C} \bar{\psi}^T , \quad \text{with} \quad \mathbf{C}^{-1} \gamma^\mu \mathbf{C} = -(\gamma^\mu)^T . \quad (3.15)$$

Notice that the evaluation point of the spinor field is not involved in the transformation.

The explicit realisation of C depends on the chosen γ matrix basis. In Dirac basis

$$\mathbf{C} = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} , \quad \mathbf{C}^{-1} = \mathbf{C}^T = \mathbf{C}^\dagger = -\mathbf{C} .$$

That this satisfies the requirement in (3.15), or equivalently

$$\gamma^\mu \mathbf{C} = -\mathbf{C} (\gamma^\mu)^T , \quad (3.16)$$

is easily seen, recalling that $(\gamma^\mu)^T = (-1)^\mu \gamma^\mu$.

⁹ Indeed, in retrospect, if electromagnetic gauge transformations are considered, complex conjugation seems a quite natural starting point for the implementation of charge conjugation. Under these the fields transform as $\psi \mapsto e^{-ie\alpha(x)}\psi$ where e is the electric charge of the field, thus the complex conjugated field behaves as if its charge has the opposite sign.

Time reversal

The discussion of time reversal is postponed to the quantised theory where some important general observations can be carried out more naturally.

3.3 Solutions of the Dirac equation

As already shown, each component of a spinor obeying Dirac equation, satisfies also Klein-Gordon equation. As a first step, we can thus express the most general solution of Dirac's equation as a superimposition of positive and negative energy plane waves:

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} \left(b_{\alpha}(k) \underbrace{u^{(\alpha)}(k)e^{-ikx}}_{\psi_k^{(+)}(x)} + d_{\alpha}^*(k) \underbrace{v^{(\alpha)}(k)e^{ikx}}_{\psi_k^{(-)}(x)} \right) \Big|_{k=(\omega_{\mathbf{k}},\mathbf{k})}, \quad (3.17)$$

$$\bar{\psi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} \left(d_{\alpha}(k) \bar{v}^{(\alpha)}(k)e^{-ikx} + b_{\alpha}^*(k) \bar{u}^{(\alpha)}(k)e^{ikx} \right) \Big|_{k=(\omega_{\mathbf{k}},\mathbf{k})}, \quad (3.18)$$

where, as usual, $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ for a particle of mass $m \neq 0$.

In the rest frame $k = (m, \mathbf{0})$, the Dirac equation for these plane waves gives

$$\begin{aligned} (i\rlap{\not{\partial}} - m) \psi_k^{(+)}(x) = 0 &\Rightarrow (\rlap{\not{k}} - m) u(m, \mathbf{0}) = m(\gamma^0 - \mathbb{I}_4) u(m, \mathbf{0}) = 0, \\ (i\rlap{\not{\partial}} - m) \psi_k^{(-)}(x) = 0 &\Rightarrow (\rlap{\not{k}} + m) v(m, \mathbf{0}) = m(\gamma^0 + \mathbb{I}_4) v(m, \mathbf{0}) = 0. \end{aligned}$$

There are two linearly independent solutions for each of these two equations. In the Dirac representation of the γ matrices they are

$$u^{(1)}(m, \mathbf{0}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)}(m, \mathbf{0}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^{(1)}(m, \mathbf{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{(2)}(m, \mathbf{0}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where the normalisation has been fixed by

$$\bar{u}^{(\alpha)} u^{(\beta)} = \delta^{\alpha\beta}, \quad \bar{v}^{(\alpha)} v^{(\beta)} = -\delta^{\alpha\beta}, \quad \bar{u}^{(\alpha)} v^{(\beta)} = 0, \quad \bar{v}^{(\alpha)} u^{(\beta)} = 0.$$

Notice that the transformation properties of spinors described in section 3.1 make these normalisation conditions Lorentz invariant.

To get $u^{(\alpha)}(k)$ and $v^{(\alpha)}(k)$ in a generic reference frame, a boost must be made via

$$S(\Lambda) = \exp\left(-\frac{i}{2}\eta^j \sigma^{0j}\right), \quad \boldsymbol{\eta} = \frac{\mathbf{k}}{\|\mathbf{k}\|} \operatorname{atanh} \frac{\|\mathbf{k}\|}{\omega_{\mathbf{k}}}.$$

However, the boosted spinors can also be obtained also observing that

$$(\rlap{\not{k}} - m)(\rlap{\not{k}} + m) = k^2 - m^2 = 0,$$

thus for plane waves

$$u^{(\alpha)}(k) = (\not{k} + m) u^{(\alpha)}(m, \mathbf{0}) , \quad v^{(\alpha)}(k) = (\not{k} - m) v^{(\alpha)}(m, \mathbf{0}) ,$$

are the sought after solutions. Indeed, these are for sure solutions of the Dirac equation. To see they are the most general ones, it is sufficient to observe that they are linearly independent.¹⁰ After applying the normalisation conditions, the explicit expression for these solutions reads

$$u^{(\alpha)}(k) = \frac{1}{\sqrt{2m}} \frac{\not{k} + m}{\sqrt{m + \omega_{\mathbf{k}}}} u^{(\alpha)}(m, \mathbf{0}) = \frac{1}{\sqrt{2m}} \begin{pmatrix} \sqrt{m + \omega_{\mathbf{k}}} e_{\alpha} \\ \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{\sqrt{m + \omega_{\mathbf{k}}}} e_{\alpha} \end{pmatrix} ,$$

$$v^{(\alpha)}(k) = \frac{1}{\sqrt{2m}} \frac{-\not{k} + m}{\sqrt{m + \omega_{\mathbf{k}}}} v^{(\alpha)}(m, \mathbf{0}) = \frac{1}{\sqrt{2m}} \begin{pmatrix} \frac{\boldsymbol{\sigma} \cdot \mathbf{k}}{\sqrt{m + \omega_{\mathbf{k}}}} e_{\alpha} \\ \sqrt{m + \omega_{\mathbf{k}}} e_{\alpha} \end{pmatrix} ,$$

where e_{α} are the canonical basis vectors of \mathbb{R}^2 . Analogously, for the conjugate spinors,

$$\bar{u}^{(\alpha)}(k) = \bar{u}^{(\alpha)}(m, \mathbf{0}) \frac{1}{\sqrt{2m}} \frac{\not{k} + m}{\sqrt{m + \omega_{\mathbf{k}}}} ,$$

$$\bar{v}^{(\alpha)}(k) = \bar{v}^{(\alpha)}(m, \mathbf{0}) \frac{1}{\sqrt{2m}} \frac{-\not{k} + m}{\sqrt{m + \omega_{\mathbf{k}}}} .$$

The degeneracy of positive and negative solutions is related to spin degrees of freedom. In particular, direct computation in Dirac basis yields for the z -component of the spin

$$\Sigma^3 = \Sigma^{12} = \frac{1}{2} \text{diag}(+1, -1, +1, -1) ,$$

thus the basis chosen above is such that the spinors $u^{(\alpha)}(m, \mathbf{0})$ and $v^{(\alpha)}(m, \mathbf{0})$ have the z -component of the spin equal to $(-1)^{\alpha-1}/2$. Another observable, related to spin, that can be used to remove the degeneracy is *helicity*, i.e. spin projection along the direction of motion. The basis in which helicity is diagonal is called *helicity basis*.

¹⁰ The number of degrees of freedom cannot depend on the reference frame chosen.

Chapter 4

Quantised Free Fields

4.1 Review of the operator formalism

Consider the two point Green function in the case of a scalar field theory

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle .$$

Where $|\Omega\rangle$ denotes the vacuum of the interacting theory and $\phi(x)$ is the field operator in the Heisenberg picture. If $\phi(\mathbf{x})$ is its analogue in the Schrödinger picture, then

$$\phi(x) = e^{iHt} \phi(\mathbf{x}) e^{-iHt} .$$

Let λ be the self-interaction constant, such that $\lambda = 0$ corresponds $V = 0$. An example of interaction satisfying these requirements is $V(\phi) = \frac{\lambda}{4!} \phi^4$. If $\lambda = 0$, H is just the free Hamiltonian H_0 . The operator

$$\phi_I(t, \mathbf{x}) := \phi(t, \mathbf{x})|_{\lambda=0} = e^{iH_0(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH_0(t-t_0)} .$$

is referred to as field operator in the interaction picture. The following expression for the two point function yields

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) \exp[-i \int_{-T}^T dt H_I(t)] \} | 0 \rangle}{\langle 0 | T \exp[-i \int_{-T}^T dt H_I(t)] \} | 0 \rangle} , \quad (4.1)$$

where $|0\rangle$ is the vacuum of the free theory. Contrary to H , the interaction potential density in the interaction picture

$$H_I(t) = e^{iH_0(t-t_0)} H_{\text{int}} e^{-iH_0(t-t_0)} ,$$

has an explicit time dependence. In ϕ^4 -theory

$$H_{\text{int}} = \frac{\lambda}{4!} \int d^3x \phi^4(\mathbf{x}) \quad \text{and} \quad H_I(t) = \frac{\lambda}{4!} \int d^3x \phi_I^4(x) .$$

4.2 Canonical quantisation of the Dirac field¹

With the quantisation ψ and $\bar{\psi}$ become operator valued fields. Nevertheless, they can be expanded in plane waves, this time with operatorial coefficients $b_\alpha^{(\dagger)}, d_\alpha^{(\dagger)}$

$$\psi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} (b_\alpha(k)u^{(\alpha)}(k)e^{-ikx} + d_\alpha^\dagger(k)v^{(\alpha)}(k)e^{ikx})|_{k=(\omega_{\mathbf{k}},\mathbf{k})} , \quad (4.2)$$

$$\bar{\psi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} (d_\alpha(k)\bar{v}^{(\alpha)}(k)e^{-ikx} + b_\alpha^\dagger(k)\bar{u}^{(\alpha)}(k)e^{ikx})|_{k=(\omega_{\mathbf{k}},\mathbf{k})} . \quad (4.3)$$

As usual, for a particle of mass m , $\omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2}$.

The operatorial coefficients have the following physical roles

- $b_\alpha(k)$ destroy a particle of 4-momentum k ,
- $d_\alpha(k)$ destroy an antiparticle of 4-momentum k ,
- $b_\alpha^\dagger(k)$ create a particle of 4-momentum k ,
- $d_\alpha^\dagger(k)$ create an antiparticle of 4-momentum k .

The u and v are used as in the previous chapter, to construct 4-momentum eigenstates:

$\psi^{(+)}(x) = e^{-ikx}u(k)$ is a positive energy solution of the Dirac equation,

$\psi^{(-)}(x) = e^{ikx}v(k)$ is a negative energy solution of the Dirac equation.

4.2.1 Anticommutation relations

The properties of the quantised Dirac field are easily described in terms of some relations between the operators $b_\alpha^{(\dagger)}, d_\alpha^{(\dagger)}$, which are analogous to the commutation relations imposed for the corresponding operators in the scalar field case. To identify these conditions, first of all observe that both the Dirac equation and its counterpart for the conjugate spinor

$$(i\rlap{\not{D}} - m)\psi = 0 , \quad \bar{\psi}(i\overleftarrow{\not{D}} + m) = 0 ,$$

¹ Marco Rigobello

follow, as Euler-Lagrange equations, from one of the Lagrangian densities²

$$\tilde{\mathcal{L}} = \bar{\psi}(i\overleftrightarrow{\not{\partial}} - m)\psi, \quad \mathcal{L} = \frac{i}{2} [\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi = \bar{\psi} \left(\frac{i}{2}\overleftrightarrow{\not{\partial}} - m \right) \psi. \quad (4.4)$$

Both Lagrangians vanish on-shell. Notice also how $\mathcal{L} = (\tilde{\mathcal{L}} + \tilde{\mathcal{L}}^\dagger)/2$ is real by construction, whereas $\tilde{\mathcal{L}}$ is not. Given the invariance under spacetime translations of \mathcal{L} , Noether's theorem provides the stress energy tensor as a conserved current:

$$T^\mu_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(\partial_\nu\psi) + (\partial_\nu\bar{\psi})\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} - \mathcal{L}g^\mu_\nu = \frac{i}{2}\bar{\psi}\gamma^\mu\overleftrightarrow{\partial}_\nu\psi, \quad \partial_\mu T^\mu_\nu = 0.$$

This local conservation, together with the appropriate vanishing behaviour of the components of T^μ_ν at spacial infinity, implies the global conservation of the 4-momentum. Inserting the Fourier decomposed fields (4.2)–(4.3) one finds

$$P_\nu = \int d^3\mathbf{x} T^0_\nu(0, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} k_\nu \sum_{\alpha=1,2} (b_\alpha^\dagger(k)b_\alpha(k) - d_\alpha(k)d_\alpha^\dagger(k)) \Big|_{k=(\omega_{\mathbf{k}}, \mathbf{k})}.$$

(In)Stability The previous is not the final expression of P because of the unphysical and possibly divergent vacuum contribution. It is a standard procedure to remove this. Defining the vacuum state of the theory³ $|0\rangle$ by means of

$$b_\alpha(k)|0\rangle = 0, \quad d_\alpha(k)|0\rangle = 0,$$

it is clear that the only vacuum contribution comes from the terms $\langle 0|d_\alpha(k)d_\alpha^\dagger(k)|0\rangle$. This is easily quantified and removed if canonical commutation relations (CCR) are imposed on the ladder operators b_α and d_α :

$$\begin{aligned} [b_\alpha(k), b_\beta^\dagger(q)] &= (2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \delta_{\alpha\beta}, \\ [d_\alpha(k), d_\beta^\dagger(q)] &= (2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \delta_{\alpha\beta}, \end{aligned} \quad (4.5)$$

all the other commutators between pairs made of b and/or d vanish.

Indeed, using the fact that these commutators are c -numbers,

$$\begin{aligned} d_\alpha(k)d_\alpha^\dagger(k) - \langle 0|d_\alpha(k)d_\alpha^\dagger(k)|0\rangle &= d_\alpha(k)d_\alpha^\dagger(k) - \langle 0|[d_\alpha(k), d_\alpha^\dagger(k)]|0\rangle \\ &= d_\alpha(k)d_\alpha^\dagger(k) - [d_\alpha(k), d_\alpha^\dagger(k)] \langle 0|0\rangle \\ &= d_\alpha^\dagger(k)d_\alpha(k). \end{aligned}$$

² Here and in the following the notation $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$ is used.

³ The notation $|0\rangle$ is customary and by no means implies that the vacuum is the null vector (neutral element) of the vector space in which quantum states live.

Nonetheless these commutation relations result in *instability*, because then b and d terms contribute with the opposite sign⁴ to the (vacuum normalised) energy,

$$H - \langle 0|H|0\rangle = m \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\alpha=1,2} (b_{\alpha}^{\dagger}(k)b_{\alpha}(k) - d_{\alpha}^{\dagger}(k)d_{\alpha}(k)) \Big|_{k=(\omega_{\mathbf{k}},\mathbf{k})} .$$

Translation Invariance Still, one wants the relations between b_{α} and d_{α} operators to imply that the conserved charges under translations (P_{μ}) are, as usual, the generators of these transformations

$$\psi(x+a) = e^{iPa}\psi(x)e^{-iPa} , \quad (4.6)$$

that is

$$\partial_{\mu}\psi(x) = i[P_{\mu}, \psi] , \quad \partial_{\mu}\bar{\psi}(x) = i[P_{\mu}, \bar{\psi}] .$$

Inserting the Fourier decomposition of the fields and of the momentum, and assuming that the two degrees of freedom represented by b and d respectively are independent one of the other, the previous are equivalent to

$$\begin{aligned} \sum_{\beta} [b_{\beta}^{\dagger}(q)b_{\beta}(q), b_{\alpha}(k)] &= -(2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k}-\mathbf{q}) b_{\alpha}(k) , \\ \sum_{\beta} [d_{\beta}(q)d_{\beta}^{\dagger}(q), d_{\alpha}^{\dagger}(k)] &= (2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k}-\mathbf{q}) d_{\alpha}^{\dagger}(k) . \end{aligned}$$

Using the identity

$$[AB, C] = A[B, C] - [C, A]B ,$$

it can be shown that (4.6) is satisfied if commutation relations (4.5) are imposed. However, this is not the only possibility. Actually, using the identity

$$[AB, C] = A\{B, C\} - \{C, A\}B ,$$

the previous equations read

$$\begin{aligned} \sum_{\beta} \left(b_{\beta}^{\dagger}(q)\{b_{\beta}(q), b_{\alpha}(k)\} - \{b_{\alpha}(k), b_{\beta}^{\dagger}(q)\}b_{\beta}(q) \right) &= -(2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k}-\mathbf{q}) b_{\alpha}(k) , \\ \sum_{\beta} \left(d_{\beta}(q)\{d_{\beta}^{\dagger}(q), d_{\alpha}^{\dagger}(k)\} - \{d_{\alpha}^{\dagger}(k), d_{\beta}(q)\}d_{\beta}^{\dagger}(q) \right) &= (2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k}-\mathbf{q}) d_{\alpha}^{\dagger}(k) . \end{aligned}$$

⁴ Clearly for any operator O , the operator $O^{\dagger}O$ is non-negative definite, because for all ϕ

$$\langle \phi|O^{\dagger}O|\phi\rangle = \langle O\phi|O\phi\rangle \geq 0 .$$

These expressions make it immediate to check that (4.6) is satisfied also by imposing the following canonical anticommutation relations (CAR)

$$\begin{aligned} \{b_\alpha(k), b_\beta^\dagger(q)\} &= (2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \delta_{\alpha\beta} , \\ \{d_\alpha(k), d_\beta^\dagger(q)\} &= (2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{k} - \mathbf{q}) \delta_{\alpha\beta} , \end{aligned} \quad (4.7)$$

all the other anticommutators between pairs made of b and/or d vanish.

The above relations imply that the only non-vanishing equal-time anticommutator for the Dirac operator fields is

$$\{\psi_\xi(t, \mathbf{x}), \psi_\eta^\dagger(t, \mathbf{y})\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_{\xi\eta} . \quad (4.8)$$

Moreover, the previous anticommutation relations remove the instability problem. Now

$$H - \langle 0|H|0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} (b_\alpha^\dagger(k)b_\alpha(k) + d_\alpha^\dagger(k)d_\alpha(k)) \Big|_{k=(\omega_{\mathbf{k}}, \mathbf{k})} ,$$

thus after subtracting the vacuum energy, the energy operator is positive semidefinite.

Propagator Causality Another aspect, not mentioned yet, is related to the causality of particles propagation. Some calculations lead to

$$\{\psi_\xi(x), \bar{\psi}_\eta(y)\} = (i\rlap{\not{\partial}}_x + m) i\Delta(x - y) ,$$

where, using the sign function $\varepsilon(x) := x/|x|$,

$$\Delta(x) = -i \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \varepsilon(k^0) e^{-ikx} ,$$

which vanishes when x is spacelike, that is $x^2 < 0$. If instead commutators were used all the way, the result would become

$$[\psi_\xi(x), \bar{\psi}_\eta(y)] = (i\rlap{\not{\partial}}_x + m) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} [e^{-ik(x-y)} + e^{ik(x-y)}] \Big|_{k=(\omega_{\mathbf{k}}, \mathbf{k})} ,$$

which does not vanish for spacelike separations, $(x - y)^2 < 0$. This results in a violation of causality. Notice also that specular violations would happen if scalar fields were to be quantised with anticommutators.

4.2.2 Normal ordering

The procedure encountered before, of subtracting to the operator O representing a given observable \mathcal{O} its unphysical vacuum expectation value, is a standard one and is formalised by the *normal ordering* operation, denoted by $:O:$. Once the observable is written as a polynomial of ladder operators, this consists in bringing all the annihilation operators on the right and all the creation operators on the left, with the prescription that, inside normal ordering, all these operators commute in the bosonic case and anticommute in the fermionic one.⁵ Note this last point is very important to avoid nuisances like

$$b_{\alpha}^{\dagger}(k) b_{\alpha}(k) = : -b_{\alpha}(k) b_{\alpha}^{\dagger}(k) : = b_{\alpha}^{\dagger}(k) b_{\alpha}(k) - (2\pi)^3 \frac{\omega_{\mathbf{k}}}{m} \delta^{(3)}(\mathbf{0}) ,$$

where the leftmost equality is correct, whereas, for the rightmost one, anticommutation was wrongly used inside the normal ordering symbol. Indeed, the normal ordering is an operation defined on the free algebra of creation and annihilation operators, not on their CCR or CAR algebra. For further information see [this] Stack Exchange answer.

As an example, the physical (i.e. normal ordered) four momentum operator is

$$: P^{\mu} : = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} k^{\mu} \sum_{\alpha=1,2} (b_{\alpha}^{\dagger}(k) b_{\alpha}(k) + d_{\alpha}^{\dagger}(k) d_{\alpha}(k)) \Big|_{k=(\omega_{\mathbf{k}}, \mathbf{k})} .$$

4.2.3 Spin statistics theorem

The procedure followed in this section to quantise the Dirac field sets a precedent, which generalises to the so-called *Spin-Statistics theorem*. It declares a relation between a particle's spin and the type of statistics by which it is described. It thus also identifies the kind of algebra obeyed by the free field operators representing the particle. Schematically:

$$\begin{aligned} \text{half-integer spin} &\sim \text{Fermi-Dirac statistics} \sim \text{CAR algebra } \{ \cdot, \cdot \} \\ \text{integer spin} &\sim \text{Bose-Einstein statistics} \sim \text{CCR algebra } [\cdot, \cdot] \end{aligned}$$

4.2.4 Discrete transformations of the quantised Dirac field

The key relation for translating transformations of a classical field to the ones of the corresponding quantum operator field ϕ is contained in (2.66) and its inverse (2.67) that in the following, will be specialised to the cases of parity, charge conjugation and time reversal. The corresponding expressions of U in terms of creation and annihilation

⁵ That is, with fermions a minus sign must be added for each exchange of two operators.

operators are given.⁶

Parity

The operatorial version of parity transformations follows by (3.10), that gives $\psi^{\mathcal{P}} = \eta_{\mathcal{P}} \gamma^0 (\psi \circ \Lambda_{\mathcal{P}}^{-1})$, and by (2.67),

$$\mathcal{P} \psi(x) \mathcal{P}^\dagger = (\eta_{\mathcal{P}} \gamma^0)^{-1} \psi(\tilde{x}) = \eta_{\mathcal{P}}^* \gamma^0 \psi(\tilde{x}) , \quad (4.9)$$

where $\tilde{x}^\mu = (\Lambda_{\mathcal{P}})^\mu_\nu x^\nu = x_\nu = (x^0, -\mathbf{x})$. Of course, if such a phase can be fixed to be real, that is $\eta_{\mathcal{P}} = \pm 1$, then $\eta_{\mathcal{P}}^*$ can be replaced by⁷ $\eta_{\mathcal{P}}$. Note that, by (2.66), (4.9) is equivalent to

$$\mathcal{P}^\dagger \psi(x) \mathcal{P} = \eta_{\mathcal{P}} \gamma^0 \psi(\tilde{x}) .$$

Introducing the decomposition of the momentum space fields, (2.67) can be applied directly to the ladder operator, obtaining

$$\mathcal{P} b_\alpha(k) \mathcal{P}^\dagger = \eta_{\mathcal{P}}^* b_\alpha(\tilde{k}) , \quad \mathcal{P} d_\alpha(k) \mathcal{P}^\dagger = -\eta_{\mathcal{P}}^* d_\alpha(\tilde{k}) ,$$

where $\tilde{k}^\mu = k_\mu = (k^0, -\mathbf{k})$. Notice how particle and antiparticle get *opposite phases* under parity, in correspondence with their opposite γ^0 eigenvalues, already recognised in the classical description of parity. A solution of the previous equations is

$$\mathcal{P} = \exp i \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} \left\{ b_\alpha^\dagger(k) \left(\lambda b_\alpha(k) + \frac{\pi}{2} b_\alpha(\tilde{k}) \right) - d_\alpha^\dagger(k) \left[(\lambda + \pi) d_\alpha(k) + \frac{\pi}{2} d_\alpha(\tilde{k}) \right] \right\} ,$$

with λ arbitrary, related to the arbitrariness of the phase $\eta_{\mathcal{P}} = e^{i(\lambda + \pi/2)}$. One may check that $\mathcal{P} P^\mu \mathcal{P}^\dagger = P_\mu$. Furthermore,

$$e^{iHa} b_\alpha(k) e^{-iHa} = e^{-ik^0 a} b_\alpha(k) , \quad e^{iHa} d_\alpha(k) e^{-iHa} = e^{-ik^0 a} d_\alpha(k) ,$$

so that

$$e^{iHa} \mathcal{P} e^{-iHa} = \mathcal{P} ,$$

that is

$$[H, \mathcal{P}] = 0 .$$

⁶ See, pg.28 of the excellent Sozzi's book [16] for a discussion of the parity phase. Such a book also contains a detailed analysis of the phenomenological implications of the discrete symmetries.

⁷ Eq.(4.9) corresponds to Eq.(3-177) in the magnificent Itzykson-Zuber's book [7], with the difference that $\eta_{\mathcal{P}}^*$ is replaced by $\eta_{\mathcal{P}}$, which is considered as an arbitrary phase. As we will see, a similar thing is done also in the case of the charge conjugation transformation. This may happen, for example, if the right-hand side of (2.67) is accidentally replaced by the right-hand side of (2.66).

Charge conjugation

In the case of charge conjugation comparing the prescription (2.67) with (3.14), we get $\mathcal{C}\psi(x)\mathcal{C}^\dagger = \psi^c = \eta_{\mathcal{C}}^* \mathbf{C}^{-1} \tilde{\psi}$, where $\mathcal{C} = \mathcal{C}(b, b^\dagger, d, d^\dagger)$ is an unitary operator and $\tilde{\psi}$ denotes the transformation of ψ which is the inverse of $\psi \rightarrow \bar{\psi}^T$. On the other hand, $\gamma_0 = \gamma_0^{-1}$, so that the latter inverse transformation coincides with the transformation itself. Therefore, we have

$$\mathcal{C}\psi(x)\mathcal{C}^\dagger = \psi^c = \eta_{\mathcal{C}}^* \mathbf{C}^{-1} \bar{\psi}^T .$$

By $\gamma^{i^2} = -\mathbb{I}_4$ it follows that in the Dirac representation $\mathbf{C}^{-1} = i\gamma^0\gamma^2 = -\mathbf{C} = -i\gamma^2\gamma^0$. The above is equivalent to⁸

$$\mathcal{C}^\dagger\psi(x)\mathcal{C} = \psi^c = \eta_{\mathcal{C}} \mathbf{C} \bar{\psi}^T .$$

Note that to the action of \mathcal{C} has no effect on the coordinates. A possible expression for $\mathcal{C} = \mathcal{C}(b, b^\dagger, d, d^\dagger)$ is

$$\mathcal{C} = \mathcal{C}_\varphi \tilde{\mathcal{C}} ,$$

with

$$\begin{aligned} \mathcal{C}_\varphi &= \exp \left[-i\varphi_{\mathcal{C}} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} (b_\alpha^\dagger(k)b_\alpha(k) - d_\alpha^\dagger(k)d_\alpha(k)) \right] , \\ \tilde{\mathcal{C}} &= \exp \left[i\frac{\pi}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\alpha=1,2} (b_\alpha^\dagger(k) - d_\alpha^\dagger(k)) (b_\alpha(k) - d_\alpha(k)) \right] , \end{aligned}$$

where \mathcal{C}_φ has the only effect of returning the phase $\eta_{\mathcal{C}} = e^{i\varphi_{\mathcal{C}}}$.

A similar procedure can be carried out in order to find, in the case of the vector potential density A^μ , the charge conjugation operator $\mathcal{C} = \mathcal{C}(a, a^\dagger)$, satisfying $\mathcal{C}A^\mu(x)\mathcal{C}^\dagger = (A^\mu)^c = -A^\mu$.

Time reversal

We now discuss the properties of time reversal. The first step is to prove that, in any infinite dimensional representation of the (full) Lorentz group acting on a Hilbert space of quantum states, the time reversal operator is antiunitary.

Let us first recall that $|\phi_a\rangle$ denotes the eigenket of the field operator in the Schrödinger

⁸ It should be observed that in the Itzykson-Zuber's book [7], in the operatorial version of the charge conjugation are used, instead of their inverse, the same $\eta_{\mathcal{C}}$ and \mathbf{C} as in the classical case. Nevertheless, at pg. 153, it is specified that in the Dirac representation the \mathbf{C} in Eq.(3-183) is now $\mathbf{C} = i\gamma^0\gamma^2$, that is the inverse of the one in Eq.(2-97), in agreement with the standard notation. In this respect, it seems that the errata at page 2 of <http://www.lpthe.jussieu.fr/~zuber/corrize.pdf> should be corrected.

representation $\phi(0, \mathbf{x})$ with eigenvalue $\phi_a(0, \mathbf{x})$, that is

$$\phi(0, \mathbf{x})|\phi_a\rangle = \phi_a(0, \mathbf{x})|\phi_a\rangle .$$

Note that the Schrödinger representation of the field operator depends on the reference time, arbitrarily chosen to be $t_0 = 0$. Let us consider

$$\langle\phi_f, t_f|\phi_i, t_i\rangle .$$

This is the probability amplitude that a state that at time t_i is in the eigenstate $|\phi_i\rangle$ of the Schrödinger operator $\phi(0, \mathbf{x})$ evolves, at time t_f , to the state $|\phi_f\rangle$. This means that $|\psi(t_i)\rangle$, the state in the Schrödinger representation at the time t_i , is $|\phi_i\rangle$. Therefore,

$$|\psi(t_i)\rangle = |\phi_i\rangle .$$

On the other hand, $|\psi(t_f)\rangle = U(t_f, t_i)|\psi(t_i)\rangle$, so that, if the Hamiltonian is time independent, then the dynamical evolution of $|\psi(t_i)\rangle$ up to time t_f is

$$|\psi(t_f)\rangle = e^{-\frac{i}{\hbar}H(t_f-t_i)}|\psi_i(t_i)\rangle = e^{-\frac{i}{\hbar}H(t_f-t_i)}|\phi_i(t_i)\rangle .$$

Therefore,

$$\begin{aligned} \langle\phi_f, t_f|\phi_i, t_i\rangle &= \langle\phi_f|\psi(t_f)\rangle = \langle\phi_f|e^{-\frac{i}{\hbar}H(t_f-t_i)}|\psi_i(t_i)\rangle \\ &= \langle\phi_f|e^{-\frac{i}{\hbar}H(t_f-t_i)}|\phi_i\rangle , \end{aligned}$$

where the first equality is a consequence of the definition of $\langle\phi_f, t_f|\phi_i, t_i\rangle$. Comparing the left- and right-hand sides, we get $|\phi_a, t_a\rangle = e^{iH(t_a-t_0)}|\phi_a\rangle$. Thus, choosing $t_0 = 0$, we have⁹

$$|\phi_a, t_a\rangle = e^{iHt_a}|\phi_a\rangle .$$

It follows that

$$\begin{aligned} \phi(t_a, \mathbf{x})e^{iHt_a}|\phi_a\rangle &= e^{iHt_a}\phi(0, \mathbf{x})|\phi_a\rangle = e^{iHt_a}\phi_a(0, \mathbf{x})|\phi_a\rangle \\ &= \phi_a(0, \mathbf{x})|\phi_a, t_a\rangle , \end{aligned}$$

that is $|\phi_a, t_a\rangle$ is the instantaneous eigenstate of $\phi(t_a, \mathbf{x}) = e^{iHt_a}\phi(0, \mathbf{x})e^{-iHt_a}$, which is the field operator in the Heisenberg picture. In other words, we have

$$\phi(t_a, \mathbf{x})|\phi_a, t_a\rangle = \phi_a(0, \mathbf{x})|\phi_a, t_a\rangle .$$

⁹ Note that the sign at the exponent is the opposite of the one in the expression for the time evolution of a state in the Schrödinger representation.

Since we also have $\phi(0, \mathbf{x})|\phi_a\rangle = \phi_a(0, \mathbf{x})|\phi_a\rangle$, it follows that the spectrum of the Heisenberg and Schrödinger operators coincide, which is obvious because such representations are related by a unitary transformation.

Let us now show that the time reversal is represented by an antiunitary operator. The discussion is carried out in a QFT setting, but the claim is valid also in non-relativistic quantum mechanics and is essentially a consequence of the positive definiteness of the Hamiltonian (energy). Consider a generic quantised field ϕ , which may have both vector and spinor indices. Suppose that the time reversal operator \mathcal{T} represents a symmetry of the theory, i.e.

$$[\mathcal{T}, H] = 0 .$$

Under a \mathcal{T} transformation the associated classical field will transform to $\phi(x) \rightarrow D\phi(-t, \mathbf{x})$, with D some matrix acting on the indices of ϕ . Then, applying the time reversed field to the vacuum of the theory $|0\rangle$ yields

$$\begin{aligned} \mathcal{T}\phi(t, \mathbf{x})\mathcal{T}^\dagger |0\rangle &= D^{-1} \phi(-t, \mathbf{x}) |0\rangle && \text{by (2.67)} \\ &= D^{-1} e^{-iHt} \phi(0, \mathbf{x}) |0\rangle && \text{time evolution, } H|0\rangle = 0 \text{ and } [D, H] = 0 \\ &= \sum_n e^{-iE_n t} d_n(\mathbf{x}) |n\rangle && \text{inserting } \sum_n |n\rangle\langle n| , \end{aligned}$$

where $d_n(\mathbf{x}) = D^{-1} \langle n | \phi(0, \mathbf{x}) | 0 \rangle$. Let us suppose that \mathcal{T} is a unitary operator, and therefore linear. Then, using the fact that it commutes with H , we also have

$$\begin{aligned} \mathcal{T}\phi(t, \mathbf{x})\mathcal{T}^\dagger |0\rangle &= e^{iHt} \mathcal{T}\phi(0, \mathbf{x})\mathcal{T}^\dagger e^{-iHt} |0\rangle && \text{time evolution, } \mathcal{T}(iHt) = (iHt)\mathcal{T} \\ &= e^{iHt} \mathcal{T}\phi(0, \mathbf{x})\mathcal{T}^\dagger |0\rangle && H|0\rangle = 0 \\ &= \sum_n e^{iE_n t} c_n(\mathbf{x}) |n\rangle && \text{inserting } \sum_n |n\rangle\langle n| , \end{aligned}$$

where $c_n(\mathbf{x}) = \langle n | \mathcal{T}\phi(0, \mathbf{x})\mathcal{T}^\dagger | 0 \rangle$. The two derivations above give contradicting results. The only hypothesis which can be relaxed is the linearity of the time reversal operator \mathcal{T} . Indeed, \mathcal{T} is *antiunitary*. Some properties of antiunitary operators, used in the remaining of this section, are now investigated. An antilinear operator A is such that

$$A(\alpha |\phi\rangle + \beta |\psi\rangle) = \alpha^* A |\phi\rangle + \beta^* A |\psi\rangle , \quad \alpha, \beta \in \mathbb{C} .$$

This behaviour requires some attention in certain circumstances, e.g.

$$A |\phi\rangle = A \sum_n |n\rangle \langle n | \phi \rangle = \sum_n (A |n\rangle) \langle n | \phi \rangle^* .$$

For a linear operator L , the adjoint L^\dagger is defined by $\langle L^\dagger \psi | \phi \rangle = \langle \psi | L \phi \rangle$.

In order for the definition to be well posed, it must comply with the properties of the

Hermitian scalar product, specifically

$$\langle \psi | \alpha(T\phi) \rangle = \alpha \langle T^\dagger \psi | \phi \rangle = \langle T^\dagger \psi | \alpha\phi \rangle = \langle \psi | T(\alpha\phi) \rangle .$$

Clearly this does hold if T is linear, but not if it is antilinear. For the antilinear case the previous definition is modified to (this property is used at pg. 155 of [7])

$$\langle A^\dagger \psi | \phi \rangle = \langle \psi | A\phi \rangle^* .$$

A unitary operator is a linear operator U satisfying the property $\langle U\psi | U\phi \rangle = \langle \psi | \phi \rangle$, for all possible pairs ϕ, ψ .

Again, this property is not satisfied by antilinear operators, but its analogue can be easily guessed: an antiunitary operator is an antilinear operator V , satisfying

$$\langle V\psi | V\phi \rangle = \langle \psi | \phi \rangle^* = \langle \phi | \psi \rangle .$$

Such an operator can be written as $V = UK$, where U is a unitary operator, and K is the complex conjugation operation.

For both unitary and antiunitary operators the following holds

$$T^\dagger T = TT^\dagger = \mathbb{I} .$$

The key relation that characterises \mathcal{T} is $\mathcal{T}i\mathcal{T}^\dagger = -i\mathbb{I}$. Therefore, returning to our analysis of time reversal, if $[\mathcal{T}, H]$ vanishes, then

$$\mathcal{T}e^{-iH(t_2-t_1)}\mathcal{T}^\dagger = e^{\mathcal{T}[-iH(t_2-t_1)]\mathcal{T}^\dagger} = e^{-iH(t_1-t_2)} .$$

Notice the relation between the imaginary unit “ i ” and time in quantum mechanics. Moreover, the transition amplitude from the state $|\phi_i\rangle$ at time t_i to $|\phi_f\rangle$ at time t_f is equal to the transition amplitude from the state $|\mathcal{T}\phi_f\rangle$ at time t_i to $|\mathcal{T}\phi_i\rangle$ at time t_f

$$\langle \phi_f, t_f | \phi_i, t_i \rangle = \langle \mathcal{T}\phi_i, t_f | \mathcal{T}\phi_f, t_i \rangle .$$

Indeed,

$$\begin{aligned} \langle \phi_f, t_f | \phi_i, t_i \rangle &= \langle \phi_f | e^{-iH(t_f-t_i)} | \phi_i \rangle \\ &= \langle \mathcal{T}^\dagger \mathcal{T}\phi_f | e^{-iH(t_f-t_i)} | \mathcal{T}^\dagger \mathcal{T}\phi_i \rangle \\ &= \langle \mathcal{T}\phi_f | \mathcal{T}e^{-iH(t_f-t_i)}\mathcal{T}^\dagger | \mathcal{T}\phi_i \rangle^* \\ &= \langle \mathcal{T}\phi_f | e^{-iH(t_i-t_f)} | \mathcal{T}\phi_i \rangle^* \\ &= \langle \mathcal{T}\phi_i | e^{-iH(t_f-t_i)} | \mathcal{T}\phi_f \rangle = \langle \mathcal{T}\phi_i, t_f | \mathcal{T}\phi_f, t_i \rangle . \end{aligned}$$

Before deriving the implementation of time reversal for the Dirac field, observe that the 3-momentum \mathbf{P} , and the angular momenta (orbital, spin and total) \mathbf{J} , \mathbf{L} , \mathbf{S} , change sign under time reversal. As a consequence, helicity, which is the projection of the spin along the momentum, is left unchanged. For this reason, in the following, the *helicity basis* for the spinors will be used, instead of the usual spin basis. This choice affects both creation–annihilation operators $b^{(\dagger)}$, $d^{(\dagger)}$ and basis spinors u, v . Let us parametrise the action of time reversal via

$$\mathcal{T} b^\dagger(k, \varepsilon) \mathcal{T}^\dagger = \eta_{\mathcal{T}}^* b^\dagger(\tilde{k}, \varepsilon) e^{i\zeta_b(k, \varepsilon)}, \quad \mathcal{T} d^\dagger(k, \varepsilon) \mathcal{T}^\dagger = \eta_{\mathcal{T}} d^\dagger(\tilde{k}, \varepsilon) e^{-i\zeta_d(k, \varepsilon)},$$

where $\tilde{k}^\mu = (k^0, -\mathbf{k})$, $\eta_{\mathcal{T}}$ is a phase and ε runs over the two possible helicity eigenstates of fermions and antifermions. Then, using the antilinearity property

$$\mathcal{T} \psi(x) \mathcal{T}^\dagger = \eta_{\mathcal{T}} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{\omega_{\mathbf{k}}} \sum_{\varepsilon=1,2} [b(\tilde{k}, \varepsilon) e^{-i\zeta_b(k, \varepsilon)} u_{(\varepsilon)}^*(k) e^{ikx} + d^\dagger(\tilde{k}, \varepsilon) e^{-i\zeta_d(k, \varepsilon)} v_{(\varepsilon)}^*(k) e^{-ikx}].$$

Let $\psi(x) \rightarrow \psi'(x) = \eta_{\mathcal{T}} S_{\mathcal{T}} \psi(-t, \mathbf{x})$, where $\eta_{\mathcal{T}}$ is a phase and $S_{\mathcal{T}}$ a matrix, the transformation of the classical Dirac field under time reversal. We then have

$$\mathcal{T} \psi(t, \mathbf{x}) \mathcal{T}^\dagger = \eta_{\mathcal{T}}^* S_{\mathcal{T}}^{-1} \psi(-t, \mathbf{x}).$$

Substituting $\mathbf{k} \rightarrow -\mathbf{k}$ in the previous integral, it is easy to see that

$$S_{\mathcal{T}}^{-1} u_{(\varepsilon)}(\tilde{k}) = e^{-i\zeta_b(\tilde{k}, \varepsilon)} u_{(\varepsilon)}^*(\tilde{k}), \quad S_{\mathcal{T}}^{-1} v_{(\varepsilon)}(\tilde{k}) = e^{-i\zeta_d(\tilde{k}, \varepsilon)} v_{(\varepsilon)}^*(\tilde{k}).$$

Now, γ^0 is Hermitian while the γ^i are anti-Hermitian, thus $\tilde{k}^* = \not{k}^T$ and the complex conjugate of Dirac equation reads

$$(\not{k}^T - m) u_{(\varepsilon)}^*(\tilde{k}) = 0, \quad (\not{k}^T + m) v_{(\varepsilon)}^*(\tilde{k}) = 0.$$

Finally, left multiplying by $\gamma^5 \mathbf{C}$ and using (3.16),

$$(\not{k} - m) \gamma^5 \mathbf{C} u_{(\varepsilon)}^*(\tilde{k}) = 0, \quad (\not{k} + m) \gamma^5 \mathbf{C} v_{(\varepsilon)}^*(\tilde{k}) = 0.$$

From these equations it is natural to guess $S_{\mathcal{T}} = i\mathbf{C}\gamma^5$ which in Dirac basis is just

$$S_{\mathcal{T}} = -i\gamma^1\gamma^3.$$

To be definite about this guess, one should verify that it respects the helicity of the states. The check requires an explicit expression for the spinor helicity basis and can be found in section 3-4 of [7].

4.2.5 Fermionic bilinears

Fermionic bilinears are crucial objects to express any physical observable related to fermions. They are expressions of the type

$$B(x) = : \bar{\psi}(x) \mathbf{M}^B \psi(x) : ,$$

where \mathbf{M}^B is a generic 4×4 matrix.

To classify all the possible bilinears we have to complete our overview of the $O(1,3)$ Clifford algebra. Indeed, any 4×4 matrix can be expanded on the 16 matrices $\{\Gamma^a\}$, providing the basis of the Clifford algebra¹⁰:

$$\Gamma^S = I , \quad \Gamma_\mu^V = \gamma_\mu , \quad \Gamma_{\mu\nu}^T = \sigma_{\mu\nu} , \quad \Gamma_\mu^A = \gamma_5 \gamma_\mu , \quad \Gamma^P = i\gamma_5 .$$

S, V, T, A, P stand for the behaviour under Lorentz transformations of the corresponding bilinear, respectively: scalar, vector, tensor, axial vector (pseudovector), pseudoscalar.

object	\mathbf{M}^B	$B(x)$	$B'(x)$ after Λ
scalar	Γ^S	$S(x)$	$S(\Lambda^{-1}x)$
vector	$(\Gamma^V)^\mu$	$V^\mu(x)$	$\Lambda^\mu_\nu V^\nu(\Lambda^{-1}x)$
tensor	$(\Gamma^T)^{\mu\nu}$	$T^{\mu\nu}(x)$	$\Lambda^\mu_\rho \Lambda^\nu_\sigma T^{\rho\sigma}(\Lambda^{-1}x)$
pseudovector	$(\Gamma^A)^\mu$	$A^\mu(x)$	$\det(\Lambda) \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x)$
pseudoscalar	Γ^P	$P(x)$	$\det(\Lambda) P(\Lambda^{-1}x)$

For discrete symmetries, defining $\tilde{x}^\mu = (x^0, -\mathbf{x})$, the transformation laws are

$B(x)$	\mathcal{P}	\mathcal{C}	\mathcal{T}	\mathcal{PCT}
$S(x)$	$S(\tilde{x})$	$S(x)$	$S(-\tilde{x})$	$S(-x)$
$V^\mu(x)$	$V_\mu(\tilde{x})$	$-V^\mu(x)$	$V_\mu(-\tilde{x})$	$-V^\mu(-x)$
$T^{\mu\nu}(x)$	$T_{\mu\nu}(\tilde{x})$	$-T^{\mu\nu}(x)$	$-T_{\mu\nu}(-\tilde{x})$	$T^{\mu\nu}(-x)$
$A^\mu(x)$	$-A_\mu(\tilde{x})$	$A^\mu(x)$	$A_\mu(-\tilde{x})$	$-A^\mu(-x)$
$P(x)$	$-P(\tilde{x})$	$P(x)$	$-P(-\tilde{x})$	$P(-x)$

¹⁰ The i factor in Γ^P is there to make the pseudoscalar bilinear Hermitian in the quantised theory.

The proof of the previous transformation laws is easily accomplished combining relations (3.3), (3.4) and (3.6), eventually inserting \mathbb{I}_4 in the form $S(\Lambda)S^{-1}(\Lambda)$, and for bilinears involving γ^5 also recalling that

$$\epsilon_{i_1 \dots i_n} A_{j_1}^{i_1} \cdots A_{j_n}^{i_n} = \det A \epsilon_{j_1 \dots j_n} ,$$

where ϵ is the Levi-Civita symbol: $\epsilon_{1\dots n} = 1$, $\epsilon_{1\dots n} = \epsilon_{[1\dots n]}$.

The basis elements of the Clifford algebra given at the beginning of this section satisfy

- (i) $(\Gamma^a)^2 = \pm I$.
- (ii) For any $\Gamma^a \neq I$, $\exists \Gamma^b : \{\Gamma^a, \Gamma^b\} = 0$.
- (iii) $\text{Tr}(\Gamma^a) = 0$, $\forall \Gamma^a \neq I$.
- (iv) For each pair (Γ^a, Γ^b) , $\exists \Gamma^c : \Gamma^a \Gamma^b = \eta \Gamma^c$, $\eta \in \{\pm 1, \pm i\}$. Also, $a \neq b \Rightarrow \Gamma^c \neq I$.
- (v) $\{\Gamma^a\}$ are linearly independent.

To prove (iii), choose the Γ^b of (ii). Then using (i) and the cyclic property of the trace:

$$\text{Tr}(\Gamma^a) = \pm \text{Tr}(\Gamma^a (\Gamma^b)^2) = \mp \text{Tr}(\Gamma^b \Gamma^a \Gamma^b) = \mp \text{Tr}((\Gamma^b)^2 \Gamma^a) = - \text{Tr}(\Gamma^a) .$$

For (v), suppose that $\sum_{b=1}^{16} \lambda_b \Gamma^b = 0$. Given $a \in \{1, \dots, 16\}$, for any $b \neq a$ let $\tilde{\Gamma}^b$ denote the $\eta \Gamma_c = \Gamma^a \Gamma^b$ of (iv). Then

$$0 = \text{Tr} \left(\Gamma^a \sum_{b=1}^{16} \lambda_b \Gamma^b \right) = \text{Tr} \left(\lambda_a (\Gamma^a)^2 + \sum_{b \neq a} \lambda_b \tilde{\Gamma}^b \right) = \pm 4 \lambda_a \implies \lambda_a = 0 .$$

In particular, (v) proves the original claim, that any 4×4 matrix can be written as a linear combination of Γ^a .

Chapter 5

Interacting Quantised Fields

5.1 Locality and PCT in QFT¹

Locality A local quantum field theory is characterised by:

- ▷ *no action at a distance*: all influences propagate at finite speed;
- ▷ *causal locality*: spacelike separated regions of spacetime behave like independent subsystems.

A Lagrangian density is said local if it depends on *finitely many derivatives of the fields* at any point of spacetime. Local Lagrangian densities are expected² to yield local quantum field theories under quantisation, at least at the perturbative level.

An example of a non-local term is given by

$$A(x_1, \dots, x_n)\phi(x_1) \dots \phi(x_n) , \quad (5.1)$$

for some $A(x_1, \dots, x_n)$. Such a term represents a self-interaction that, unlike the local one $\phi^n(x)$, involves the field at different space-time points. By making an expansion of each ϕ with respect the same point, the term (5.1) becomes a series with infinitely many derivatives of ϕ .

As we will see in proving the Jona-Lasinio theorem, an example of non-local interaction is provided by the effective action.

PCT Theorem. The PCT theorem is a result by Pauli, Zumino and Schwinger which states that a Lorentz invariant local QFT is $\mathcal{PCT}(\Theta)$ invariant. Two main consequences are

¹ Marco Rigobello

² It is not yet proven rigorously.

- (i) each particle has an antiparticle, with same mass and spin,
- (ii) a neutral particle coincides with its antiparticle.³

Consider a local QFT described by a Lorentz invariant local Lagrangian density

$$\mathcal{L}(x) = F[\phi, \psi, A_\mu, \dots] ,$$

and the action

$$\mathcal{S} = \int d^4y \mathcal{L}(y) .$$

Then, if conditions (i)-(ii) are verified,

$$\Theta \mathcal{L}(x) \Theta^\dagger = \mathcal{L}(-x) \quad \Rightarrow \quad \Theta \mathcal{S} \Theta^\dagger = \mathcal{S} .$$

Heuristically, for a Hermitian⁴ ($\phi^\dagger = \phi$) scalar field the proof follows these steps

- (i) $\lambda \rightarrow \lambda^*$, the constants of the theory are complex conjugated by \mathcal{T} ,
- (ii) $x \rightarrow -x$, the argument of each field changes sign,
- (iii) $\partial_\mu \rightarrow -\partial_\mu$, according to the previous point,
- (iv) $A_\mu(x) \rightarrow -A_\mu(-x)$, like $\partial_\mu \alpha(x)$,
- (v) $\bar{\psi}_a T^{\mu_1 \dots \mu_P} \psi_b \rightarrow (-1)^P \bar{\psi}_b T^{\mu_1 \dots \mu_P} \psi_a$, fermionic bilinears are subject to one sign flip for each Lorentz index coming from γ^μ or ∂_μ , and the order of the fermionic fields is inverted by \mathcal{C} .

Since \mathcal{L} is scalar the total number of tensor indices is even (they are all contracted), and since normal ordering is implied in the Lagrangian, we can commute bosonic fields and arrange them in the opposite order as they appear in \mathcal{L} . Therefore, we have

$$\Theta \mathcal{L}(x) \Theta^\dagger = \mathcal{L}^\dagger(-x) .$$

Using the Hermiticity⁵ of \mathcal{L} , this allows concluding that if the vacuum is invariant under

³ The wave functions of particle and antiparticle are related by complex conjugation, which is an effect of the charge conjugation. Then, a (truly) neutral particle is a particle that remains invariant under charge conjugation. Therefore, a neutral particle should not have electric charge, colour charge etc. Examples are photon, Higgs and Z bosons. A case of neutral spin-1/2 particle is the hypothetical neutralino which is a Majorana fermion, that is its wave function is real.

⁴ The generalisation to the non-Hermitian case $\phi^\dagger \neq \phi$ is straightforward, simply use the fields

$$\phi_1 = (\phi + \phi^\dagger)/\sqrt{2} , \quad \phi_2 = (\phi - \phi^\dagger)/(i\sqrt{2}) .$$

⁵ Actually, it is sufficient that \mathcal{L} is equivalent to an Hermitian Lagrangian, as in the case of \mathcal{L}_{dirac} , which is not Hermitian, but it leads to the same equations as $(\mathcal{L}_{dirac} + \mathcal{L}_{dirac}^\dagger)/\sqrt{2}$.

Θ , so will be the dynamics.

For further information on these topics consult [3].

5.2 Källén-Lehmann representation

Consider the following quantities in Minkowski spacetime

$$\begin{aligned} G^{(2)}(x-y) &= \langle \Omega | T \phi(x) \phi(y) | \Omega \rangle , \\ W^{(2)}(x-y) &= \langle \Omega | \phi(x) \phi(y) | \Omega \rangle , \\ G_R^{(2)}(x-y) &= \theta(x^0 - y^0) \langle \Omega | [\phi(x), \phi(y)] | \Omega \rangle . \end{aligned} \quad (5.2)$$

$G^{(2)}(x-y)$ is the exact propagator, $W^{(2)}(x-y)$ the Wightman function, $G_R^{(2)}(x-y)$ the retarded Green function.

Let us first consider the above functions in the case of the free theory. By

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{ipx} + a_{\mathbf{p}}^\dagger e^{-ipx} \right) \Big|_{p^0=\omega_{\mathbf{p}}} ,$$

we see that only the term that survives in $\langle 0 | \phi(x) \phi(y) | 0 \rangle$ is

$$\langle 0 | a_{\mathbf{p}} a_{\mathbf{q}}^\dagger | 0 \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) ,$$

so that

$$W_0^{(2)}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} e^{-ip(x-y)} .$$

To find the Fourier transform of $W_0^{(2)}(x-y)$, note that

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) \Big|_{p^0>0} = \int \frac{d^4p}{(2\pi)^4} (2\pi) \theta(p^0) \delta(p^2 - m^2) ,$$

so that the Wightman function in momentum space reads

$$\tilde{W}_0^{(2)}(p) = 2\pi \theta(p^0) \delta(p^2 - m^2) .$$

The fact that the commutator is a c -number implies that it coincides with its vacuum expectation value. Therefore, in the free case,

$$[\phi(x), \phi(y)] = W_0^{(2)}(x-y) - W_0^{(2)}(y-x) .$$

In particular,

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) \Big|_{p^0=\omega_{\mathbf{p}}} \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(e^{-ip(x-y)} \Big|_{p^0=\omega_{\mathbf{p}}} - e^{-ip(x-y)} \Big|_{p^0=-\omega_{\mathbf{p}}} \right). \end{aligned} \quad (5.3)$$

It is easy to check that for $x^0 > y^0$ we have

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle \stackrel{x^0 > y^0}{=} - \int \frac{d^3 p}{(2\pi)^3} \int_{\Gamma} \frac{dp^0}{2\pi i} \frac{1}{p^2 - m^2} e^{-ip(x-y)}, \quad (5.4)$$

where Γ is the contour coinciding with the real axis of the $(\text{Re } p^0, \text{Im } p^0)$ plane, except for the points $\pm\omega_{\mathbf{p}}$, climbed over the complex upper half-plane. To see this, note that, by Jordan's lemma, the integral in (5.4) is the same of the one in which the contour integral Γ is closed in the lower half-plane. By a trivial residue calculus, this reproduces, for $x^0 > y^0$, the original expression of the commutator $[\phi(x), \phi(y)]$.

Even if in the case $x^0 < y^0$ the right-hand side of (5.4) no longer corresponds to $[\phi(x), \phi(y)]$, it is interesting to observe that

$$- \int \frac{d^3 p}{(2\pi)^3} \int_{\Gamma} \frac{dp^0}{2\pi i} \frac{1}{p^2 - m^2} e^{-ip(x-y)} \stackrel{x^0 < y^0}{=} 0,$$

that follows by noticing that for $x^0 > y^0$ such integral is equivalent to the one in which the contour Γ is closed in the upper half-plane. We then have

$$G_{0R}^{(2)}(x-y) := \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \int_{\Gamma} \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip(x-y)}. \quad (5.5)$$

It is instructive to derive the same result by showing that

$$(\partial^2 + m^2)\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = -i\delta^{(4)}(x-y). \quad (5.6)$$

On the other hand,

$$G_{0R}^{(2)}(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \tilde{G}_{0R}^{(2)}(p),$$

so that

$$(-p^2 + m^2)\tilde{G}_{0R}^{(2)}(p) = -i,$$

that is

$$\tilde{G}_{0R}^{(2)}(p) = -\frac{i}{-p^2 + m^2}.$$

Therefore,

$$G_{0R}^{(2)}(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)},$$

where the contour in the p^0 plane of the two last integrals is Γ .

Let us make some observations

- (i) Eq.(5.6) shows that $G_{0R}^{(2)}(x-y)$ is a Green's function of the Klein-Gordon operator. The fact it vanishes for $x^0 < y^0$ means that is a *retarded* Green's function.
- (ii) One should remember that the contour integral Γ in the p^0 plane is the real axis except for the points $\pm\omega_{\mathbf{p}}$, climbed over the complex upper half-plane.
- (iii) The choice of the contour can be remembered by modifying the integrand. So, for example, the Feynman propagator $\Delta_F(x-y) = -iG_0^{(2)}(x-y)$ can be expressed as an integral in p^0 that coincides with the full real axis, but now with m^2 shifted by $i\epsilon$. This corresponds to the case in which the contour integral in the p^0 plane, coincides with the real axis except for the point $-\omega_{\mathbf{p}}$, climbed below the complex lower half-plane, and the point $\omega_{\mathbf{p}}$, climbed over the complex upper half-plane. In other words, the Feynman propagator is⁶

$$\Delta_F(x-y) = -i\langle 0|T\phi(x)\phi(y)|0\rangle = \lim_{\epsilon \rightarrow 0} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}.$$

In the following the limit $\epsilon \rightarrow 0$ will be understood.

- (iv) The Feynman propagator can be expressed in terms of the Hankel function $H_1^{(2)}$ and of the modified Bessel function K_1

$$\Delta_F(x-y) = \begin{cases} -\frac{1}{4\pi} \delta(s) + \frac{m}{8\pi\sqrt{s}} H_1^{(2)}(m\sqrt{s}), & s \geq 0, \\ -\frac{im}{4\pi^2\sqrt{-s}} K_1(m\sqrt{-s}), & s < 0, \end{cases}$$

where $s = (x-y)^2$.

- (v) It is useful to write the Feynman propagator directly as in integration in $d^3 p$

$$\Delta_F(x-y) = -i \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \left(\theta(x^0 - y^0) e^{-ip(x-y)} + \theta(y^0 - x^0) e^{ip(x-y)} \right).$$

⁶ Some authors call $i\Delta_F(x-y)$ Feynman propagator, so that, it coincides with the free two-point Green's function.

In an interacting theory

$$\begin{aligned}
W^{(2)}(x-y) &= \sum_n \langle \Omega | \phi(x) | n \rangle \langle n | \phi(y) | \Omega \rangle \\
&= \sum_n e^{-ip_n(x-y)} |\langle \Omega | \phi(0) | n \rangle|^2 \\
&= \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) |\langle \Omega | \phi(0) | n \rangle|^2, \quad (5.7)
\end{aligned}$$

where we have used $P^\mu |\Omega\rangle = 0$, and inserted a complete set of states $|n\rangle$, $P^\mu |n\rangle = p_n^\mu |n\rangle$. Keep in mind that here we are considering exact, non-perturbative properties of the theory. In particular, the states $\{|n\rangle\}$ are physical, thus on-shell. This set includes the vacuum, single particle states and multi particle states. For any k , the set of k -particle states generates a subspace of the Hilbert space of the theory. The sum over n is schematic and includes integrals over relative momenta. In the following we assume that the vacuum contribution vanishes, $\langle \Omega | \phi(x) | \Omega \rangle = 0$. This requirement can always be satisfied, eventually performing a shift of ϕ . For the sake of simplicity, in the following it is assumed that no bound states are present.

The implicit assumption of the uniqueness of the vacuum, together with $P^0 |\Omega\rangle = 0$, imply that the states $|n\rangle \neq |\Omega\rangle$ have positive energy. The $E_n = 0$ case is excluded since $|\Omega\rangle$ is the only state with vanishing energy, whereas the existence of negative E_n would mean that $|\Omega\rangle$ is not the fundamental state and thus is instable. As a result, $\delta^{(4)}(p-p_n)$ vanishes if $p^0 < 0$ and a factor $\theta(p^0)$ can be safely inserted in defining the distribution $\rho(p^2)$ via

$$(2\pi) \theta(p^0) \rho(p^2) = (2\pi)^4 \sum_n \delta^{(4)}(p-p_n) |\langle \Omega | \phi(0) | n \rangle|^2, \quad (5.8)$$

where the p^2 dependence is a consequence of Lorentz invariance. The states $|n\rangle$ are physical and p_n^2 is the total invariant mass squared. Then (except for the vacuum which satisfies $p^2 = 0$), making the assumption that $m \neq 0$, one has $p_n^2 > 0$. As a result, $\delta^{(4)}(p-p_n)$ vanishes also for the four-momenta p_μ such that $p^2 < 0$. Similarly $\rho(p^2) = 0$ when $p^2 < 0$. Therefore,

$$\begin{aligned}
\tilde{W}^{(2)}(p) &= 2\pi \theta(p^0) \rho(p^2) \\
&= \int_0^\infty d\mu^2 2\pi \delta(p^2 - \mu^2) \theta(p^0) \rho(\mu^2) \\
&= \int_0^\infty d\mu^2 \rho(\mu^2) \tilde{W}_0^{(2)}(p; \mu^2). \quad (5.9)
\end{aligned}$$

Consider now the general case of a Lagrangian involving various scalar fields of different

masses. In this case

$$\sum_{\substack{\text{single} \\ \text{particle} \\ \text{states}}} = \sum_j \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{k,j}} = \sum_j \int \frac{d^4k}{(2\pi)^4} 2\pi\theta(k^0)\delta(k^2 - m_j^2), \quad (5.10)$$

where j runs over particles species and

$$\omega_{k,j} = \sqrt{\mathbf{k}^2 + m_j^2}.$$

If states of more than one particle are considered ω can assume a continuum of possible values. For example, for a two particle state

$$\omega = \sqrt{\mathbf{k}_1^2 + m_1^2} + \sqrt{\mathbf{k}_2^2 + m_2^2}.$$

In the case $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, ω assumes continuous values, bounded from below by $m_1 + m_2$. Correspondingly the lowest possible value that k^2 can take is

$$(k^2)_{\text{lowest}} = (m_1 + m_2)^2. \quad (5.11)$$

Now note that, by (5.8) and (5.10), we have

$$(2\pi)\theta(p^0)\rho(p^2) = \sum_j \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m_j^2)\theta(k^0)(2\pi)^4\delta^{(4)}(p-k)|\langle\Omega|\phi(0)|k,j\rangle|^2 \\ + 2\pi\theta(p^0)\sigma(p^2),$$

where $\sigma(p^2)$ represents the contribute of multi-particle states; ρ can be rewritten as

$$\rho(p^2) = \sigma(p^2) + \sum_j \delta(p^2 - m_j^2)Z_j, \quad (5.12)$$

where

$$Z_j := |\langle\Omega|\phi(0)|k,j\rangle|^2.$$

Z_j is a Lorentz scalar, thus its dependence upon k^μ can involve only $k^2 = m_j^2$. Moreover, $\sigma(p^2)$ vanishes when $p^2 < 4m_1^2$, where m_1 is the mass of the lightest particle. We can now find the expression for $\tilde{G}^{(2)}(p)$ by replacing $W_0^{(2)}(p; \mu^2)$ in (5.9) by

$$\tilde{G}_0^{(2)}(p; \mu^2) = \frac{i}{p^2 - \mu^2 + i\epsilon}.$$

In particular, using

$$\rho(\mu^2) = \sigma(\mu^2) + \sum_j \delta(\mu^2 - m_j^2)Z_j,$$

and by (5.12), we get

$$\tilde{G}^{(2)}(p) = \sum_j \frac{iZ_j}{p^2 - m_j^2 + i\epsilon} + \int_{4m_1^2}^{\infty} d\mu^2 \sigma(\mu^2) \frac{i}{p^2 - \mu^2 + i\epsilon} . \quad (5.13)$$

$\tilde{G}^{(2)}(p)$ can be expressed by means of the complex function

$$\Gamma[s] := \sum_j \frac{iZ_j}{s - m_j^2} + \int_{4m_1^2}^{\infty} d\mu^2 \sigma(\mu^2) \frac{i}{s - \mu^2} ,$$

for s approaching the real p^2 value from the upper complex direction

$$\tilde{G}^{(2)}(p) = \Gamma(s = p^2 + i\epsilon) .$$

$\Gamma[s]$ has poles for $s = m_j^2$ and a cut along the real axis from $4m_1^2$ to ∞ , with discontinuity

$$\Gamma(r + i\epsilon) - \Gamma(r - i\epsilon) = 2\pi\sigma(r) .$$

Consider now a single scalar field in a potential density $V(\phi, \lambda)$, vanishing for $\lambda = 0$. In the $\lambda = 0$ case $\tilde{G}^{(2)}(p)$ reduces to

$$\tilde{G}_0^{(2)}(p) = \frac{i}{p^2 - m_0^2 + i\epsilon} ,$$

where m_0 denotes the mass in the Lagrangian. In the $\lambda \neq 0$ case, instead

$$\tilde{G}^{(2)}(p) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{4m^2}^{\infty} d\mu^2 \sigma(\mu^2) \frac{i}{p^2 - \mu^2 + i\epsilon} .$$

If $m^2 \neq m_0^2$ mass renormalisation is introduced. If $Z \neq 1$ a renormalisation of ϕ is performed, in such a way that $|\langle \Omega | \phi_{\text{phys}}(0) | k \rangle|^2 = 1$, where

$$\phi_{\text{phys}} := \sqrt{Z} \phi .$$

To further investigate the role of Z , it is useful to write down the Källén-Lehmann representation in the case of $\langle \Omega | [\phi(x), \phi(y)] | \Omega \rangle$. The derivation of such a representation is completely analogous to the one that leads to Eq.(5.13)

$$\langle \Omega | [\phi(x), \phi(y)] | \Omega \rangle = iZ\Delta(x - y; m) + i \int_{4m^2}^{\infty} d\mu^2 \sigma(\mu^2) \Delta(x - y; \mu) , \quad (5.14)$$

where

$$i\Delta(x - y; m) = W_0^{(2)}(x - y) - W_0^{(2)}(y - x) = \int \frac{d^4q}{(2\pi)^3} \epsilon(q^0) \delta(q^2 - m^2) e^{-iq(x-y)} ,$$

with $\epsilon(x) = \theta(x) - \theta(-x)$ the sign function. By taking the time derivative of (5.14) and then computing it at $t = 0$, we get⁷

$$1 = Z + \int_{4m^2}^{\infty} d\mu^2 \sigma(m^2) ,$$

that, due to the positivity of σ , implies

$$0 \leq Z < 1 .$$

Note that here we are considering the interacting case, the free case corresponds to $\sigma = 0$, that is $Z = 1$.

It is worth stressing that the preceding discussion concerns the exact theory. As a consequence, even if the above wave-function renormalisation also arises in considering the UV divergences discussed in the renormalisation chapter, a comparison between the non-perturbative and the perturbative wave function renormalisations requires particular attention.

5.3 LSZ reduction formula⁸

The Lehmann-Symanzik-Zimmermann reduction formula is a fundamental result in QFT. It shows that the building blocks to compute the transition amplitudes, and therefore the cross sections, are the vacuum expectation values of time ordered operators. In the following we follow the analysis in Srednicki's book, Quantum Field Theory.⁹

Let us consider a field of the form

$$\phi(x) = \int d\tilde{k} (a(k)e^{-ikx} + a^\dagger(k)e^{ikx}) , \quad (5.15)$$

where we used

$$d\tilde{k} = \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} , \quad k^0 = \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} > 0 .$$

⁷ Recall that $[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$, $\pi := \partial_0\phi$, so that,

$$\partial_{x_0}[\phi(x), \phi(y)]|_{x_0=y_0=0} = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}) .$$

Since in the free case $[\phi(x), \phi(y)] = i\Delta(x - y)$, this also implies $\partial_{x_0}\Delta(x - y)|_{x_0=y_0=0} = -\delta^{(3)}(\mathbf{x} - \mathbf{y})$, that can be verified by explicit calculation.

⁸ Luca Teodori

⁹ Srednicki's book uses the metric $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ which is the opposite of the one used here and in most of the QFT literature. Since $x^\mu = (x^0, \mathbf{x})$, it follows that with Srednicki's choice $x_\mu = (-x_0, \mathbf{x})$. In the following we continue to use the standard QFT notation, so to compare with the Srednicki book, the reader should keep in mind that, for example, $(A_\mu B^\mu)_{\text{Srednicki}} = -(A_\mu B^\mu)$.

In this section we use the normalisation choice for $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ done in Srednicki's book. Besides the usual condition for the ground state $|0\rangle$, $\langle 0|0\rangle = 1$, we use $|\mathbf{k}\rangle = a^\dagger(\mathbf{k})|0\rangle$ and the Lorentz invariant normalisation for the commutation relations¹⁰

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k}') ,$$

which implies the Lorentz invariant orthonormality condition

$$\langle k|k'\rangle = (2\pi)^3 2\omega_{\mathbf{k}} \delta^{(3)}(\mathbf{k} - \mathbf{k}') .$$

Note that since k^0 is fixed by the value of \mathbf{k} , it follows that the notation $|k\rangle$ contains the same information of $|\mathbf{k}\rangle$. Let us express the creation operator in terms of ϕ

$$a^\dagger(\mathbf{k}) = -i \int d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \phi(x) , \quad (5.16)$$

where $\overleftrightarrow{\partial}_0 \equiv \overrightarrow{\partial}_0 - \overleftarrow{\partial}_0$. We associate to each momentum \mathbf{k}_i the operator

$$a_i^\dagger \equiv a_i^\dagger(\mathbf{k}_i) := \int d^3k f(\mathbf{k}, \mathbf{k}_i) a^\dagger(\mathbf{k}) , \quad f_i(\mathbf{k}) \equiv f(\mathbf{k}, \mathbf{k}_i) \propto \exp\left(-\frac{(\mathbf{k} - \mathbf{k}_i)^2}{4\sigma^2}\right) , \quad (5.17)$$

creating a state with a gaussian-distributed momenta localised near \mathbf{k}_i . In \mathbf{x} space this corresponds to a particle localised near the origin.¹¹ Note that with the dynamical evolution, the wave packet spreads. If we move on to the interacting theory, the $a^\dagger(\mathbf{k})$ will no longer be time independent, so from Eq.(5.17) one sees that also the a_i^\dagger will depend on time. So, having a scattering experiment in mind, one can be interested at the initial state and the final state as two particles widely separated in the far past and in the far future respectively; in formulas

$$|i\rangle = \lim_{t \rightarrow -\infty} a_1^\dagger(t) a_2^\dagger(t) |0\rangle , \quad |f\rangle = \lim_{t \rightarrow +\infty} a_1^\dagger(t) a_2^\dagger(t) |0\rangle .$$

By (5.16) and (5.17) one finds

$$a_i^\dagger = -i \int d^3k f_i(\mathbf{k}) \int d^3x e^{-ikx} \overleftrightarrow{\partial}_0 \phi(x) . \quad (5.18)$$

¹⁰ It is worth stressing that such a Lorentz invariant normalisation is also used in the Itzykson-Zuber book [7], whereas in the book by Peskin and Schroeder [11] the operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are the ones in (5.15) divided by $\sqrt{2\omega_{\mathbf{k}}}$.

¹¹ Recall that the Fourier transform of a Gaussian is, up to a possible phase, still a Gaussian. For example,

$$\frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{(x-x_0)^2}{4\sigma^2}} e^{-ikx} = \sqrt{2\sigma} e^{ix_0k} e^{-\sigma^2 k^2} ,$$

whose modulo square describes a particle localised near the origin.

To derive the LSZ reduction formula, we consider the following steps

$$\begin{aligned}
 a_i^\dagger(+\infty) - a_i^\dagger(-\infty) &= \int_{-\infty}^{+\infty} dt \partial_0 a_i^\dagger(t) \\
 &= -i \int d^3k f_i(\mathbf{k}) \int d^4x \partial_0(e^{-ikx} \overleftrightarrow{\partial}_0 \phi(x)) \\
 &= -i \int d^3k f_i(\mathbf{k}) \int d^4x e^{-ikx} (\partial_0^2 + \mathbf{k}^2 + m^2)\phi(x) \\
 &= -i \int d^3k f_i(\mathbf{k}) \int d^4x e^{-ikx} (\square + m^2)\phi(x) ,
 \end{aligned}$$

where in the first line we used the fundamental theorem of calculus, in the second line we used (5.18), on the third line we did the derivative and used $(k^0)^2 = \mathbf{k}^2 + m^2$; now notice that we can replace \mathbf{k}^2 by $-\overleftarrow{\nabla}^2$, so on the fourth line we changed $\overleftarrow{\nabla}^2$ with $\overrightarrow{\nabla}^2 \equiv \nabla^2$ by performing a double integration by parts (the boundary term always vanishes). Notice that since in a free theory $\phi(x)$ satisfies the Klein Gordon equation, the last line would be zero and this is reminiscent of the fact that in a free theory a_i^\dagger is time independent.

We then have

$$a_i^\dagger(-\infty) = a_i^\dagger(+\infty) + i \int d^3k f_i(\mathbf{k}) \int d^4x e^{-ikx} (\square + m^2)\phi(x) , \quad (5.19)$$

$$a_i(+\infty) = a_i(-\infty) + i \int d^3k f_i(\mathbf{k}) \int d^4x e^{ikx} (\square + m^2)\phi(x) , \quad (5.20)$$

where the second line is simply the Hermitian conjugate of the first. Turning back to our two particle states, we can write the following amplitude (we will use the prime to denote the operators relating to the final state)

$$\begin{aligned}
 \langle f|i \rangle &= \langle 0|a'_1(+\infty)a'_2(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)|0 \rangle \\
 &= \langle 0|T\{a'_1(+\infty)a'_2(+\infty)a_1^\dagger(-\infty)a_2^\dagger(-\infty)\}|0 \rangle ,
 \end{aligned} \quad (5.21)$$

where, since the operators are already time ordered, in the last equality we inserted the T product. Such an insertion simplifies the expression. In fact by (5.19) and (5.20) in (5.21), it follows that the T product moves all the $a_i(-\infty)$ to the right and all the $a_i^\dagger(+\infty)$ to the left, where they annihilate the vacuum. Since now the wave packets do not play any particular role, we can take the limit $\sigma \rightarrow 0$ with the effect of shrinking the wave packets, $a_i^\dagger \rightarrow \int d^3k \delta^{(3)}(\mathbf{k} - \mathbf{k}_i) a^\dagger(k) = a^\dagger(\mathbf{k}_i)$ since¹² $f_i(\mathbf{k}) \rightarrow \delta^{(3)}(\mathbf{k} - \mathbf{k}_i)$.

¹² Recall that

$$\lim_{\sigma \rightarrow 0} \int_{\mathbb{R}} dx \frac{1}{2\sqrt{\pi}\sigma} e^{-(x-x_0)^2/(4\sigma^2)} f(x) = f(x_0) ,$$

which holds for any test function $f(x)$. This shows that the tempered distribution δ is not a regular tempered distribution, that is a distribution F_g such that

$$F_g(f) = \int_{\mathbb{R}} dx g(x) f(x) ,$$

We then get the LSZ reduction formula

$$\begin{aligned} \langle f|i\rangle &= \int d^4x_1 e^{-ik_1x_1}(\square_1 + m^2) \int d^4x_2 e^{-ik_2x_2}(\square_2 + m^2) \int d^4x'_1 e^{ik'_1x'_1}(\square'_1 + m^2) \\ &\quad \times \int d^4x'_2 e^{ik'_2x'_2}(\square'_2 + m^2) \langle 0|T\phi(x_1)\phi(x_2)\phi(x'_1)\phi(x'_2)|0\rangle . \end{aligned}$$

Such a result generalises to the case in which there are n particles in the initial state and n' in the final state

$$\begin{aligned} \langle f|i\rangle &= i^{n+n'} \prod_j \int d^4x_j e^{-ik_jx_j}(\square_j + m^2) \prod_k \int d^4x'_k e^{ik'_kx'_k}(\square'_k + m^2) \\ &\quad \times \langle 0|T\phi(x_1)\dots\phi(x_n)\phi(x'_1)\dots\phi(x'_{n'})|0\rangle . \end{aligned} \quad (5.22)$$

An interesting related topic concerns the dispersion relations for multi-particle states. In the following we assume

- ▷ the vacuum state $|0\rangle$ is unique;
- ▷ $P^\mu|0\rangle = 0$.

Let us consider the first excited state. Note that in the $(-|\mathbf{P}| \cup |\mathbf{P}|, E)$ plane the dispersion relation $E(\mathbf{P}^2) - \mathbf{P}^2 = m^2$ is an isolated hyperbola with minimum for $\mathbf{P} = \mathbf{0}$

$$\text{1-particle state: } E(0) = m .$$

However, the dispersion relation for the n -particle state, $n \geq 2$, is no longer an isolated hyperbola. Rather, it corresponds to the surface bounded by the hyperbola passing through $(0, 2nm)$. To see this it is sufficient to consider the case of the two-particle state. Let $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ be the total three-momentum. Since the total energy is

$$E^2 = (E_1 + E_2)^2 = 2m^2 + \mathbf{P}_1^2 + \mathbf{P}_2^2 + 2\sqrt{(m^2 + \mathbf{P}_1^2)(m^2 + \mathbf{P}_2^2)} ,$$

it follows that, unlike in the case of one-particle state, the value of total energy when $\mathbf{P}^2 = 0$ may now be greater than $2m$, that is

$$\text{2-particle state: } E(0) \geq 2m .$$

Let us now consider the vacuum expectation value of $\phi(x)$. We have $\langle 0|\phi(x)|0\rangle = \langle 0|e^{iPx}\phi(0)e^{-iPx}|0\rangle = \langle 0|\phi(0)|0\rangle$. On the other hand, we want that $a_i^\dagger(\pm\infty)$ creates a one-particle state, whereas $\langle 0|\phi(0)|0\rangle \neq 0$ would imply that $a_i^\dagger|0\rangle$ contains $|0\rangle$. For this

for any test function $f(x)$.

reason we do the shift

$$\phi(x) \rightarrow \phi(x) - v ,$$

$$v = \langle 0|\phi(0)|0\rangle .$$

Let us investigate the case of a one-particle state. Let p be its four-momentum and consider

$$\langle p|\phi(x)|0\rangle = \langle p|e^{iPx}\phi(0)e^{-iPx}|0\rangle = e^{ipx} \langle p|\phi(0)|0\rangle .$$

Note that by Lorentz invariance, $\langle p|\phi(0)|0\rangle$ depends on p only through $p^2 = m^2$. In the limit of asymptotic times, the creation/annihilation operators should behave as free operators, so that we should have $\langle p|\phi(0)|0\rangle = 1$. This can be obtained by a finite rescaling of ϕ .

Finally, we consider the case of multi-particle states. Let P be the total four-momentum and denote by n all the other quantum numbers. We have

$$\langle p, n|\phi(x)|0\rangle = \langle p, n|e^{iPx}\phi(0)e^{-iPx}|0\rangle = e^{ipx} \langle p, n|\phi(0)|0\rangle .$$

Again, since we want that the only effect of $a_i^\dagger(\pm\infty)$ is the creation of a one-particle state, the matrix element $\langle p, n|a_i^\dagger(\pm\infty)|0\rangle$ should vanish, that maybe the case even if $\langle p, n|\phi(0)|0\rangle \neq 0$. However, this is a subtle point, and rather than evaluating $\langle p, n|a_i^\dagger(\pm\infty)|0\rangle$, we should consider $\langle \psi|a_i^\dagger(\pm\infty)|0\rangle$, where $|\psi\rangle$ is a normalisable state

$$|\psi\rangle = \sum_n \int d^3p \psi_n(\mathbf{p}) |p, n\rangle ,$$

with $\psi_n(\mathbf{p})$ some wave packets. We have

$$\begin{aligned} \langle \psi|a_i^\dagger(t)|0\rangle &= -i \sum_n \int d^3p \psi_n^*(\mathbf{p}) \int d^3k f_i(\mathbf{k}) \int d^3x \left(e^{-ikx} \overleftrightarrow{\partial}_0 e^{ipx} \right) A_n(\mathbf{p}) \\ &= \sum_n \int d^3p \psi_n^*(\mathbf{p}) \int d^3k f_i(\mathbf{k}) \int d^3x (p^0 + k^0) e^{i(p-k)x} A_n(\mathbf{p}) , \end{aligned} \quad (5.23)$$

where

$$A_n(\mathbf{p}) = \langle p, n|\phi(0)|0\rangle ,$$

and $p^0 = \sqrt{\mathbf{p}^2 + M^2}$, $k^0 = \sqrt{\mathbf{p}^2 + m^2}$ and $M^2 = p^2$. Finally, using the integral representation of $\delta^{(3)}(\mathbf{k} - \mathbf{p})$, we get

$$\langle \psi|a_i^\dagger(t)|0\rangle = \sum_n \int d^3p (2\pi)^3 (p^0 + k^0) \psi_n^*(\mathbf{p}) f_i(\mathbf{p}) e^{i(p^0 - k^0)t} A_n(\mathbf{p}) ,$$

Now note that by $M \geq 2m > m$ it follows that

$$p^0 > k^0 ,$$

so that, for asymptotic times, the term $e^{i(p^0-k^0)t}$ oscillates very rapidly. In particular, by the Riemann-Lebesgue Lemma,¹³ it follows that

$$\langle \psi | a_i^\dagger(\pm\infty) | 0 \rangle = 0 .$$

With all these conditions we made legitimate all the steps that brought us to (5.22). We can say that from the LSZ reduction formula we found out that the Green functions are the building blocks of the scattering amplitudes, so our effort in trying to find out the Green functions of a certain process that will be done in next sections is completely justified.

¹³The key point in the proof of such a Lemma is to note that if $f \in C^\infty$ vanishes at $\pm\infty$, then, integrating by parts, we get

$$\int_{\mathbb{R}} d\omega f(\omega) e^{-i\omega t} = \left(-\frac{i}{t}\right)^N \int_{\mathbb{R}} d\omega \frac{d^N f(\omega)}{d\omega^N} e^{-i\omega t} ,$$

for any positive integer N . This means that for $t \rightarrow \pm\infty$ the integral $\int_{\mathbb{R}} d\omega f(\omega) e^{-i\omega t}$ vanishes faster than any positive power of $1/t$.

Chapter 6

Path Integral Formulation of Quantum Mechanics

6.1 Hamilton-Jacobi Theory¹

As we will see, the path integral formulation of quantum mechanics has been introduced by Dirac who investigated the quantum analogue of the canonical transformations. Here we shortly review the main topics concerning canonical transformations and the Hamilton-Jacobi theory.

In classical Physics a dynamical system of n particles can be described in the Lagrange formalism by a function $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where $\mathbf{q} = (q_1, \dots, q_n)$ are the spatial coordinates, $\dot{\mathbf{q}} = (\partial q_1/\partial t, \dots, \partial q_n/\partial t)$ are the generalised velocities and t denotes time. The equations of motion are derived by applying the minimal action principle, which states that if the configurations at time $t = t_1$ and $t = t_2$ are fixed at the points $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$, the trajectories followed by the system connecting these two points are the ones that minimise the action

$$S = \int_{t_1}^{t_2} dt L(\mathbf{q}, \dot{\mathbf{q}}, t) . \quad (6.1)$$

The Euler-Lagrange equations are derived after a few calculations, yielding the well-known result

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} . \quad (6.2)$$

It is also possible to describe the system in terms of the coordinates \mathbf{q} and the generalised momenta $\mathbf{p} = \partial L(\mathbf{q}, \dot{\mathbf{q}}, t)/\partial \dot{\mathbf{q}}$ in place of the velocities, shifting from the Lagrange point of view to the Hamilton formalism. As it is often done in Thermodynamics, one performs a Legendre transformation to shift from a function depending on a set of

¹ Federico De Bettin

independent variables, in this case $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, to one that depends on a different one, $H(\mathbf{q}, \mathbf{p}, t)$. The new function is the Hamilton function

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^n p_i \dot{q}_i - L . \quad (6.3)$$

One may immediately realise that the independent variables of such function are indeed \mathbf{q} and \mathbf{p} by calculating its total differential

$$dH = \sum_i \left(\dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i \right) .$$

From such expression, the *Hamilton equations*, which are the equations of motion, are readily derived by keeping in mind the Euler-Lagrange equations (6.2)

$$\dot{q}_i = \frac{\partial H}{\partial p_i} , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} . \quad (6.4)$$

By defining the *Poisson brackets*

$$\{\cdot, \cdot\} : function \times function \longrightarrow function \quad (6.5)$$

$$\{f, g\} = \sum_k \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} \right) ,$$

one may write the Hamilton equations as

$$\dot{q}_i = \{H, q_i\} , \quad \dot{p}_i = \{H, p_i\} . \quad (6.6)$$

Actually, the total derivative with respect to time of any function $f = f(\mathbf{q}, \mathbf{p}, t)$ can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} ,$$

by taking into account the Hamilton equations (6.4). It follows that a constant of motion, also known as *first integral*, I , must be such that

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \{H, I\} = 0 ,$$

so that if it does not depend explicitly on t it must satisfy

$$\{H, I\} = 0 .$$

It is important to notice that the Poisson brackets satisfy the relations required by the

Lie brackets, i.e. for any function of $\mathbf{q}, \mathbf{p}, t$ and $a, b, c \in \mathbb{C}$

$$\begin{aligned} \{f, f\} &= 0 && \textit{antisymmetric} \\ \{af_1 + bf_2, cg\} &= (ac)\{f_1, g\} + (bc)\{f_2, g\} && \textit{bilinear} \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0 && \textit{Jacobi - identity} , \end{aligned}$$

meaning that the conjugate variables \mathbf{q} and \mathbf{p} form a Lie algebra. In particular, the Poisson brackets of \mathbf{q} and \mathbf{p} yield

$$\{q_i, q_j\} = 0 , \quad \{p_i, p_j\} = 0 , \quad \{q_i, p_j\} = \delta_{ij} . \quad (6.7)$$

6.1.1 The Action as a Function of Coordinates

One may write the action S in terms of the Hamilton function through the explication of L in (6.3) as

$$S = \int_{t_1}^{t_2} \left(\sum_i p_i dq_i - H(q, p) dt \right) . \quad (6.8)$$

The equations of motion can be derived again by applying the principle of least action. The difference with respect to the Lagrange formalism is that to get to the Hamilton equations, one must vary independently both the coordinates and the momenta. In the Lagrange case, instead, one had to vary only the coordinates, and the generalised velocities would vary accordingly. The \mathbf{q} and \mathbf{p} must actually be considered as independent variables, then.

To show this, we take the variation of the action, keeping the integration extreme t_1 and t_2 and the initial and final points of the trajectories $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$ fixed

$$\delta S = \int_{t_1}^{t_2} \sum_i \left(\delta p_i dq_i + p_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i dt - \frac{\partial H}{\partial p_i} \delta p_i dt \right) .$$

After integrating by parts the second term in the integral, one gets

$$\delta S = \sum_i \left[\int_{t_1}^{t_2} \delta p_i \left(dq_i - \frac{\partial H}{\partial p_i} dt \right) + p_i \delta q_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_i \left(dp_i + \frac{\partial H}{\partial q_i} dt \right) \right] , \quad (6.9)$$

which is null only if the Hamilton equations (6.4) are satisfied, since $\delta \mathbf{q} = 0$ at the extreme of integration by hypothesis.

The action is a function of the trajectories of a system described by a Lagrange or Hamilton function. It is possible, then, to work on shell and see its behaviour when varying from a physical trajectory to another. To do this, we fix the initial point in space and time of a trajectory, so that the variation $\delta \mathbf{q}(t_1) = 0$; we fix the time of

arrival t_2 , but not the final configuration, so that $\delta \mathbf{q}(t_2) = \delta \mathbf{q} \neq 0$. By looking at (6.9) and recalling that we are working on shell, so that the Hamilton equations are indeed satisfied, one finds

$$\delta S = \sum_i p_i \delta q_i .$$

This means that, on shell, S can be considered as a function of the coordinates at the time at the upper extremum of integration, since such equation implies that

$$\frac{\partial S}{\partial q_i} = p_i . \quad (6.10)$$

If the final time t_2 is also considered as variable, we know by (6.1) that

$$\frac{dS}{dt} = L(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) .$$

In the meantime, though, if one works on shell as before,

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i .$$

By confronting these two equations and using (6.10) one can make sense of the partial derivative of the action with respect to time, as

$$\frac{\partial S}{\partial t} = L - \sum_i p_i \dot{q}_i .$$

It is then a remarkable result that, by using (6.3), one can write

$$\frac{\partial S}{\partial t} = -H , \quad (6.11)$$

from which the Hamilton-Jacobi equation will be derived.

Equations (6.10) and (6.11) can be used to write the total differential of the action as a function of coordinates and time in the upper extremum of the integral (6.8):

$$dS = \sum_i p_i dq_i - H dt . \quad (6.12)$$

The function $S = S(q, t)$ with such differential is called *Hamilton principal function* and, as deducible from how it has been constructed, it differs from the action S by a constant coming from the lower extremum of the integral.

Now, if the initial points in space and time are also considered to be variables, the total

differential of S will have the form

$$dS = \sum_i p_i^{(2)} dq_i^{(2)} - H^{(2)} dt^{(2)} - \sum_i p_i^{(1)} dq_i^{(1)} + H^{(1)} dt^{(1)}, \quad (6.13)$$

which is the subtraction of (6.12) calculated in the upper and lower integration extreme of the action. Only the orbits for which (6.13) is an exact differential can be physical.

6.1.2 Canonical Transformations

The n coordinates \mathbf{q} can be mapped to another n new coordinates \mathbf{Q} by a possibly time dependent diffeomorphism

$$Q_i = Q_i(\mathbf{q}, t).$$

Such transformation, in the Lagrange formalism, induces a transformation of the generalised velocities

$$\dot{Q}_i = \sum_j \frac{\partial Q_i}{\partial q_j} \dot{q}_j + \frac{\partial Q_i}{\partial t}.$$

Such transformations, the only possible ones in the Lagrange formalism, are called *punctual transformations*; they leave both the Euler-Lagrange and the Hamilton equations invariant and are performed on shell. As we discussed in the previous section, though, in the Hamilton case \mathbf{q} and \mathbf{p} are independent variables. Thanks to this fact, changes of variables can be made also off shell, provided they leave the Hamilton equations invariant. Such transformations are called *canonical*, and they have many more possibilities than punctual transformations, since they are of the form

$$Q_i = Q_i(\mathbf{q}, \mathbf{p}, t), \quad P_i = P_i(\mathbf{q}, \mathbf{p}, t). \quad (6.14)$$

The physical meaning of coordinates and momenta may be lost after a canonical transformation, since the two can be mixed to create an entirely new variable. Momenta may no longer be the usual mechanical momenta and one may actually transform coordinates into momenta and viceversa.

For a transformation to be canonical, the condition is that for any hamiltonian H ,

$$\dot{Q}_i = \frac{\partial \tilde{H}}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial \tilde{H}}{\partial Q_i},$$

where \tilde{H} will be a new Hamilton function related to the original H and the particular transformation used. Its explicit expression will be found later.

Since the Hamilton equations can be derived using the principle of least action, as we

found in (6.9), we want that both

$$\delta S = \delta \int \left(\sum_i p_i dq_i - H dt \right) = \delta \int \left(\sum_i P_i dQ_i - \tilde{H} dt \right) = 0 ,$$

yielding equivalent equations of motion, i.e. equations that can be mapped to each other using (6.14) or its inverse. But this requirement is satisfied only if the two expressions in the integral differ by an exact differential dF , that is

$$\sum_i p_i dq_i - H dt = \sum_i P_i dQ_i - \tilde{H} dt + dF ,$$

where, then,

$$dF = \sum_i p_i dq_i - \sum_i P_i dQ_i - (H - \tilde{H}) dt . \quad (6.15)$$

It is clear that in this form $F = F(\mathbf{q}, \mathbf{Q}, t)$, and that it defines a canonical transformation, as

$$\frac{\partial F}{\partial q_i} = p_i , \quad \frac{\partial F}{\partial Q_i} = -P_i , \quad \tilde{H} = \frac{\partial F}{\partial t} + H . \quad (6.16)$$

Here, \mathbf{q} and \mathbf{Q} are taken as independent variables, while the momenta \mathbf{p} and \mathbf{P} are defined through F , as well as the new Hamilton function \tilde{H} . If F does not depend explicitly on time, $\tilde{H}(\mathbf{Q}, \mathbf{P}) = H(\mathbf{q}(\mathbf{Q}, \mathbf{P}, t), \mathbf{p}(\mathbf{Q}, \mathbf{P}, t))$.

By exploiting Legendre transformations, one may express, for example, $F = F(\mathbf{q}, \mathbf{P}, t)$ just by taking $F \rightarrow F + \sum_i P_i Q_i$ in (6.15). This way

$$\frac{\partial F}{\partial q_i} = p_i , \quad \frac{\partial F}{\partial P_i} = Q_i , \quad \tilde{H} = \frac{\partial F}{\partial t} + H . \quad (6.17)$$

The function F , in both cases, is known as the *generating function* of the transformation.

Now one may take into consideration the exact differential of the action found in (6.13) with fixed initial and final times: t and $t + \tau$ respectively. The expression then becomes

$$dS = \sum_i p_{i,t+\tau} dq_{i,t+\tau} - \sum_i p_{i,t} dq_{i,t} .$$

Such is to be compared to (6.15) to realise that S is actually the generator of a canonical transformation. The flux of the Hamilton equations is indeed a diffeomorphism for any fixed t . The opposite of the action, $-S$, is then the generator of such change of variables, which is a canonical transformation. This fact is crucial in Dirac's article introducing the path integration formalism in quantum mechanics [2].

Canonical transformations preserve the Poisson brackets. In particular it can be checked

that

$$\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{P_i, Q_j\} = \delta_{ij}, \quad (6.18)$$

and that for any two functions $f(\mathbf{q}, \mathbf{p}, t)$ and $g(\mathbf{q}, \mathbf{p}, t)$

$$\{f(\mathbf{Q}, \mathbf{P}, t), g(\mathbf{Q}, \mathbf{P}, t)\} = \{f, g\}(\mathbf{Q}, \mathbf{P}, t), \quad (6.19)$$

where the left-hand side is derived with respect to \mathbf{Q} and \mathbf{P} , while the right-hand one with respect to \mathbf{q} and \mathbf{p} . Equation (6.18) can be proved via direct computation, while (6.19) follows by applying the chain rule to the left-hand side of the equation and then using (6.18).

Simply thanks to this consideration, one may realise that the shift from first quantisation in configuration space to second quantisation in Fock space is simply a canonical transformation of variables. The Poisson brackets are indeed invariant under such transformation.

6.1.3 The Hamilton-Jacobi Equation

By taking equation (6.11) and writing all the momenta in terms of equation (6.10) one finds the Hamilton Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}; t\right) = 0. \quad (6.20)$$

Such is a first order partial differential equation for $S(\mathbf{q}, t)$. We are interested in finding the general solution of the equation, that must depend on the initial conditions of the system, that will be denoted as $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, so that

$$S = S(t, \mathbf{q}; \boldsymbol{\alpha}) + A,$$

where all the α_i and A are arbitrary constants. One can use $S(t, \mathbf{q}; \boldsymbol{\alpha}) = S(t, \mathbf{q}, \boldsymbol{\alpha})$ as the generator of a canonical transformation by considering $\boldsymbol{\alpha}$ as the new momenta of the system. In this way one can use the equations (6.17) to get

$$\frac{\partial S}{\partial q_i} = p_i, \quad \frac{\partial S}{\partial \alpha_i} = \beta_i, \quad \tilde{H} = \frac{\partial S}{\partial t} + H. \quad (6.21)$$

But since the Hamilton-Jacobi equation (6.20) is satisfied by S , we have $\tilde{H} = 0$. The new Hamilton equations, then, are

$$\dot{\alpha}_i = 0, \quad \dot{\beta}_i = 0,$$

meaning that all α and β are constants of motion, which is coherent with the fact that α are arbitrary constants.

On the other hand, the second equation in (6.21) can be inverted in such a way to express the coordinates \mathbf{q} in terms of α , β and t . The expressions for the momenta can then be found by computing $p_i = \partial S / \partial q_i$. This procedure is always applicable provided that one can actually solve equation (6.20).

In the case in which the Hamilton function does not depend on time explicitly, $\{H, H\} = 0$, meaning that it is a constant of motion. Along a given trajectory, then, one may write $H(\mathbf{q}(t), \mathbf{p}(t)) = E$, so that the action becomes

$$S = \sum_i \int_{t_1}^{t_2} p_i dq_i - Et = S_0(\mathbf{q}) - Et . \quad (6.22)$$

By plugging this expression in (6.20), the Hamilton-Jacobi equation is calculated along a trajectory of energy E and becomes

$$H\left(\mathbf{q}, \frac{\partial S_0}{\partial \mathbf{q}}\right) = E . \quad (6.23)$$

S_0 is the *Hamilton's characteristic function*.

In the case in which a pair of conjugate variables, say q_1 and $\frac{\partial S}{\partial q_1}$, enter in the expression of the Hamilton-Jacobi equation only through a term $\psi\left(q_1, \frac{\partial S}{\partial q_1}\right)$, so that H becomes

$$H = H\left(\psi\left(q_1, \frac{\partial S}{\partial q_1}\right), q_2, \dots, q_n, \frac{\partial S}{\partial q_2}, \dots, \frac{\partial S}{\partial q_n}\right) ,$$

S is a sum of a term depending on q_1 and $\frac{\partial S}{\partial q_1}$ and a term depending on the other variables

$$S = S_1(q_1) + S'(q_2, \dots, q_n, t) . \quad (6.24)$$

By plugging this expression in (6.20) one finds

$$\frac{\partial S'}{\partial t} + H\left(\psi\left(q_1, \frac{\partial S_1}{\partial q_1}\right), q_2, \dots, q_n, \frac{\partial S'}{\partial q_2}, \dots, \frac{\partial S'}{\partial q_n}\right) = 0 ,$$

which must be an identity for any value of q_1 . Such condition is verified if $\psi\left(q_1, \frac{\partial S_1}{\partial q_1}\right)$ is a constant of motion. Equation (6.24), then, can be partitioned in two equations, one

of which is an ordinary differential equation that, in principle, can be easily solved:

$$\psi \left(q_1, \frac{\partial S_1}{\partial q_1} \right) = \alpha_1 ,$$

$$\frac{\partial S'}{\partial t} + H \left(\alpha_1, q_2, \dots, q_n, \frac{\partial S'}{\partial q_2}, \dots, \frac{\partial S'}{\partial q_n} \right) = 0 ,$$

where α_1 is an arbitrary constant. In the ideal case, through a good choice of coordinates, one would want all the variables to separate in order to transform the initial equation (6.20) into n ordinary differential equations. At that point, the action could be expressed as

$$S = \sum_i S_i(q_i; \alpha_1, \dots, \alpha_n) - E(\alpha_1, \dots, \alpha_n)t , \quad (6.25)$$

where the energy in terms of the arbitrary constants can be found using (6.23), by substituting the Hamilton's characteristic function $S_0 \rightarrow \sum_i S_i$.

6.2 On Dirac's article where the path integral is formulated for the first time

In the following we introduce the path integral formulation. Such a formulation has been introduced by Dirac in [2]. As we will see, all the main points of the path integral formulation of quantum mechanics were explicitly stated. It also includes a section on the "Applications to Field Dynamics". Dirac's reasoning takes off from the observation that the canonical transformations $(q, p) \rightarrow (Q, P)$ generated by Hamilton's principal function $S(q, Q, t)$ treats q and Q as independent variables. Dirac's idea is to introduce a quantum mechanical analogue of canonical transformations and identify a quantum version of the Hamilton-Jacobi relations

$$p = \frac{\partial S}{\partial q} , \quad P = -\frac{\partial S}{\partial Q} .$$

In other words, Dirac's idea was to find the operator version of such relations, that is

$$\hat{p} = \frac{\widehat{\partial S}}{\partial q} , \quad \hat{P} = -\frac{\widehat{\partial S}}{\partial Q} . \quad (6.26)$$

To this aim, he introduced a coordinate representation $|Q\rangle$ "independent" of $|q\rangle$. This raises naturally the problem of determining the "mixed" matrix elements $\langle q|Q\rangle$.

However, in Ramond textbook there are some inaccuracies on this matter. In equation

(2.1.36), Ramond uses the identification

$$\hat{p}|q\rangle = -i\hbar \frac{\partial}{\partial q} |q\rangle . \quad (6.27)$$

It is important to pay attention to the action of a given operator on “bras” and “kets”. The determination of such an action is immediate when the state is described corresponds to a physical observable represented by a self-adjoint operator, e.g. $\hat{p}|p'\rangle = p'|p'\rangle$ and $\hat{q}|q'\rangle = q'|q'\rangle$. On the other hand the relation

$$\hat{p}\psi(q, t) = -\frac{i}{\hbar} \partial_q \psi(q, t) ,$$

does not imply (6.27). We take the opportunity to show that the correct version of (6.27) has the opposite sign to make some observations on related topics, such as on the bra and ket representation of states.

Let us first note that since in quantum mechanics the time t is not an observable,² it follows that, in any representation, the Schrödinger equation has the same form.³ So, for example,

$$H(\hat{p}, \hat{q}, t)|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle . \quad (6.28)$$

In virtue of the definition of the coordinate representation based on the identity

$$\psi_s(q) = \langle q|s\rangle ,$$

we have

$$\langle q|\hat{p} = -i\hbar \frac{\partial}{\partial q} \langle q| , \quad \langle q'|\hat{q} = q' \langle q'| .$$

In particular, being $\langle q|\psi(t)\rangle = \psi(q, t)$, the usual representation of the Schrödinger differential equation

$$i\hbar \frac{\partial}{\partial t} \psi(q, t) = H(-i\hbar \partial_q, q, t) \psi(q, t) ,$$

is directly recovered from (6.28)

$$\langle q|H(\hat{p}, \hat{q}, t)|\psi(t)\rangle = H(-i\hbar \partial_q, q, t) \langle q|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} \langle q|\psi(t)\rangle .$$

The fact that the relation (6.27) does not indeed have the right sign, follows immediately

² Recall that in quantum field theory both time and spatial coordinates are parameters.

³ For related issues see section 1.4.5 of Kleinert’s book [17]. The field theory textbooks by Kleinert are very useful references. For example, sections 7.17 and 7.18, concern a well written and detailed analysis of Wick’s theorem. Another useful book by Kleinert is [18].

by observing that $\langle p|q\rangle = \bar{\psi}_p(q) = e^{-\frac{i}{\hbar}pq}/\sqrt{2\pi}$, so that, by

$$\frac{\partial}{\partial q}|q\rangle = \lim_{a \rightarrow 0} \frac{|q+a\rangle - |q\rangle}{a} ,$$

we have

$$\langle p|\frac{\partial}{\partial q}|q\rangle = \lim_{a \rightarrow 0} \frac{\langle p|q+a\rangle - \langle p|q\rangle}{a} = \frac{\partial}{\partial q}\bar{\psi}_p(q) = -\frac{i}{\hbar}p\bar{\psi}_p(q) ,$$

Therefore,

$$\langle p'|\hat{p}|q\rangle = \langle p'|i\hbar\frac{\partial}{\partial q}|q\rangle = p'\langle p'|q\rangle ,$$

that is⁴

$$\langle p'|\hat{p}|q\rangle = \langle p'|p'|q\rangle = p'\langle p'|q\rangle ,$$

to be compared with (6.27) that would imply

$$\langle p'|i\hbar\frac{\partial}{\partial q}|q\rangle = -p'\langle p'|q\rangle .$$

We then have

$$\begin{aligned} \langle q|\hat{p}|Q\rangle &= -i\hbar\frac{\partial}{\partial q}\langle q|Q\rangle , \\ \langle q|\hat{P}|Q\rangle &= +i\hbar\frac{\partial}{\partial Q}\langle q|Q\rangle . \end{aligned}$$

The steps outlined above show the discrepancy between the signs of (2.1.37) and (2.1.38) in the Ramond text and equations (3) and (5) in the original Dirac paper (whose reading is strongly recommended).

In the following, we will show the method adapted by Dirac to derive (3) and (5). We consider

$$\langle q'|\hat{O}|Q'\rangle = \int dq'' \langle q'|\hat{O}|q''\rangle \langle q''|Q'\rangle = \int dQ'' \langle q'|Q''\rangle \langle Q''|\hat{O}|Q'\rangle . \quad (6.29)$$

Observe that projecting both sides of

$$\hat{O}|s\rangle = |s'\rangle ,$$

on the $\{|q\rangle\}$ basis one gets

$$\int dq' \langle q|\hat{O}|q'\rangle \langle q'|s\rangle = \langle q|s'\rangle ,$$

⁴ Good references concerning the bra-ket formalism are [19] and its references 3, 4, 5, 26. Another useful reference is [20]. Finally the discussion at [this] Stack Exchange post may be useful.

which is equivalent to

$$\int dq' \langle q | \hat{O} | q' \rangle \psi_s(q') = \psi_{s'}(q) .$$

Comparing this expression with the one defining the action of the operator \hat{O}_q , that is the operator representing the observable O in configuration space

$$\hat{O}_q \psi_s(q) = \psi_{s'}(q) ,$$

one finds

$$\langle q | \hat{O} | q' \rangle = \hat{O}'_q \delta(q - q') , \quad (6.30)$$

where $\hat{O}'_q \delta(q - q')$ is the distribution defined in a way that, for every test function f in the Schwartz space $\mathcal{S}(\mathbb{R})$

$$\int dq' \hat{O}'_q \delta(q - q') f(q') = \int dq' \delta(q - q') \hat{O}'_q f(q') .$$

Now, keeping in mind that

$$\int dq' \partial_{q'}^n \delta(q - q') f(q') = (-1)^n \partial_q^n f(q) ,$$

it follows that if

$$\hat{O}_q = \sum_{k \geq 0} f_k(q) \partial_q^k ,$$

then

$$\hat{O}'_q = \sum_{k \geq 0} (-1)^k f_k(q) \partial_q^k .$$

In particular, if we take the operator \hat{O}'_q as the momentum operator, we get

$$\langle q | \hat{p} | q' \rangle = i\hbar \frac{\partial}{\partial q'} \delta(q - q') . \quad (6.31)$$

Summarising, by

$$\langle q | \hat{p} = -i\hbar \partial_q \langle q | , \quad \hat{p} | q' \rangle = i\hbar \partial_{q'} | q' \rangle ,$$

one gets

$$\langle q | \hat{p} | q' \rangle = -i\hbar \partial_q \delta(q - q') = i\hbar \partial_{q'} \delta(q - q') ,$$

or, equivalently,

$$\int dq \langle q | \hat{p} | q' \rangle f(q) = i\hbar \partial_{q'} f(q') , \quad \int dq' \langle q | \hat{p} | q' \rangle f(q') = -i\hbar \partial_q f(q) .$$

Consider now the action of an operator on a “bra”

$$\langle s | \hat{O} = \langle s' | .$$

Proceeding as before, projecting onto the configuration space basis, one gets

$$\int dq' \langle s | q' \rangle \langle q' | \hat{O} | q \rangle = \langle s' | q \rangle ,$$

that is

$$\int dq' \bar{\psi}_s(q') \langle q' | \hat{O} | q \rangle = \bar{\psi}_{s'}(q) ,$$

that, compared with

$$\bar{O}_q \bar{\psi}_s(q) = \bar{\psi}_{s'}(q) ,$$

yields

$$\langle q' | \hat{O} | q \rangle = \bar{O}'_q \delta(q - q') , \quad (6.32)$$

where, again, $\bar{O}'_q \delta(q - q')$ is the distribution such that for every $f \in \mathcal{S}(\mathbb{R})$

$$\int dq' \bar{O}'_q \delta(q - q') f(q') = \int dq' \delta(q - q') \bar{O}'_q f(q') .$$

Thus, if

$$\hat{O}_q = \sum_{k \geq 0} f_k(q) \partial_q^k ,$$

then

$$\bar{O}'_q = \sum_{k \geq 0} (-1)^k \bar{f}_k(q) \partial_q^k .$$

Notice that this result is trivial and could be obtained simply observing that \bar{O}'_q is the complex conjugate of \hat{O}'_q .

In some circumstances it is important to remember that Dirac’s delta is a distribution. An example is given by the fact that it is important to specify if the derivative of the delta is taken with respect to the “integration variable” or to the point in which it is centered. Indeed,

$$\begin{aligned} \int dy f(y) \frac{d^k}{dy^k} \delta(x - y) &= (-1)^k \int dy \delta(x - y) \frac{d^k}{dy^k} f(y) = (-1)^k \frac{d^k}{dx^k} f(x) , \\ \int dy f(y) \frac{d^k}{dx^k} \delta(x - y) &= \frac{d^k}{dx^k} \int dy \delta(x - y) f(y) = \frac{d^k}{dx^k} f(x) . \end{aligned}$$

To make the distinction between integration variable, say x , and the point in which is

centered, it would be better to use the notation

$$\delta_{x_0}(x) ,$$

or

$$(\delta_{x_0}, u) = u(x_0) ,$$

$\forall u \in \mathcal{S}(\mathbb{R})$, with $\mathcal{S}(\mathbb{R})$ the Schwartz space⁵. Analogously, the determination of the sign $\alpha = \pm 1$ in

$$\partial_x^k \delta(x-y) = \alpha \delta(x-y) \partial_x^k , \quad \partial_y^k \delta(x-y) = \alpha \delta(x-y) \partial_y^k .$$

depends on the context. From the relations (13.3) and (6.31) it follows that

$$\langle q' | \hat{p} | Q' \rangle = \int dq'' \langle q' | \hat{p} | q'' \rangle \langle q'' | Q' \rangle = -i\hbar \frac{\partial}{\partial q'} \langle q' | Q' \rangle . \quad (6.33)$$

The computation of $\langle q' | \hat{P} | Q' \rangle$ is similar. By (6.32) we have

$$(6.34)$$

$$\langle q' | \hat{P} | Q' \rangle = \int dQ'' \langle q' | Q'' \rangle \langle Q'' | \hat{P} | Q' \rangle = i\hbar \frac{\partial}{\partial Q'} \langle q' | Q' \rangle . \quad (6.35)$$

The wrong signs of equations (2.1.37) and (2.1.38) in Ramond textbook are balanced by the minus sign in the mixed matrix element considered by Ramond, that is $\langle q | Q \rangle = e^{-i/\hbar G(q,Q)}$. On the other hand, while handling the “correspondence” between $\langle q | Q \rangle$ and $e^{i/\hbar \int_T dt' L}$, the wrong signs by Ramond are put in evidence by the fact that the quantum

⁵ Recall that $\mathcal{S}(\mathbb{R}^n)$ is the space of complex functions $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\|u\|_{\alpha,\beta} \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta u(x)| < \infty , \quad \forall \alpha, \beta \in \mathbb{N}^n ,$$

where α is the multi-index $\alpha := \{\alpha_1, \dots, \alpha_n\}$, $\alpha_k \in \mathbb{N}$, and

$$D^\alpha := \prod_{k=1}^n \frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}} , \quad x^\alpha := \prod_{k=1}^n x_k^{\alpha_k} .$$

In other words, such functions correspond to the $u(x_1, \dots, x_n)$ in $\mathcal{C}^\infty(\mathbb{R}^n)$ going to 0 at $\pm\infty$ faster than the inverse of any polynomial in (x_1, \dots, x_n) . The δ distribution belongs to the space of tempered distributions $\mathcal{S}'(\mathbb{R})$, the dual space of $\mathcal{S}(\mathbb{R})$. It does not belong to the subspace of $\mathcal{S}'(\mathbb{R})$ of regular tempered distributions F_f , which are the ones represented by a function, that is $F_f(u) := (f, u) = \int_{\mathbb{R}} dx f(x)u(x)$, $\forall u \in \mathcal{S}(\mathbb{R})$. The standard notation $\int_{\mathbb{R}} dx \delta(x-x_0)u(x) = u(x_0)$ should be interpreted as a symbolic notation for

$$(\delta_{x_0}, u) := \lim_{\nu \rightarrow \infty} \int_{\mathbb{R}} dx f_\nu(x-x_0)u(x) = u(x_0) ,$$

with $\{f_\nu\}$ a suitable sequence of piecewise continuous functions on \mathbb{R} . For an introduction to distributions see, e.g., [12].

analogue of the relations

$$p = \frac{\partial S}{\partial q}, \quad P = -\frac{\partial S}{\partial Q},$$

should have the opposite sign too, contrary to what Dirac obtained.

Dirac justifies the “correspondence”

$$\langle q|Q\rangle \sim e^{i/\hbar \int_T^t dt' L},$$

by the following reasoning

“The equations of motion of the classical theory cause the dynamical variables to vary in such a way that their values q_t, p_t at any time t are connected with their values q_T, p_T at any other time T by a contact transformation, which may be put into the form (1) with $q, p = q_t, p_t$; $Q, P = q_T, p_T$ and S equal to the time integral of the Lagrangian over the range T to t . In the quantum theory the q_t, p_t will still be connected with the q_T, p_T by a contact transformation and there will be a transformation function $\langle q_t|q_T\rangle$ connecting the two representations in which the q_t and the q_T are diagonal respectively.”

In the following we will show that Dirac’s argument is related to the Heisenberg representation of operators. Let O be an observable and \hat{O} the associated operator. A diagonal representation of \hat{O} is the one referred to a basis $\{|o\rangle\}$ such that any of its elements satisfies

$$\hat{O}|o\rangle = o'|o\rangle.$$

Consider again, with a slight change in notation, the probability amplitude of finding a particle at time t'' in a point q'' knowing that its position is q' at time t' . Denote by \hat{Q} the position operator. Both the initial and the final states are eigenstates of \hat{Q} , that is

$$\hat{Q}|q'\rangle = q'|q'\rangle, \quad \hat{Q}|q''\rangle = q''|q''\rangle. \quad (6.36)$$

Remember that the time evolution of a state is given by

$$|\psi(t'')\rangle = U(t'', t')|\psi(t')\rangle,$$

where

$$U(t'', t') = T \exp\left(-\frac{i}{\hbar} \int_{t'}^{t''} dt H(t)\right),$$

and T stands for time ordering. If H does not depend explicitly on time, the previous is equivalent to

$$U(t'', t') = e^{-\frac{i}{\hbar} H(t''-t')}.$$

We want to consider the amplitude

$$\langle q'' | e^{-\frac{i}{\hbar}H(t''-t')} | q' \rangle .$$

In what follows the notation

$$|o, t\rangle , \tag{6.37}$$

is used to describe a particle that at time t occupies a state in which the observable O assumes the value o . Be aware that $|o, t\rangle$ is not the time evolved “ket” of $|o\rangle$. In other words, even though

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)} |\psi(t_0)\rangle ,$$

it is clear that

$$|o, t\rangle \neq e^{-\frac{i}{\hbar}Ht} |o\rangle . \tag{6.38}$$

Indeed, the state described by $e^{-\frac{i}{\hbar}Ht} |o\rangle$ is given by the time evolution of a state that at time $t = 0$ is an eigenstate of the operator \hat{O} with eigenvalue o . This is in general completely different from (6.37).

In the following we assume that $\langle o'', t'' | o', t' \rangle$ is the probability amplitude $\langle o'' | \psi(t'') \rangle$ of having the state $|\psi(t')\rangle$ that evolves at time t'' in $|o''\rangle$. This is not obvious a priori. Indeed, $\langle o'', t'' | o', t' \rangle$ represents a scalar product. Thus, one could in principle consider also

$$\langle o'', t'' | o', t' \rangle = \int dq \langle o'', t'' | q \rangle \langle q | o', t' \rangle = \int dq \bar{\psi}_{o'', t''}(q) \psi_{o', t'}(q) . \tag{6.39}$$

The focus is to understand what $\psi_{o, t}(q) = \langle q | o, t \rangle$ corresponds to. The given interpretation of $\langle o'', t'' | o', t' \rangle$ implies that this relation can be recast in the form

$$\int dq \bar{\psi}_{o''}(q) e^{-\frac{i}{\hbar}H(t''-t')} \psi_{o'}(q) , \tag{6.40}$$

where $\psi_{o'}(q)$ ($\bar{\psi}_{o''}(q)$) is the eigenfunction, in coordinate representation, of \hat{O} with eigenvalue o' (o''). Observe that (6.39) has a symmetric interpretation of the two states $|o', t'\rangle$ and $|o'', t''\rangle$ in $\langle o'', t'' | o', t' \rangle$, whereas in (6.40) the state $\psi_{o'}(q)$ is the one effectively treated as the initial state and time evolved via the operator $\exp[-\frac{i}{\hbar}H(t'' - t')]$. To keep the interpretation symmetric it is necessary to assume the existence of an element common to the two states. This is also suggested by the fact that the amplitude we want to extract is just the complex conjugate of the probability amplitude of having a state $|o''\rangle$ at time t'' evolving in $|o'\rangle$ at time t' , an aspect related to time reversal. We will see that (6.40) implies the use of the Heisenberg representation, which is defined in relation to a time reference.

Using Schrödinger representation, the statement that at time t' the state is $|o'\rangle$ can be

written as $|\psi(t')\rangle = |o'\rangle$. Then, the time evolution reads

$$|\psi(t'')\rangle = e^{-\frac{i}{\hbar}H(t''-t')}|\psi(t')\rangle = e^{-\frac{i}{\hbar}H(t''-t')}|o'\rangle ,$$

and this leads to

$$\langle o'', t'' | o', t' \rangle = \langle o'' | \psi(t'') \rangle = \langle o'' | e^{-\frac{i}{\hbar}H(t''-t')} | o' \rangle ,$$

which, in turn, implies

$$|o, t\rangle = e^{\frac{i}{\hbar}H(t-t_0)}|o\rangle , \quad (6.41)$$

where t_0 is arbitrary. Now, note that a state in the Heisenberg representation and the one in the Schrödinger representation are related, at every t , by the relation

$$|\psi_H\rangle = e^{\frac{i}{\hbar}Ht}|\psi(t)\rangle . \quad (6.42)$$

The Heisenberg state is a “ket” not evolving in time. Moreover, the operators in the Heisenberg and Schrödinger representations are related by

$$\hat{O}_H(t) = e^{\frac{i}{\hbar}Ht}\hat{O}e^{-\frac{i}{\hbar}Ht} .$$

The above implies that if $\hat{O}|o\rangle = o|o\rangle$, then $\hat{O}_H(t)|o, t\rangle = o|o, t\rangle$. The preceding discussion makes it possible to identify at every t , the “ket” $|o, t\rangle$ with $|\psi(t_0)\rangle$. In this way $|o, t\rangle$ is the “ket” in the Heisenberg representation $|\psi_H\rangle$ and corresponds to the instantaneous eigenstate of $\hat{O}_H(t)$ of eigenvalue o . In the Schrödinger representation the corresponding state is $|\psi(t)\rangle = |o\rangle$ while the corresponding operator is \hat{O} , thus $\hat{O}|\psi(t)\rangle = o|\psi(t)\rangle$. This observation stresses the obvious fact that $|\psi_H\rangle$ depends on the time reference chosen. For example, (6.42) identifies $|\psi_H\rangle$ with $|\psi(0)\rangle$. In this regard, instead of time independence of the states in the Heisenberg picture, it would be more appropriate to talk about their invariance under time evolution. A more accurate notation is

$$|\psi_H(t_0)\rangle = e^{\frac{i}{\hbar}H(t-t_0)}|\psi(t)\rangle = |\psi(t_0)\rangle ,$$

which has the virtue of emphasising that the relation between Heisenberg states, referring to different choices of reference time, is the same relating Schrödinger states at different times, i.e.

$$|\psi_H(t_2)\rangle = e^{\frac{i}{\hbar}H(t_1-t_2)}|\psi_H(t_1)\rangle , \quad |\psi(t_2)\rangle = e^{\frac{i}{\hbar}H(t_1-t_2)}|\psi(t_1)\rangle .$$

The amplitude $\langle o'', t'' | o', t' \rangle$ can be expressed in the Heisenberg picture with an arbitrary choice of reference time. That is, if $|\psi'_H(t_i)\rangle$ and $|\psi''_H(t_i)\rangle$ are the Heisenberg states associated to $|\psi'(t_i)\rangle$ and $|\psi''(t_i)\rangle$, then $\langle \psi''_H(t_2) | \psi'_H(t_2) \rangle = \langle \psi''_H(t_1) | \psi'_H(t_1) \rangle$.

Define

$$|o', t'\rangle = e^{\frac{i}{\hbar}H(t'-t_0)}|o'\rangle, \quad |o'', t''\rangle = e^{\frac{i}{\hbar}H(t''-t_0)}|o''\rangle.$$

The preceding analysis has shown that the probability amplitude of an eigenstate at time t' of \hat{O} of eigenvalue o' evolving at time t'' in an eigenstate of eigenvalue o'' , coincides with the scalar product between Heisenberg representation states, that is

$$\langle o'' | e^{-\frac{i}{\hbar}H(t''-t')} | o' \rangle = \langle o'', t'' | o', t' \rangle.$$

Also, the correct version of (6.39) is nothing but

$$\langle o'', t'' | o', t' \rangle = \int dq \langle o'', t'' | q \rangle \langle q | o', t' \rangle = \int dq \bar{\psi}_{H o''}(t'' - t_0; q) \psi_{H o'}(t' - t_0; q), \quad (6.43)$$

where

$$\psi_{H o}(t - t_0; q) = \langle q | e^{\frac{i}{\hbar}H(t-t_0)} | o \rangle = e^{\frac{i}{\hbar}H(t-t_0)} \langle q | o \rangle,$$

is the state in Heisenberg representation, with time reference $t - t_0$. This shows the elegance of the Heisenberg picture. Notice that the interpretation of the state $|o', t'\rangle$ as a state explicitly realised at time t' , used in the first description of $\langle o'', t'' | o', t' \rangle$, is no longer implied once this is considered as scalar product between the eigenstates of the operators $\hat{O}_H(t')$ and $\hat{O}_H(t'')$.

The previous observations allow to complete Dirac's observation

“... transformation function ($q_t | q_T$) connecting the two representations in which the q_t and the q_T are diagonal respectively”

pointing out that q_t (q_T) is the basis in which $\hat{Q}_H(t)$ ($\hat{Q}_H(T)$) is diagonal. In Dirac and Ramond's notation, where $|q_t\rangle := |q, t\rangle$ and $|q_T\rangle := |Q, T\rangle$, this statement means

$$\hat{Q}_H(t)|q_t\rangle = q|q_t\rangle \quad \text{and} \quad \hat{Q}_H(T)|q_T\rangle = Q|q_T\rangle.$$

Another observation is concerned with the “well-ordered” prescription mentioned by Ramond. In the second edition Ramond added the comment

“Well-ordered means that they are separable as a function of \hat{q} times a function of \hat{Q} .”

Actually the concept can be made more general. Indeed, Dirac states that $F(\hat{q}, \hat{Q})$ is well-ordered if it is a sum of products of functions of \hat{q} and functions of \hat{Q} , i.e.

$$F(\hat{q}, \hat{Q}) = \sum_k f_k(\hat{q}) g_k(\hat{Q}).$$

As a result, using $\langle q|f(\hat{q}) = \langle q|f(q)$ and $g(\hat{Q})|Q\rangle = g(Q)|Q\rangle$, one gets

$$\langle q|F(\hat{q}, \hat{Q})|Q\rangle = F(q, Q)\langle q|Q\rangle .$$

Now note that setting

$$\langle q|Q\rangle = e^{\frac{i}{\hbar}G(q, Q)} ,$$

we have

$$\langle q|\hat{p}|Q\rangle = -i\hbar \frac{\partial}{\partial q} \langle q|Q\rangle = \frac{\partial G}{\partial q} \langle q|Q\rangle ,$$

and

$$\langle q|\hat{P}|Q\rangle = i\hbar \frac{\partial}{\partial Q} \langle q|Q\rangle = -\frac{\partial G}{\partial Q} \langle q|Q\rangle .$$

It follows that if $\partial G/\partial q$ and $\partial G/\partial Q$ were well ordered, then we would have

$$\hat{p} = \frac{\widehat{\partial G}}{\partial q} , \quad \hat{P} = -\frac{\widehat{\partial G}}{\partial Q} ,$$

that is, according to (6.26), G would be the quantum analogue of the Hamilton principal function.

6.2.1 Path integral formulation

In the article by Dirac all the steps necessary to define the path integral formalism are described far more than the ones for which Ramond textbook gives credit. After equation (2.2.1) of Ramond's book

$$\langle q'_t|q_T\rangle \sim e^{\frac{i}{\hbar} \int_T^t dt L} , \quad (6.44)$$

the author writes

“Let me emphasize that the \sim sign means just a loose connection, because to arrive at (2.1.44) Dirac had to make all kinds of assumptions with no way to justify them. In fact, we can see that an equality sign would not be correct for (2.2.1) as long as the time interval $T - t$ is finite: split up $T - t$ into N infinitesimal time intervals $t_a = t + a\epsilon$; $N\epsilon = T - t$. Let $q_a = q_{t_a}$ and use the completeness relation (2.1.33) for each t_a to write

$$\langle q'_t|q_T\rangle = \int dq_1 dq_2 \dots dq_{N-1} \langle q'_t|q_1\rangle \langle q_1|q_2\rangle \dots \langle q_{N-1}|q_T\rangle . \quad (6.45)$$

This is an exact quantum mechanical formula.”

As a matter of fact, those observations are reported in Dirac's original article. This includes the last relation which corresponds to equation (2.2.2) of Ramond and equation (11) in Dirac's work. With regard to these annotations see pages 68–69 of Dirac's article

[2]. Similarly, Ramond erroneously attributes to Feynman the fundamental relation (2.2.4)

$$\langle q'_t | q_{t+\delta t} \rangle = A \exp \left(-\frac{i}{\hbar} \delta t L(q'_t, q_{t+\delta t}) \right). \quad (6.46)$$

Indeed, after stating this relation, Ramond goes on commenting

“where L (in the spirit of the Hamilton-Jacobi theory) is taken to be a function of q'_t and $q_{t+\delta t}$, we run into no conflict with the quantum mechanical formula (2.2.2). This is exactly what Feynman did! [*Rev. Mod. Phys.* **20**, 267 (1948).] This leads to the Feynman Path Integral for the transition amplitude, using (2.2.4) and (2.2.2):

$$\langle q'_t | q_T \rangle = \lim_{\substack{N \rightarrow \infty \\ N \epsilon \text{ fixed}}} A^N \int \left(\prod_{i=1}^{N-1} dq_i \right) \exp \left(\frac{i}{\hbar} \int_T^t dt L(q, \dot{q}) \right) \equiv \int \mathcal{D}q \exp \left(\frac{i}{\hbar} S(t, T, [q]) \right), \quad (6.47)$$

where the second expression is just a fancy way of hiding our lack of knowledge about the measure; the square brackets indicate the functional relationship between S and q .”

Anyhow (6.46), corresponding to Ramond’s (2.2.4), was also present in Dirac’s paper. It is equation (9) of his paper, that is

$$\langle q_{t+dt} | q_t \rangle \text{ corresponds to } \exp[iLdt/\hbar]. \quad (6.48)$$

As a matter of fact, Dirac in his paper provided the fundamental bases of the path integral, including the classical correspondence. This is even more evident by the following description in Dirac’s paper

“The right-hand side is then a function, not only of q_T and q_t , but also of q_1, q_2, \dots, q_m , and in order to get from it a function of q_T and q_t only, which we can equate to the left-hand side, we must substitute for q_1, q_2, \dots, q_m their values given by the action principle. This process of substitution for the intermediate q ’s then corresponds to the process of integration over all values of these q ’s in (11).

Equation (11) contains the quantum analogue of the action principle, as may be seen more explicitly from the following argument. From equation (11) we can extract the statement (a rather trivial one) that, if we take specified values for q_T and q_t , then the importance of our considering any set of values for the intermediate q ’s is determined by the importance of this set of values in the integration on the right-hand side of (11). If we now make \hbar tend to zero, this statement goes over into the classical statement that, if we take specified values for q_T and q_t , then the importance of our considering any set of values for the intermediate q ’s is zero unless these values make the action function stationary. This statement is one way of formulating the classical action principle.”

As seen, the same text by Ramond shows what is otherwise evident, namely that (2.2.4) and (2.2.2) imply the path integral. The point is that these two relations are not only

reported in Dirac's paper, but Dirac already connected them describing the integration over the paths. Moreover, the level of depth of thought reached by Dirac is also further strengthened by another fundamental aspect of his work: having realised the relevance and the role of classic trajectories. In particular, it affects the commentary in which the principle of classical action can, or even it must be seen as the limit of the quantum formulation. In addition to this Dirac dedicates the final section of the work to the quantum field theory extension of the path integral, describing its salient features. In this part there is also a mention to a space-time slicing that recalls the one considered later in the ADM formulation of General Relativity.

It is therefore clear that while Feynman goes the great merit of having thoroughly developed the path integral and having introduced the diagrammatic calculation, it is also undoubted that the idea and the formulation of the foundations of the path integral are due to Dirac. For completeness it should be mentioned that probably in turn Dirac was influenced to some degree by the following two works by Jordan

P. Jordan, "Über kanonische Transformationen in der Quantenmechanik", *Zeitschrift für Physik A Hadrons and Nuclei*, Volume: 37 Issue: 4-5, (1926) 383-386,

and, quoted by Dirac,

P. Jordan, "Über kanonische Transformationen in der Quantenmechanik", *Zeitschrift für Physik A Hadrons and Nuclei*, Volume: 38, Issue: 6-7, (1926) 513-517.

6.2.2 Check of the path integral formula

Before checking the path integral representation of the transition amplitude we summarise the main points concerning the Heisenberg and Schrödinger representations discussed above. We saw that $\hat{q}_H |q, t\rangle = q |q, t\rangle$. In particular, $|q, t\rangle$ is not the state $|q\rangle$ time-evolved by $U(t, t_0)$. In the following we will assume that H is time independent (otherwise the time ordering operation is needed), and hence we can write the operator U , the states in the Heisenberg and Schrödinger representation respectively as

$$\begin{aligned} U(t, t_0) &= e^{-\frac{i}{\hbar} H(t-t_0)} , \\ |\psi_H(t_0)\rangle &= U^{-1}(t, t_0) |\psi(t)\rangle , \\ |\psi_S(t)\rangle &= U(t, t_0) |\psi_S(t_0)\rangle . \end{aligned}$$

In the Heisenberg representation we have $\hat{A}_H = e^{\frac{i}{\hbar} H(t-t_0)} \hat{A} e^{-\frac{i}{\hbar} H(t-t_0)}$, that requires the choice of a time reference t_0 . Now, instead of considering t_0 , we work with two different

reference times, t_1 and t_2 ,

$$|\psi_H(t_2)\rangle = e^{\frac{i}{\hbar}H(t_1-t_2)} |\psi_H(t_1)\rangle , \quad (6.49)$$

$$|\psi_S(t_2)\rangle = e^{\frac{i}{\hbar}H(t_1-t_2)} |\psi_S(t_1)\rangle . \quad (6.50)$$

The fact that at the time t' the state in the Schrödinger representation is $|\psi(t')\rangle = |q'\rangle$, implies

$$|\psi(t'')\rangle = e^{-\frac{i}{\hbar}H(t''-t')} |\psi(t')\rangle = e^{-\frac{i}{\hbar}H(t''-t')} |q'\rangle .$$

Now we consider a state $|q, t\rangle$ which is an eigenstate of the operator \hat{q} at time t with eigenvalue, i.e. $\hat{q} |q, t\rangle = q |q, t\rangle$, then $\langle q'', t'' | q', t' \rangle$ in the transition amplitude that bears the notion of the particle evolution at time t' and coordinate q' to q'' at t'' . On the other hand, we know that

$$\langle q'', t'' | q', t' \rangle = \langle q'' | \psi(t'') \rangle = \langle q'' | e^{-\frac{i}{\hbar}H(t''-t')} |q'\rangle ,$$

and therefore

$$|q, t\rangle = e^{+\frac{i}{\hbar}H(t-t_0)} |q\rangle ,$$

with t_0 arbitrary, which has the opposite sign as the one in the Schrödinger representation and showing that $|q, t\rangle$ is not the time evolution of the state $|q\rangle$. In particular, when we let an operator in the Heisenberg picture act on this state

$$\hat{q}_H(t) |q, t\rangle = e^{\frac{i}{\hbar}H(t-t_0)} \hat{q} e^{-\frac{i}{\hbar}H(t-t_0)} |q, t\rangle = e^{\frac{i}{\hbar}H(t-t_0)} \hat{q} |q\rangle = q e^{\frac{i}{\hbar}H(t-t_0)} |q\rangle = q |q, t\rangle ,$$

we can conclude that $|q, t\rangle$ is the eigenstate of \hat{q}_H with eigenvalue q .

The propagator $\langle q', t' | q, t \rangle$ admits various representations. For example,⁶

$$\langle q', t' | q, t \rangle = \langle q' | e^{-\frac{i}{\hbar}H(t'-t)} |q\rangle \quad (6.51)$$

$$= \sum_{m,n} \langle q' | n \rangle \langle n | e^{-\frac{i}{\hbar}H(t'-t)} |m\rangle \langle m | q \rangle \quad (6.52)$$

$$= \sum_n e^{-\frac{i}{\hbar}E_n(t'-t)} \psi_n(q') \bar{\psi}_n(q) , \quad (6.53)$$

where $H\psi_n = E_n\psi_n$.

⁶ Here the summation over the Hamiltonian eigenstates should be intended in the generalised sense, that is including the case of continuum spectrum, so that \sum_n includes the integration. In this respect, note that constancy of the Wronskian of two linearly independent solutions of the eigenvalue equation $H\psi = E\psi$, and the fact that the discrete spectrum corresponds to $\psi_n \in L^2(\mathbb{R})$, implies, in one spatial dimension, that $\bar{\psi}_n$ must be proportional to a real function. Actually, the reality of the equation $H\psi = E\psi$ implies that, besides ψ_n , even $\bar{\psi}_n$ is a solution. On the other hand, if ψ_n is a solution of $H\psi = E\psi$, then, considering the Wronskian at spatial infinity, one sees that a solution, linearly independent of ψ_n , cannot be in $L^2(\mathbb{R})$, whereas $\psi_n \in L^2(\mathbb{R})$ implies $\bar{\psi}_n \in L^2(\mathbb{R})$.

An expression in terms of operator eigenfunctions can be also derived for Green functions. Given a linear differential operator L_q , the Green function is an arbitrary solution of the equation

$$L_q G(q', q) = \delta(q' - q) .$$

The arbitrariness in the definition of $G(q', q)$ is given by the fact that if⁷

$$L_q \phi_0(q) = 0 ,$$

then $G(q', q)$ and $G(q', q) + \phi_0(q)$ satisfy the same equation. If the coefficients of L_q are independent of q , then one can choose a $G(q', q)$ which is invariant under translations

$$G(q' + a, q + a) = G(q', q) .$$

In this case it is customary to set $G(q' - q) := G(q', q)$. Consider the eigenvalue equation

$$L_q \psi_n(q) = \lambda_n \psi_n(q) ,$$

then, using the representation (6.54) of the δ distribution, we have

$$G(q' - q) = \sum_n \frac{1}{\lambda_n} \psi_n(q') \bar{\psi}_n(q) .$$

In the context of the axiomatic approach, a Green function is often called covariance. In this regard, see the paper [21]. For further information look at the excellent and advanced book [22].⁸

Note that

$$\delta(q' - q) = \sum_n \psi_n(q') \bar{\psi}_n(q) , \tag{6.54}$$

⁷ Obviously, the number of linearly independent solutions $L_q \phi_0 = 0$ is equal to the order of the differential operator L_q .

⁸ This book contains an interesting analysis of the Wick rotation. As an example in the following it is reproduced a passage from page 363, related to dimensional regularisation

“When only the Gaussian contribution to the functional integral is kept, the contour C is that appropriate to the Feynman propagator (see e.g. [3]) and runs from $-\infty$ to 0 below the negative real axis (in the complex k^0 - plane) and from 0 to ∞ above the positive real axis. If the integral (18.24) were convergent, the contour could be rotated so that it would run along the imaginary axis. One would set $k^0 = ik^n$, and (18.24) would become an integral over Euclidean momentum- n -space. Generically, however, this rotation, which is known as Wick rotation, is not legitimate. Contributions from arcs at infinity, which themselves diverge or are non-vanishing, have to be included. These contributions cannot be handled by dimensional regularisation.”

implies that the propagator can be expressed in the following form⁹

$$\langle q', t' | q, t \rangle = e^{-\frac{i}{\hbar}H(t'-t)}\delta(q' - q) = e^{-\frac{i}{\hbar}H(t'-t)} \int \frac{dp}{2\pi} e^{\frac{i}{\hbar}p(q'-q)} .$$

This is also implied by the following observation. The distribution $\delta(q' - q)$ is the probability amplitude of finding a particle with position q , in the point q' . Knowing that at time t the particle was in q , means that

$$\psi_q(Q, t) = \delta(Q - q) ,$$

which satisfies

$$\hat{Q}\psi_q(Q, t) = q\psi_q(Q, t) .$$

At time t' the wave function is

$$\psi(Q, t') = e^{-\frac{i}{\hbar}H(t'-t)}\psi_q(Q, t) = e^{-\frac{i}{\hbar}H(t'-t)}\delta(Q - q) .$$

This is the probability amplitude of finding a particle in Q at time t' , knowing that at time t it was in q . Then, by $\langle q', t' | q, t \rangle = e^{-\frac{i}{\hbar}H(t'-t)}\langle q' | q \rangle$, it is obvious that

$$\langle q', t' | q, t \rangle = \psi(q', t') = e^{-\frac{i}{\hbar}H(t'-t)}\delta(q' - q) .$$

Let us compute $\langle q', t' | q, t \rangle$. By

$$|q, t + \delta t\rangle = e^{\frac{i}{\hbar}H\delta t}|q, t\rangle = |q, t\rangle + \frac{i}{\hbar}\delta t H|q, t\rangle + \mathcal{O}((\delta t)^2) ,$$

it follows that

$$\langle q', t + \delta t | q, t \rangle = \langle q', t | q, t \rangle - \frac{i}{\hbar}\langle q', t | H | q, t \rangle \delta t + \mathcal{O}((\delta t)^2) .$$

⁹ By (6.53) and (6.54) one may explicitly check that $\langle q', t' | q, t \rangle$ solves the Schrödinger equation with respect to q', t' , and its complex conjugate with respect to q, t , with the initial condition

$$\lim_{t' \rightarrow t} \langle q', t' | q, t \rangle = \delta(q' - q) .$$

The same check can be done by applying the Schrödinger operators

$$i\hbar \frac{\partial}{\partial t'} - H(q') , \quad -i\hbar \frac{\partial}{\partial t} - H(q) ,$$

to $\langle q' | e^{-\frac{i}{\hbar}H(t'-t)} | q \rangle$.

Consider the Hamiltonian for a particle of mass 1, that is $H = \frac{p^2}{2} + V(q)$. We have

$$\begin{aligned} \langle q', t | H | q, t \rangle &= \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} + V(q) \right) \underbrace{\langle q', t | q, t \rangle}_{\delta(q'-q)} \\ &= \int \frac{dk}{2\pi} \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial q^2} + V(q) \right) e^{ik(q-q')} , \end{aligned} \quad (6.55)$$

implying that

$$\langle q', t + \delta t | q, t \rangle = \int \frac{dk}{2\pi} e^{ik(q-q')} \left(1 - \frac{i}{\hbar} \delta t H(k, q) + \mathcal{O}((\delta t)^2) \right) ,$$

where

$$H(k, q) = \frac{\hbar^2 k^2}{2} + V(q) .$$

Setting

$$q' - q = \frac{dq}{dt} \delta t = \dot{q} \delta t ,$$

leads to

$$\langle q', t + \delta t | q, t \rangle = \int \frac{dk}{2\pi} \exp \left[\frac{i\delta t}{\hbar} \left(\hbar k \dot{q} - \frac{1}{2} \hbar^2 k^2 - V(q) \right) \right] + \mathcal{O}((\delta t)^2) .$$

One can notice that integrating over k seems problematic because the integrand is oscillating.¹⁰ We can proceed by two alternatives corresponding to the following formal manipulations

- (i) insert $e^{-\epsilon k^2}$ and at the end of the computations take the limit $\epsilon \rightarrow 0$,
- (ii) consider $i\delta t$ to take real values, i.e. $t \rightarrow it$.

We choose the second option by considering $i\delta t$ a real quantity and consider the change of variables

$$k \rightarrow k' = \left(\frac{i\delta t}{\hbar} \right)^{1/2} (\hbar k - \dot{q}) .$$

We then get

$$\begin{aligned} \langle q', t + \delta t | q, t \rangle &= \frac{1}{2\pi} \exp \left[\frac{i\delta t}{\hbar} \left(\frac{1}{2} \dot{q}^2 - V(q) \right) \right] \int \frac{dk'}{\sqrt{i\delta t \hbar}} e^{-\frac{1}{2} k'^2} \\ &= \frac{1}{\sqrt{2\pi i\delta t \hbar}} \exp \left[\frac{i\delta t}{\hbar} \left(\frac{1}{2} \dot{q}^2 - V(q) \right) \right] , \end{aligned}$$

¹⁰ As we will see in deriving the Feynman-Kac formula, this problem is easily solved.

so that, for a finite time interval, we have

$$\langle q'', t'' | q', t' \rangle = \lim_{\substack{\delta t \rightarrow 0 \\ N\delta t \text{ fixed}}} \int \prod_{k=1}^{N-1} \left(\frac{dq_k}{\sqrt{2\pi i \delta t \hbar}} \right) \exp \left(\frac{i}{\hbar} \int_{t'}^{t''} L dt \right).$$

6.3 The Feynman-Kac formula

We now provide a more rigorous proof of the previous formula. The idea is to use the Trotter product formula, related to the Baker-Campbell-Hausdorff formula on which there has been a recent progress [23, 24, 25, 26, 27]. Let us first enunciate the Lie's theorem. Given two matrices A and B ,

$$e^{A+B} = \lim_{n \rightarrow \infty} \left(e^{A/n} e^{B/n} \right)^n.$$

The Trotter formula is a generalisation of Lie's theorem to the case of self-adjoint operators. More precisely

If A and B are self-adjoint operators and $A + B$ is essentially self-adjoint on the intersection of their domains, then¹¹

$$e^{-it(A+B)} = s - \lim_{n \rightarrow \infty} \left(e^{-itA/n} e^{-itB/n} \right)^n.$$

Furthermore, if A and B are bounded from below, then

$$e^{-\tau(A+B)} = s - \lim_{n \rightarrow \infty} \left(e^{-\tau A/n} e^{-\tau B/n} \right)^n.$$

Therefore, since H_0 and V are self-adjoint, we have

$$e^{-i(H_0+V)t} = \lim_{n \rightarrow \infty} \left(e^{-iH_0 t/n} e^{-iV t/n} \right)^n.$$

¹¹ If the operator sequence $\{A_n\}_{n \in \mathbb{N}}$ and A have a common domain D , then the strong limit of A_n is A if and only if

$$\|A_n \psi - A \psi\| \rightarrow 0, \quad \forall \psi \in D.$$

We then have

$$\begin{aligned}
 \langle q', t | q, 0 \rangle &= \langle q' | e^{-iHt} | q \rangle = \lim_{n \rightarrow \infty} \langle q' | \left(e^{-iH_0 t/n} e^{-iVt/n} \right)^n | q \rangle \\
 &= \lim_{n \rightarrow \infty} \langle q' | e^{-iH_0 t/n} e^{-iVt/n} I e^{-iH_0 t/n} e^{-iVt/n} \dots e^{-iH_0 t/n} e^{-iVt/n} | q \rangle \\
 &= \lim_{n \rightarrow \infty} \int dq_1 \dots dq_{n-1} \prod_{j=0}^{n-1} \langle q_{j+1} | e^{-iH_0 t/n} e^{-iVt/n} | q_j \rangle , \quad (6.56)
 \end{aligned}$$

where, in the third equality, we replaced the $n - 1$ identities I by $\int dq_j |q_j\rangle \langle q_j|$, $j = 1, \dots, n - 1$ and

$$q_0 \equiv q, \quad q_n \equiv q'.$$

Since the multiplication operator $V(q)$ is diagonal in position space, that is

$$V(\hat{q})|q_0\rangle = V(q_0)|q_0\rangle,$$

we have

$$\begin{aligned}
 \langle q_{j+1} | e^{-iH_0 t/n} e^{-iVt/n} | q_j \rangle &= \int dq \langle q_{j+1} | e^{-iH_0 t/n} | q \rangle \langle q | e^{-iVt/n} | q_j \rangle \\
 &= \int dq \langle q_{j+1} | e^{-iH_0 t/n} | q \rangle e^{-iV(q_j)t/n} \delta(q - q_j) \\
 &= \langle q_{j+1} | e^{-iH_0 t/n} | q_j \rangle e^{-iV(q_j)t/n}. \quad (6.57)
 \end{aligned}$$

There are various way to compute $\langle q_{j+1} | e^{-iH_0 t/n} | q_j \rangle$. For example, by solving the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \langle q', t | q, 0 \rangle = H \langle q', t | q, 0 \rangle,$$

with the initial condition

$$\lim_{t \rightarrow 0} \langle q', t | q, 0 \rangle = \delta(q' - q).$$

One may check that in the free case

$$\langle q', t | q, 0 \rangle = \langle q' | e^{-iH_0 t/\hbar} | q \rangle = \left(\frac{m}{2\pi i \hbar t} \right)^{1/2} e^{im(q' - q)^2 / (2\hbar t)}.$$

Collecting the above results, we get

$$\langle q' | e^{-iHt/\hbar} | q \rangle = \lim_{n \rightarrow \infty} \left(\frac{nm}{2\pi i \hbar t} \right)^{n/2} \int dq_1 \dots dq_{n-1} \exp \left\{ \frac{it}{n\hbar} \sum_{j=0}^{n-1} \left[\frac{n^2 m}{2} \left(\frac{q_{j+1} - q_j}{t} \right)^2 - V(q_j) \right] \right\},$$

which is the Feynman-Kac formula.

6.4 Path integral over coordinates and momentum

The above investigations concern the case in which the Hamiltonian corresponds to the Laplacian in addition to a potential depending on the coordinates only. To show this we consider the Hamiltonian

$$H = \frac{\hat{p}^2}{2} v(\hat{q}) .$$

Note that $v(\hat{q})$ is some function of \hat{q} and that \hat{p} and \hat{q} do not commute, and therefore an ordering prescription is required¹² and we define a symmetric ordering¹³ as

$$\langle q', t' | \frac{\hat{p}^2}{2} v(\hat{q}) | q, t \rangle = \int \frac{dl}{2\pi} \frac{\hbar^2}{2} l^2 v\left(\frac{q+q'}{2}\right) e^{il(q'-q)} .$$

With this definition of ordering one can easily show that¹⁴

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}q \mathcal{D}p e^{\frac{i}{\hbar} \int_{t'}^{t''} dt [p \frac{dq}{dt} - H(p, \langle q \rangle)]} ,$$

with $\langle q \rangle$ the average of q in a given time interval. The general rule is to consider $\mathcal{D}q$ and $\mathcal{D}p$ as independent variables like in the canonical transformation and write down the Legendre transform of H

$$L = p\dot{q} - H .$$

This way, the $\mathcal{D}p$ integration is trivial and hence one obtains the standard expression

$$\int \mathcal{D}q \exp\left(\frac{i}{\hbar} \int_{t'}^{t''} dt L\right) .$$

¹² This implies that for a classical model, whose Hamiltonian contains terms such as $p^m q^n$, there are several quantum models that differ by the ordering prescription. Note that the condition that the corresponding operator be self-adjoint imposes some constraints. The simplest example is $q^2 p^2$. A way to construct a self-adjoint operator, is to take the self-adjoint part of $\hat{q}^2 \hat{p}^2$, that is

$$q^2 p^2 \longrightarrow \hat{q}^2 \hat{p}^2 - \frac{1}{2} [\hat{q}^2, \hat{p}^2] = \hat{q}^2 \hat{p}^2 - i\{\hat{q}, \hat{p}\} .$$

A possible alternative is

$$q^2 p^2 \longrightarrow \hat{q} \hat{p}^2 \hat{q} = \hat{q}^2 \hat{p}^2 + \hat{q} [\hat{p}^2, \hat{q}] = \hat{q}^2 \hat{p}^2 - 2i\hat{q}\hat{p} .$$

The two prescriptions differ by a c -number. Higher powers lead to differences that depend on \hat{q} and \hat{p} .

¹³ This prescription is also called Weyl quantisation.

¹⁴ It should be stressed that whereas the range of integration for the path is constrained by the initial conditions $q(t'') = q''$, $q(t') = q'$, there are no constraint for the range of p .

6.5 Forced harmonic oscillator

An interesting application of the path integral formalism treated above is the computation done on the forced harmonic oscillator system. Therefore, the main goal of this section is to compute

$$\langle Q', t | Q, T \rangle_F = \int \mathcal{D}q \exp \left\{ i \int_T^t dt' \left[\frac{\dot{q}^2}{2} - \frac{1}{2}(\omega^2 - i\epsilon)q^2 + F(t')q(t') \right] \right\}, \quad (6.58)$$

with $F(t')$ a driving force and $i\epsilon$, $\epsilon > 0$, a damping term that will assure the convergence of the oscillating integrand. The implicit boundary condition is that the system has configuration Q at time T and Q' at t . Now suppose that we want to find the transition amplitude of the system starting at time $T = -\infty$ till $t = +\infty$. Some useful mathematical tools are needed and therefore we introduce the Fourier transformations

$$G(t) = \int_{\mathbb{R}} \frac{dE}{\sqrt{2\pi}} e^{iEt} \tilde{G}(E),$$

$$\tilde{G}(E) = \int_{\mathbb{R}} \frac{dt}{\sqrt{2\pi}} e^{-iEt} G(t).$$

Expressing $q(t)$ and $F(t)$ in terms of $\tilde{q}(E)$ and $\tilde{F}(E)$, we have

$$\frac{1}{2} [\dot{q}^2 - (\omega^2 - i\epsilon)q^2] = \frac{1}{2} \int_{\mathbb{R}} \frac{dE}{\sqrt{2\pi}} \frac{dE'}{\sqrt{2\pi}} e^{i(E+E')t} (-EE' - \omega^2 + i\epsilon) \tilde{q}(E) \tilde{q}(E'),$$

$$F(t)q(t) = \frac{1}{2} \int_{\mathbb{R}} \frac{dE}{\sqrt{2\pi}} \frac{dE'}{\sqrt{2\pi}} e^{i(E+E')t} (\tilde{q}(E) \tilde{F}(E') + \tilde{q}(E') \tilde{F}(E)).$$

Integrating over t and using $\delta(x - x') = \int \frac{dl}{2\pi} e^{il(x-x')}$ and then integrating over E' , one obtains for the exponent in (6.58)

$$\frac{i}{2} \int_{\mathbb{R}} dE [(E^2 - \omega^2 + i\epsilon) \tilde{q}(E) \tilde{q}(-E) + \tilde{q}(E) \tilde{F}(-E) + \tilde{q}(-E) \tilde{F}(E)]. \quad (6.59)$$

Consider the new variable

$$\tilde{q}'(E) = \tilde{q}(E) + \frac{\tilde{F}(E)}{E^2 - \omega^2 + i\epsilon},$$

and note that in q space this is just a constant shift, so that $\mathcal{D}q' = \mathcal{D}q$. In t -space we have

$$q'(t) = q(t) + \int_{\mathbb{R}} \frac{dE}{\sqrt{2\pi}} e^{iEt} \frac{\tilde{F}(E)}{E^2 - \omega^2 + i\epsilon}.$$

The previous analysis implies that, in the limits $T \rightarrow -\infty$ and $t \rightarrow +\infty$, (6.58) reads

$$\begin{aligned} \langle Q'_{+\infty} | Q_{-\infty} \rangle_F &= \exp \left(-\frac{i}{2} \int_{\mathbb{R}} dE \frac{\tilde{F}(E)\tilde{F}(-E)}{E^2 - \omega^2 + i\epsilon} \right) \\ &\times \int \mathcal{D}q \exp \left[\frac{i}{2} \int_{\mathbb{R}} dE \tilde{q}'(E)(E^2 - \omega^2 + i\epsilon)\tilde{q}'(-E) \right]. \end{aligned} \quad (6.60)$$

For $F = 0$, the first exponential on the right-hand side is 1, so that

$$\langle Q'_{+\infty} | Q_{-\infty} \rangle_F = \langle Q'_{\infty} | Q_{-\infty} \rangle_{F=0} \exp \left(-\frac{i}{2} \int_{\mathbb{R}} dE \frac{\tilde{F}(E)\tilde{F}(-E)}{E^2 - \omega^2 + i\epsilon} \right). \quad (6.61)$$

Then note that

$$\int_{\mathbb{R}} dE \frac{\tilde{F}(E)\tilde{F}(-E)}{E^2 - \omega^2 + i\epsilon} = \int_{\mathbb{R}} dt dt' F(t)D(t-t')F(t'), \quad (6.62)$$

where

$$D(t-t') = \int_{\mathbb{R}} \frac{dE}{2\pi} \frac{e^{-i(t-t')E}}{E^2 - \omega^2 + i\epsilon}. \quad (6.63)$$

To compute $D(t)$ we consider the contour integral around the poles as in the second graph

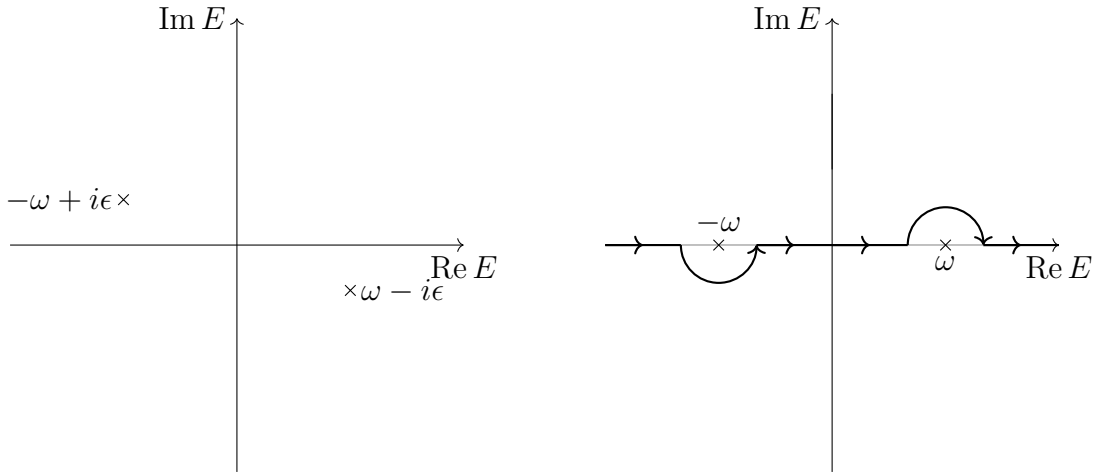


Figure 6.1

In the case $t' = 0$, we have

- ▷ if $t > 0$, then, by Jordan's lemma, one closes the integration contour by an half-circle in $\text{Im } E < 0$, so that

$$D(t) = -\frac{2\pi i}{2\pi} \frac{(E - \omega)e^{-itE}}{(E - \omega)(E + \omega)} \Big|_{E=\omega} = \frac{1}{2i\omega} \Theta(t) e^{-i\omega t}.$$

▷ If $t > 0$, then one closes the integration contour by an half-circle in $\text{Im } E > 0$, getting

$$D(t) = \frac{2\pi i}{2\pi} \frac{(E + \omega)e^{-itE}}{(E - \omega)(E + \omega)} \Big|_{E=-\omega} = \frac{1}{2i\omega} \Theta(-t)e^{i\omega t} .$$

Summarising, we have

$$D(t) = \frac{1}{2i\omega} (\Theta(t)e^{-i\omega t} + \Theta(-t)e^{i\omega t}) . \quad (6.64)$$

Note that $D(t - t')$ is a Green function, that is, it satisfies the equation

$$\left(\frac{d^2}{dt^2} + \omega^2 \right) D(t) = -\delta(t) . \quad (6.65)$$

Note that the $i\epsilon$ prescription is due to the requirement of the existence of the path integral, and consequently fixes the boundary conditions. From the physical point of view, $D(t)$ describes a signal coming from two sources, namely, as signal of positive energy (particle) states moving forward in time ($e^{-i\omega t}$) and another one of negative energy (antiparticle) moving backward in time ($e^{i\omega t}$).

In the case $F(\pm\infty) = 0$ the vacuum states are F -independent. Denoting by $|\Omega_{\pm\infty}\rangle$ such states¹⁵ we have

$$\langle \Omega_{+\infty} | \Omega_{-\infty} \rangle_F = \int dQ' dQ \langle \Omega_{+\infty} | Q'_{+\infty} \rangle \underbrace{\langle Q'_{+\infty} | Q_{-\infty} \rangle_F}_{\langle Q'_{+\infty} | Q_{-\infty} \rangle_{F=0} e^{-\frac{i}{2}\langle FDF \rangle}} \langle Q_{-\infty} | \Omega_{-\infty} \rangle , \quad (6.66)$$

where we used (6.61) and defined

$$\langle FDF \rangle := \int_{\mathbb{R}} dt dt' F(t) D(t - t') F(t') .$$

Now note that (6.66) implies

$$\langle \Omega_{+\infty} | \Omega_{-\infty} \rangle_F = \langle \Omega_{+\infty} | \Omega_{-\infty} \rangle_{F=0} e^{-\frac{i}{2}\langle FDF \rangle} . \quad (6.67)$$

If we are in the vacuum state at $t = -\infty$ and in the absence of the driving force $F = 0$, i.e. there is no driving force, and we will be in the vacuum as well even at $t = +\infty$. Therefore, if the vacuum is normalisable, then

$$\langle \Omega_{+\infty} | \Omega_{-\infty} \rangle_{F=0} = 1 ,$$

¹⁵ Recall that, as explained above, the bras and kets in the expression for the transition amplitudes, correspond to states in the Heisenberg picture. In particular, as we said, we have $\hat{O}_H(t)|o, t\rangle = o|o, t\rangle$.

and, by (6.67),

$$\langle \Omega_{+\infty} | \Omega_{-\infty} \rangle_F = e^{-\frac{i}{2} \langle F D F \rangle} .$$

Hence, the term

$$e^{-\frac{i}{2} \langle F D F \rangle} ,$$

can be interpreted as the transition amplitude of the system from initial past- to final in the future ground state in the presence of an external driving force F . To proceed with the analysis, define

$$Z[F] \equiv e^{iW[F]} := e^{-\frac{i}{2} \langle F_1 D_{12} F_2 \rangle_{12}} ,$$

with $Z[0] = 1$, and where $\langle \rangle_{12}$ denotes integration over the variables in subscript. Note that

$$D(t_1 - t_2) = i \left. \frac{\delta^2 Z[F]}{\delta F(t_1) \delta F(t_2)} \right|_{F=0} .$$

Let us now consider the probability amplitude

$$\langle Q', t_f | Q, t_i \rangle_J = \int Dq e^{i \int_{t_i}^{t_f} dt (L(q, \dot{q}) + J(t)q(t))} , \quad (6.68)$$

where $J(t)$ is an external source acting only in the time interval $[t_a, t_b]$ with

$$t_i < t_a < t_b < t_f .$$

Since the external source vanishes outside $[t_a, t_b]$, we have

$$\langle Q', t_f | Q, t_i \rangle_J = \int dq dq' \langle Q', t_f | q', t_b \rangle \langle q', t_b | q, t_a \rangle_J \langle q, t_a | Q, t_i \rangle .$$

By

$$|Q, t_i \rangle = e^{iHt_i} |Q \rangle ,$$

we get

$$\langle q, t_a | Q, t_i \rangle = \sum_n \langle q, t_a | E_n \rangle \langle E_n | Q \rangle e^{iE_n t_i} = \sum_n \langle q, t_a | E_n \rangle \phi_n^*(Q) e^{iE_n t_i} ,$$

and, similarly,

$$\langle Q', t_f | q', t_b \rangle = \sum_m \phi_m(Q') e^{-iE_m t_f} \langle E_m | q', t_b \rangle .$$

Therefore,

$$\langle Q', t_f | Q, t_i \rangle_J = \sum_{m,n} \int dq dq' e^{i(E_n t_i - E_m t_f)} \phi_m(Q') \langle E_m | q', t_b \rangle \langle q', t_b | q, t_a \rangle_J \langle q, t_a | E_n \rangle \phi_n^*(Q) .$$

On the other hand,

$$\int dqdq' \langle E_m | q', t_b \rangle \langle q', t_b | q, t_a \rangle_J \langle q, t_a | E_n \rangle = \langle E_m | E_n \rangle_J ,$$

so that

$$\langle Q', t_f | Q, t_i \rangle_J = \sum_{m,n} \phi_n^*(Q) \phi_m(Q') e^{i(E_n t_i - E_m t_f)} \langle E_m | E_n \rangle_J .$$

Let us consider the case in which $t_i \rightarrow i\infty$ and $t_f \rightarrow -i\infty$, so that

$$e^{iE_n t_i} \longrightarrow e^{-\infty \cdot E_n} , \quad e^{-iE_m t_f} \longrightarrow e^{-\infty \cdot E_m} .$$

In this case the least damped term is the one with lowest energy, that is E_0 . If there is no degeneracy, then can identify the corresponding state with the vacuum state $|\Omega\rangle$. Then, we have

$$\lim_{\substack{t_i \rightarrow i\infty \\ t_f \rightarrow -i\infty}} \langle Q', t_f | Q, t_i \rangle_J = \phi_0^*(Q) \phi_0(Q') e^{-iE_0(t_f - t_i)} \langle \Omega | \Omega \rangle_J .$$

The key point of such a construction is that by (6.68) we have been able to find the relation between the path integral and the vacuum to vacuum amplitude in the presence of an external source, that is

$$\lim_{\substack{t_i \rightarrow i\infty \\ t_f \rightarrow -i\infty}} \int Dq e^{i \int_{t_i}^{t_f} dt (L(q, \dot{q}) + J(t)q(t))} = \phi_0^*(Q) \phi_0(Q') e^{-iE_0(t_f - t_i)} \langle \Omega | \Omega \rangle_J . \quad (6.69)$$

Taking the functional derivative with respect to J we can get the n -point Green functions. In this respect, it is worth mentioning that the J dependence arises only in the vacuum-to-vacuum amplitude, and the factor $\phi_0^*(Q) \phi_0(Q') e^{-iE_0(t_f - t_i)}$ can be absorbed by a normalisation, which is the path integral itself with $J = 0$. In particular, we have

$$\begin{aligned} (-i)^n \frac{\delta \langle \Omega | \Omega \rangle_J}{\delta J(t_1) \dots \delta J(t_n)} \Big|_{J=0} &= \langle \Omega | T q(t_1) \dots q(t_n) | \Omega \rangle \\ &= \frac{(-i)^n}{Z[0]} \frac{\delta Z[J]}{\delta J(t_1) \dots \delta J(t_n)} \Big|_{J=0} , \end{aligned} \quad (6.70)$$

where

$$Z[J] = \lim_{\substack{t_i \rightarrow i\infty \\ t_f \rightarrow -i\infty}} \int Dq e^{i \int_{t_i}^{t_f} dt (L(q, \dot{q}) + J(t)q(t))} .$$

6.6 A formal anticipation of perturbation theory

A general lesson of the above analysis is the following formal structure that, as we will see, extends to quantum field theory. We consider the path integral in Euclidean space

$$\int \mathcal{D}X \exp(-\langle X\hat{O}X - XY \rangle) \sim \exp(-\langle Y\hat{O}^{-1}Y \rangle / 2) \int \mathcal{D}X \exp(-\langle X\hat{O}X \rangle), \quad (6.71)$$

where \hat{O} is a second order operator. We will see that the path integral of Gaussian integrals, as the one in the right-hand side of (6.71), is a constant proportional to $(\det \hat{O})^{-1/2}$. Therefore,

$$\int \mathcal{D}X \exp(-\langle X\hat{O}X - XY \rangle) \sim (\det \hat{O})^{-1/2} \exp(-\langle Y\hat{O}^{-1}Y \rangle / 2).$$

The extension to the case with a potential density $V(X)$ can be treated by using the Schwinger trick

$$\int \mathcal{D}X \exp(-\langle X\hat{O}X + V(X) - XY \rangle) = \exp \langle V(\delta/\delta Y) \rangle \int \mathcal{D}X \exp(-\langle X\hat{O}X - XY \rangle).$$

Therefore,

$$\begin{aligned} & \int \mathcal{D}X \exp(-\langle X\hat{O}X + V(X) - XY \rangle) \\ & \sim (\det \hat{O})^{-1/2} \exp \langle V(\delta/\delta Y) \rangle \exp(-\langle Y\hat{O}^{-1}Y \rangle / 2). \end{aligned} \quad (6.72)$$

One then can consider the perturbation theory by expanding $\exp \langle V(\delta/\delta Y) \rangle$, that is

$$\begin{aligned} & \int \mathcal{D}X \exp(-\langle X\hat{O}X + V(X) - XY \rangle) \\ & \sim (\det \hat{O})^{-1/2} \sum_{k=0}^{\infty} \frac{1}{k!} \langle V(\delta/\delta Y) \rangle^k \exp(-\langle Y\hat{O}^{-1}Y \rangle / 2). \end{aligned} \quad (6.73)$$

6.7 Path integral for quadratic Lagrangians

Another commonly used notation for the transition amplitude is

$$K(x_2, t_2 | x_1, t_1) \equiv \langle x_2, t_2 | x_1, t_1 \rangle = \int_{x(t_1)=x_1}^{x(t_2)=x_2} \mathcal{D}[x(t)] \exp \left(\frac{i}{\hbar} \int_{t_1}^{t_2} dt L(x, \dot{x}, t) \right). \quad (6.74)$$

Note that in the classical approximation

$$K(x_2, t_2 | x_1, t_1) \sim \exp \left(\frac{i}{\hbar} \int_{t_1}^{t_2} dt L(x_{cl}, \frac{dx_{cl}}{dt}, t) \right) .$$

In the following We will consider the path integral in the case of the quadratic Lagrangian

$$L(x, \dot{x}, t) = a(t)x^2 + b(t)\dot{x}^2 + c(t)x\dot{x} + d(t)x + e(t)\dot{x} + f(t) .$$

Explicit calculations of the transition amplitude, for models with such a Lagrangian, are possible only in the case the coefficients in L are constants. We will see that in this case

$$K(x_2, t_2 | x_1, t_1) = A(t_2, t_1) \exp \left(\frac{i}{\hbar} \int_{t_1}^{t_2} dt L(x_{cl}, \dot{x}_{cl}, t) \right) , \quad (6.75)$$

where $A(t_2, t_1) = A(t_2 - t_1)$. Consider an arbitrary path

$$x(t) = x_{cl}(t) + y(t) , \quad y(t_1) = y(t_2) = 0 ,$$

insert this in the integrand of (6.74) and perform a Taylor expansion

$$\begin{aligned} & \int_{t_1}^{t_2} dt L(x_{cl} + y, \dot{x}_{cl} + \dot{y}, t) = \\ & = \int_{t_1}^{t_2} dt \left[L + \frac{\partial L}{\partial x} y + \frac{\partial L}{\partial \dot{x}} \dot{y} + \frac{1}{2} \left(\frac{\partial^2 L}{\partial x^2} y^2 + 2 \frac{\partial^2 L}{\partial x \partial \dot{x}} y \dot{y} + \frac{\partial^2 L}{\partial \dot{x}^2} \dot{y}^2 \right) + 0 \right] \Big|_{(x_{cl}, \dot{x}_{cl}, t)} . \end{aligned}$$

Notice the zero term in the expansion due to the fact that L is quadratic and therefore ‘‘Taylor exact’’. Note that

$$\int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} y + \frac{\partial L}{\partial \dot{x}} \dot{y} \right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right) y = 0 ,$$

which is a consequence of the fact that this equation is just the Euler-Lagrange equations evaluated at $x_{cl}(t)$. Therefore, we are left with an expression where the path integral depends on t_1 and t_2 but not on x_1 and x_2

$$K(x_2, t_2 | x_1, t_1) = e^{\frac{i}{\hbar} S[x_{cl}(t)]} \int_{y(t_1)=0}^{y(t_2)=0} \mathcal{D}y(t) \exp \left[\frac{i}{\hbar} \int_{t_1}^{t_2} dt (a(t)y^2 + b(t)\dot{y}^2 + c(t)y\dot{y}) \right] . \quad (6.76)$$

Therefore,

$$K(x_2, t_2 | x_1, t_1) = A(t_2, t_1) \exp \left(\frac{i}{\hbar} \int_{t_1}^{t_2} dt L(x_{cl}, \dot{x}_{cl}, t) \right) .$$

As we said, if $a(t)$, $b(t)$ and $c(t)$ are time independent, then

$$A(t_2, t_1) = A(0, t_2 - t_1) .$$

To show this, one first expresses

$$\int_{t_1}^{t_2} dt (ay^2(t) + by^2(t) + cy(t)\dot{y}(t)) ,$$

as an integral on the interval $[t_1 + \Delta t, t_2 + \Delta t]$ with the arguments of x , \dot{x} , y and \dot{y} shifted by $-\Delta t$. Then, defining $y'(t) = y(t - \Delta t)$, one gets

$$\begin{aligned} A(t_2, t_1) &= \int_{y'(t_1+\Delta t)=0}^{y'(t_2+\Delta t)=0} \mathcal{D}y'(t) \exp \left[\frac{i}{\hbar} \int_{t_1}^{t_2} dt (ay'(t)^2 + by'(t)^2 + cy'(t)\dot{y}'(t)) \right] \\ &= A(t_2 + \Delta t, t_1 + \Delta t) . \end{aligned} \quad (6.77)$$

Let us keep x_1 and t_1 fixed and consider

$$K_{(x_1, t_1)}(x, t) \equiv K(x, t | x_1, t_1) ,$$

as a wave function. Actually, this is the probability amplitude of finding the particle at x at time t , knowing that it was at x_1 at time t_1 . Therefore, $K(x_2, t_2 | x_1, t_1)\psi(x_1, t_1)$ is the probability amplitude of finding the particle at t_2 knowing that it had the probability amplitude $\psi(x_1, t_1)$ of staying at x_1 at $t = t_1$ and hence

$$\psi(x_2, t_2) = \int dx_1 K(x_2, t_2 | x_1, t_1) \psi(x_1, t_1) .$$

This also implies the result already derived above, that is $K_{(x_1, t_1)}(x, t) = \delta(x - x_1)$. Since $K(x_2, t_2 | x_1, t_1)$ is a wave function, we have that it must satisfy the group property

$$K(x_3, t_3 | x_1, t_1) = \int dx_2 K(x_3, t_3 | x_2, t_2) K(x_2, t_2 | x_1, t_1) .$$

This is sufficient to find the propagator in the case of the free particle $L(x, \dot{x}) = \frac{m}{2}\dot{x}^2$

$$K(x_2, t_2 | x_1, t_1) = A(t_2 - t_1) \exp \left(\frac{i}{\hbar} \int_{t_1}^{t_2} \frac{m}{2} \dot{x}_{cl}^2 dt \right) = A(t_2 - t_1) \exp \left(\frac{i}{\hbar} \frac{m}{2} \frac{(x_2 - x_1)^2}{t_2 - t_1} \right) .$$

and one may check that the group property yields

$$A(t) = \sqrt{\frac{m}{2\pi i \hbar t}} ,$$

so that

$$\psi(x, t) = K(x, t|0, 0) = \sqrt{\frac{m}{2\pi i\hbar t}} \exp\left(\frac{i m x^2}{\hbar 2 t}\right).$$

6.8 Bohm-Aharonov effect¹⁶

Consider a non-relativistic particle of charge q , propagating from (\mathbf{x}_1, t_1) to (\mathbf{x}_2, t_2) , in the presence of a fixed magnetic field background $\mathbf{B} = \nabla \times \mathbf{A}$, where \mathbf{A} is the vector potential. Such a system is described by adding the term $q\dot{\mathbf{x}} \cdot \mathbf{A}/c$ to the free Lagrangian L_0

$$L_0 \rightarrow L_0 + \frac{q}{c} \dot{\mathbf{x}} \cdot \mathbf{A}.$$

Correspondingly, the action gets the additional term

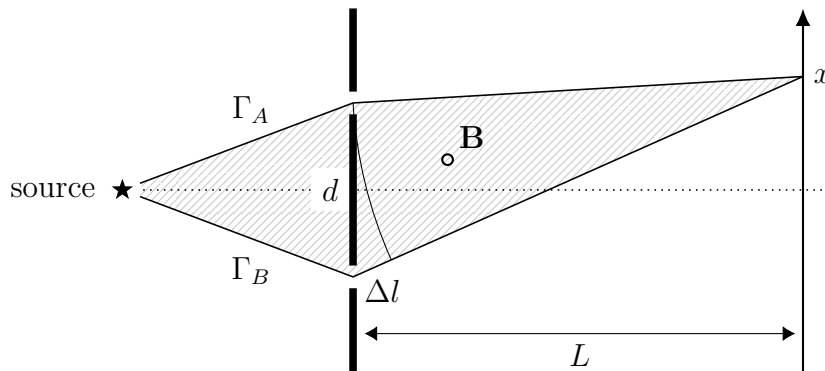
$$\frac{q}{c} \int_{t_1}^{t_2} \dot{\mathbf{x}} \cdot \mathbf{A} dt = \frac{q}{c} \int_{\Gamma} \mathbf{A} \cdot d\mathbf{x}.$$

Each path Γ between the two extrema ($d\Gamma/dt =: \mathbf{v}$) contributes to the transition amplitude with a factor

$$\Phi_{\Gamma}(2|1) = \exp\left[\frac{i}{\hbar} \left(\int_{t_1}^{t_2} \frac{1}{2} m \|\mathbf{v}\|^2 dt + \frac{q}{c} \int_{\Gamma} \mathbf{A} \cdot d\mathbf{x} \right)\right] = \Phi_{\Gamma}^0(2|1) \Phi_{\Gamma}^{\text{int}}(2|1),$$

where $\Phi_{\Gamma}^0(2|1)$ is the phase factor relative to the path Γ contributing to the free propagator.

Let us now isolate two paths and study their quantum interference in the free case. In order to do this, a double slit experiment can be prepared.



The dominant contribution to the transition amplitude from the source to a generic point x on the screen, comes from the two classical paths Γ_A and Γ_B , corresponding to the particle propagating along straight segments and passing through one of the two slits. Before considering the case with the magnetic field, it is instructive to make

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a qualitative analysis of the phases associated to such paths by considering the wave function of a free particle with momentum p

$$\psi(x, t) = e^{i(px - Et)/\hbar} ,$$

where $E = p^2/(2m)$. The associated de Broglie wavelength λ and frequency ν are

$$\lambda = \frac{h}{p} , \quad \nu = \frac{E}{h} .$$

In the limit of distant screen, $L \gg d$, the phase difference between these contributions

$$\Delta^0 = \theta_A - \theta_B ,$$

can be computed simply as

$$\Delta^0 = \frac{\Delta l}{\lambda} 2\pi \approx \frac{dx}{\lambda L} 2\pi .$$

Since the paths Γ_A and Γ_B are the classical ones, these provide the dominant contributions to the probability amplitude. In other words, in good approximation Δ^0 will be the phase of the transition amplitude for a free particle to travel from the source to the point x of the screen.

Let us denote by θ'_A and θ'_B the phases associated to $\Phi_{\Gamma_A}^0(2|1)$ $\Phi_{\Gamma_B}^0(2|1)$ respectively. If a very localised magnetic field \mathbf{B} , like the one shown in picture, is introduced as a small perturbation by means of a narrow solenoid, the dominant paths can still be approximated by Γ_A and Γ_B . The phases corresponding to such paths are now shifted.

$$\begin{aligned} \theta'_A &\longrightarrow \theta_A + \frac{q}{\hbar c} \int_{\Gamma_A} \mathbf{A} \cdot d\mathbf{x} , \\ \theta'_B &\longrightarrow \theta_B + \frac{q}{\hbar c} \int_{\Gamma_B} \mathbf{A} \cdot d\mathbf{x} , \end{aligned}$$

so that the phase of the transition amplitude relative to such a dominant contribution to the transition amplitude will be shifted by

$$\frac{q}{\hbar c} \left(\int_{\Gamma_B} - \int_{\Gamma_A} \right) \mathbf{A} \cdot d\mathbf{x} ,$$

due to the additional term in the action. Defining $\Gamma = \Gamma_B - \Gamma_A$ and Σ as a surface such that $\partial\Sigma = \Gamma$, the phase shift is seen to be proportional to the magnetic flux through Σ

$$\oint_{\Gamma} \mathbf{A} \cdot d\mathbf{x} = \iint_{\Sigma} \mathbf{B} \cdot d\mathbf{\Sigma} .$$

This flux is clearly non-vanishing even in our case, in which the magnetic field is confined

to a region not penetrated by the considered paths. It follows that turning on the magnetic field, it will produce a new interference on the screen, due to such a phase difference. This is the *Bohm-Aharonov* effect which has been tested with experiments. It is then a manifestation of the fact that at the quantum level the electromagnetic potential has a direct physical significance, indeed the particle travelling along its path feels the effect of the vector potential \mathbf{A} which is present everywhere. Thanks to Stokes theorem, this effect can be alternatively interpreted as a non-local action of the magnetic field \mathbf{B} on particles moving along the classical path.

Chapter 7

Path Integral Formulation of Quantum Field Theory

Before we start handling the formalism of path integral in quantum field theory, it is necessary to recall some mathematical tools, needed in order to make the manipulations in this chapter accessible and clear.

7.1 Functional derivative

Consider the formal expansion of a functional $G[f]$ as

$$G[f] = G_0 + \int dx G_1(x)f(x) + \frac{1}{2!} \int dx_1 dx_2 G_2(x_1, x_2)f(x_1)f(x_2) + \dots, \quad (7.1)$$

where G_0 is a constant and $G_n(x_1, \dots, x_n)$ are symmetric functions. The analogue of the difference quotient limit defining the derivative is

$$\frac{\delta G[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (G[f(\cdot) + \epsilon \delta(\cdot - x)] - G[f(\cdot)]), \quad (7.2)$$

where “.” in $f(\cdot)$ and $\delta(\cdot - x)$ stands for the argument of a given f which is integrated out after computing the functional G . With such definition one can make explicit calculations, e.g.

$$\begin{aligned} & \frac{\delta}{\delta f(x)} \int dx_1 dx_2 G_2(x_1, x_2)f(x_1)f(x_2) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int dx_1 dx_2 G_2(x_1, x_2) [(f(x_1) + \epsilon \delta(x_1 - x))(f(x_2) + \epsilon \delta(x_2 - x)) - f(x_1)f(x_2)] \\ &= 2 \int dx_1 G_2(x_1, x)f(x_1), \end{aligned}$$

that can be obtained immediately by observing that (7.2) implies

$$\frac{\delta f(y)}{\delta f(x)} = \delta(y - x) . \quad (7.3)$$

There is a subtle point concerning such a relation that provides a key difference with respect to the standard derivative. Namely, we have

$$\frac{\partial x_j}{\partial x_k} = \delta_{jk} \text{ is dimensionless ,} \quad \frac{\delta f(y)}{\delta f(x)} = \delta(y - x) \text{ has dimension } L^{-1} .$$

Therefore, a characteristic property of the functional derivative is the following dimensional discrepancy

$$\left[\frac{\delta g(y)}{\delta f(x)} \right] \neq \left[\frac{g(y)}{f(x)} \right] .$$

Such an observation is related to the $\delta^{(D)}(0)$ divergences in quantum field theory. Noticing that in momentum space

$$\delta^{(D)}(0) = \int \frac{d^D x}{(2\pi)^D} ,$$

one treats such a singularity¹ by setting

$$\delta^{(D)}(0) = \frac{V(\mathbb{R}^D)}{(2\pi)^D} ,$$

with $V(\mathbb{R}^D)$ the space-time volume, treated as finite, and removed at the end of the calculations.

Using (7.3) one may easily check

$$G_n(x_1, \dots, x_n) = \frac{\delta^n G[f]}{\delta f(x_1) \dots \delta f(x_n)} \Big|_{f=0} .$$

Thus, (7.1) is identified as the natural functional generalisation of Taylor expansion. It should be noted that for our purposes it is not necessary that the expansion (7.1) of $G[f]$ is convergent. This is thought of simply as a formal expansion. An useful property of the functional derivative is

$$\frac{1}{i} \frac{\delta}{\delta J(x)} e^{i\langle J\phi \rangle} = \phi(x) e^{i\langle J\phi \rangle} .$$

For further details on the functional derivative a suggested source is [28, Appendix A].

¹ As we will see, $\delta^{(D)}(0)$ singularities in quantum field theory appear in the functional determinants and in gluing legs of the same vertex in a Feynman diagram, which is a singularity removable by normal ordering.

7.2 Identification of N -point functions in the path integral formalism²

In the previous section it was shown that the amplitude $\langle q_b, T | q_a, 0 \rangle$ can be expressed in terms of the path integral. Let us move on to the general case of a quantum system described by a set of coordinates $q = \{q^k\}$ and conjugated momenta $p = \{p^k\}$. Denote by $q_a = \{q_a^k\}$ the set of initial values of the coordinates and by $q_b = \{q_b^k\}$ the final one. Then

$$\langle q_b, T | q_a, 0 \rangle = \left(\prod_k \int \mathcal{D}q(t) \mathcal{D}p(t) \right) \exp \left[i \int_0^T dt \left(\sum_k p^k \dot{q}^k - H(q, p) \right) \right]. \quad (7.4)$$

Notice that the coordinates trajectories $q(t)$ have fixed values at the extremes, $q(0) = q_a$ e $q(T) = q_b$, whereas the $p(t)$ do not have any constraint. The measure in (7.4) corresponds at any time instant to

$$\prod_k \int \frac{dq^k dp^k}{2\pi\hbar}. \quad (7.5)$$

To take into account that such a measure is the one at every time, we can also re-write it in the form

$$\prod_{t \in [0, T]} \prod_k \int \frac{dq^k(t) dp^k(t)}{2\pi\hbar}. \quad (7.6)$$

The coordinates q^k can be interpreted as a scalar field $\phi(\mathbf{x})$. The change $q \rightarrow \phi$ takes place by replacing the discrete index k with the continuous 3-dimensional “index” \mathbf{x} , that is

$$k \in I \longrightarrow \mathbf{x} \in \mathbb{R}^3, \quad q^k \longrightarrow \phi(\mathbf{x}),$$

with I some index set. We then have

$$q^k(t) \longrightarrow \phi(x).$$

The corresponding Hamiltonian is then

$$H = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla\phi)^2 + V(\phi) \right]. \quad (7.7)$$

Before considering the path integral formulation of quantum field theory, it is worth recalling some aspects of its operator formulation. First, one considers the Lagrange function

$$L(t) = \int d^3x \mathcal{L}(\phi, \partial\phi),$$

² Umberto Natale

and define the conjugate momentum³

$$\pi(t, \mathbf{x}) = \frac{\delta L(t)}{\delta(\partial_0 \phi(t, \mathbf{x}))} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(t, \mathbf{x}))} .$$

One then imposes the equal time canonical commutation relations

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y}) . \quad (7.8)$$

Note that all the others *equal time* commutation relations vanish

$$[\phi(t, \mathbf{x}), \phi(t, \mathbf{y})] = 0 , \quad [\pi(t, \mathbf{x}), \pi(t, \mathbf{y})] = 0 .$$

The vanishing of the last two commutation relations is not a property that extends to different times, that is we have

$$[\phi(x), \phi(y)] \neq 0 , \quad [\pi(x), \pi(y)] \neq 0 .$$

For example, in the case of the free theory, we have

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^4 k}{(2\pi)^4} 2\pi \delta(k^2 - m^2) \theta(k^0) \left[e^{-ik(x-y)} - e^{ik(x-y)} \right] \\ &= \int \frac{d^4 k}{(2\pi)^3} \delta(k^2 - m^2) \epsilon(k^0) e^{-ik(x-y)} , \end{aligned}$$

which is an odd, Lorentz invariant solution of the Klein-Gordon equation. Furthermore, since the above commutator vanishes for $x^0 = y^0$, we see that, by Lorentz invariance, it also vanishes outside the light-cone, that is for $(x - y)^2 < 0$. Also note that, by (7.8),

$$\partial_0 [\phi(x), \phi(y)]|_{x^0=y^0} = -i\delta^{(3)}(\mathbf{x} - \mathbf{y}) .$$

To derive the path integral formulation of quantum field theory, recall that, in the case of quantum mechanics, we inserted in $\langle q', t | q, 0 \rangle$ infinitely many copies of the completeness relation

$$\int dq |q\rangle \langle q| = I .$$

Then, we should find the quantum field analogue, $|\phi\rangle$ of $|q\rangle$. The latter is the eigenket

³ Following the standard notation, by

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(t, \mathbf{x}))} ,$$

we mean

$$\frac{\partial \mathcal{L}}{\partial y} ,$$

with $y = \partial_0 \phi(t, \mathbf{x})$.

of \hat{Q} , that is

$$\hat{Q}|q\rangle = q|q\rangle ,$$

so that

$$\hat{\phi}(\mathbf{x})|\phi_a\rangle = \phi_a(\mathbf{x})|\phi_a\rangle .$$

In some text-books the path integral is formulated considering the eigenkets of the annihilation part $\hat{\phi}_-$ of $\hat{\phi} = \hat{\phi}_- + \hat{\phi}_+$, that is

$$\hat{\phi}_-(\mathbf{x})|\phi_a\rangle = \phi_a(\mathbf{x})|\phi_a\rangle ,$$

where $|\phi_a\rangle$ is the quantum field theory analogue of the coherent states for a single harmonic oscillator, that is

$$|\phi_a\rangle = \exp\left(\int d^3x \hat{\phi}_+(\mathbf{x})\phi_a(\mathbf{x})\right)|0\rangle .$$

In the case of the momentum, we have

$$\hat{\pi}(\mathbf{x})|\pi_a\rangle = \pi_a(\mathbf{x})|\pi_a\rangle ,$$

and the analogue of $\langle \mathbf{p}|\mathbf{x}\rangle = \exp(-i\mathbf{p}\cdot\mathbf{x})$ reads

$$\langle \pi|\phi\rangle = \exp\left(-i\int d^3x \pi(\mathbf{x})\phi(\mathbf{x})\right) .$$

Now, note that the analog of the transition amplitude (7.4) is

$$\langle \phi_b|e^{-iHT}|\phi_a\rangle = \int \mathcal{D}\phi \mathcal{D}\pi \exp\left[i\int_0^T d^4x \left(\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right] ,$$

where the path integral is constrained by $\phi(0, \mathbf{x}) = \phi_a(\mathbf{x})$ and $\phi(T, \mathbf{x}) = \phi_b(\mathbf{x})$. Since the Hamiltonian (7.7) is quadratic in π , one has

$$\langle \phi_b(\mathbf{x})|e^{-iHT}|\phi_a(\mathbf{x})\rangle = \int \mathcal{D}\phi \exp\left(i\int_0^T d^4x \mathcal{L}\right) , \quad (7.9)$$

where

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi/2 - V(\phi) .$$

Similarly to the path integral expression of the amplitude $\langle q', t'|q, t\rangle$, also in this case it is possible to extract the time evolution operator from the bracket

$$\langle \phi_b|e^{-iHT}|\phi_a\rangle = e^{-iHT}\langle \phi_b|\phi_a\rangle = e^{-iHT}\delta(\phi_b - \phi_a) ,$$

with $\delta(\phi_b - \phi_a)$ the functional δ distribution whose argument is the difference between

two fields. $\delta(\phi_b - \phi_a)$ can be thought of as an infinite product

$$\delta(\phi_b - \phi_a) = \prod_{\mathbf{x} \in \mathbb{R}^3} \delta(\phi_b(\mathbf{x}) - \phi_a(\mathbf{x})) ,$$

whose integral representation is

$$\delta(\phi_b - \phi_a) = \int \mathcal{D}\pi(\mathbf{x}) \exp \left[i \int d^3x \pi(\mathbf{x})(\phi_b(\mathbf{x}) - \phi_a(\mathbf{x})) \right] .$$

Note that, the above analysis implies

$$\begin{aligned} \langle \phi_b | \phi_a \rangle &= \int \mathcal{D}\pi(\mathbf{x}) \langle \phi_b | \pi \rangle \langle \pi | \phi_a \rangle \\ &= \int \mathcal{D}\pi(\mathbf{x}) \exp \left[i \int d^3x \pi(\mathbf{x})(\phi_b(\mathbf{x}) - \phi_a(\mathbf{x})) \right] \\ &= \delta(\phi_b - \phi_a) . \end{aligned}$$

Excluding the temporal dependence of the integration interval and on the boundary conditions, (7.9) is manifestly Lorentz invariant. However, the N -point function will correspond to the time integration interval $[-T, T]$ in the limit $T \rightarrow \infty(1 - i\epsilon)$. That is, we will see that in the path integral formalism the time ordered vacuum expectation values of N quantum fields, also called N -point functions or Green's functions, correspond to

$$\langle \Omega | T \phi(x_1) \dots \phi(x_N) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \phi(x_1) \dots \phi(x_N) \exp(i \int_{-T}^T d^4x \mathcal{L})}{\int \mathcal{D}\phi \exp(i \int_{-T}^T d^4x \mathcal{L})} . \quad (7.10)$$

Here the measure is the product of Lebesgue measures at each space-time point⁴

$$\mathcal{D}\phi = \prod_{x \in \mathbb{R}^4} d\phi(x) ,$$

which is the field analogue of (7.5) in the case the momentum integration has been carried out.

⁴ As discussed in the case of the Wightman's axioms, quantum fields are distributions. On the other hand, even classical fields are distributions, that is the Dirac δ also appears in the Poisson brackets. As such, even the fields in the path integral should be treated as distributions. In particular, the space of ϕ 's, on which one integrate in the path integral, is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^4)$. In this respect, the external source J should belong to the Schwarz space of test functions $\mathcal{S}(\mathbb{R}^4)$. The standard reference for such issues is the book by Glimm and Jaffe [4].

Consider the expression

$$\int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2) \exp\left(i \int_{-T}^T d^4x \mathcal{L}\right), \quad (7.11)$$

with $x_1^0, x_2^0 \in [-T, T]$. Contrary to (7.9) a symmetric time interval around $t = 0$ has been chosen, thus now $\phi(-T, \mathbf{x}) = \phi_a(\mathbf{x})$ and $\phi(T, \mathbf{x}) = \phi_b(\mathbf{x})$.

Let us investigate the relation between (7.11) and $\langle \Omega | T\phi(x_1)\phi(x_2) | \Omega \rangle$. To this end observe that

$$\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x}) \int_{[V]} \mathcal{D}\phi(x),$$

where the constraint $[V]$ in the last integral is given by the usual $\phi(-T, \mathbf{x}) = \phi_a(\mathbf{x})$ and $\phi(T, \mathbf{x}) = \phi_b(\mathbf{x})$, but also by

$$\phi(x_1^0, \mathbf{x}) = \phi_1(\mathbf{x}), \quad \phi(x_2^0, \mathbf{x}) = \phi_2(\mathbf{x}). \quad (7.12)$$

The equality holds because the constraints in (7.12) are neutralised by the two successive (leftmost) integrations. The utility of such a decomposition is that, now by construction, the subset of functions over which the third (innermost) integral is computed is populated only by functions that at the times x_1^0 and x_2^0 coincide with $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$. As a consequence, applying this decomposition to (7.11), $\phi(x_1)$ and $\phi(x_2)$ can be substituted by $\phi_1(\mathbf{x}_1)$ and $\phi_2(\mathbf{x}_2)$, respectively, and brought outside the integral.

Assuming $x_1^0 < x_2^0$, (7.11) is equivalent to

$$\begin{aligned} & \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x}) \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \int_{[V]} \mathcal{D}\phi(x) e^{i \int_{-T}^T d^4x \mathcal{L}} \\ &= \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x}) \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) \langle \phi_b, T | \phi_2, x_2^0 \rangle \langle \phi_2, x_2^0 | \phi_1, x_1^0 \rangle \langle \phi_1, x_1^0 | \phi_a, -T \rangle, \end{aligned} \quad (7.13)$$

where two relations have been used. The first one is

$$\int_{[V]} \mathcal{D}\phi(x) e^{i \int_{-T}^T d^4x \mathcal{L}} = \int_{[V_1]} \mathcal{D}\phi(x) e^{i \int_{-T}^{x_1^0} d^4x \mathcal{L}} \int_{[V_2]} \mathcal{D}\phi(x) e^{i \int_{x_1^0}^{x_2^0} d^4x \mathcal{L}} \int_{[V_3]} \mathcal{D}\phi(x) e^{i \int_{x_2^0}^T d^4x \mathcal{L}},$$

with integration bounds

$$\begin{aligned} [V_1] : & \quad \phi(-T, \mathbf{x}) = \phi_a(\mathbf{x}) \quad \text{and} \quad \phi(x_1^0, \mathbf{x}) = \phi_1(\mathbf{x}), \\ [V_2] : & \quad \phi(x_1^0, \mathbf{x}) = \phi_1(\mathbf{x}) \quad \text{and} \quad \phi(x_2^0, \mathbf{x}) = \phi_2(\mathbf{x}), \\ [V_3] : & \quad \phi(x_2^0, \mathbf{x}) = \phi_2(\mathbf{x}) \quad \text{and} \quad \phi(T, \mathbf{x}) = \phi_b(\mathbf{x}). \end{aligned}$$

The second one is (7.9), which we rewrite in the form

$$\langle \phi_\beta, t_2 | \phi_\alpha, t_1 \rangle = \int \mathcal{D}\phi(x) e^{i \int_{t_1}^{t_2} d^4x \mathcal{L}} ,$$

This is consistent since $|\phi, t\rangle$ refers to the Heisenberg picture, thus

$$|\phi, t\rangle = e^{iHt} |\phi\rangle ,$$

where $|\phi\rangle$ is the state in the Schrödinger picture.

For the sake of clarity, in the rest of this section the operators in the Schrödinger and Heisenberg pictures will be denoted by adding to the fields the subscript S and H respectively. As we saw, by definition $\phi_S(\mathbf{x}_1)$ and $\phi_S(\mathbf{x}_2)$ satisfy

$$\phi_S(\mathbf{x}_1) |\phi_1\rangle = \phi_1(\mathbf{x}_1) |\phi_1\rangle , \quad \phi_S(\mathbf{x}_2) |\phi_2\rangle = \phi_2(\mathbf{x}_2) |\phi_2\rangle .$$

It is thus possible to transform $\phi_1(\mathbf{x}_1)$ and $\phi_2(\mathbf{x}_2)$ in (7.13), in operators acting on $|\phi_1\rangle$ and $|\phi_2\rangle$ respectively. With this substitution (7.13) becomes

$$\begin{aligned} \int \mathcal{D}\phi_1 \int \mathcal{D}\phi_2 \langle \phi_b | e^{-iH(T-x_2^0)} \phi_S(\mathbf{x}_2) |\phi_2\rangle \langle \phi_2 | e^{-iH(x_2^0-x_1^0)} \phi_S(\mathbf{x}_1) |\phi_1\rangle \langle \phi_1 | e^{-iH(x_1^0+T)} |\phi_a\rangle \\ = \langle \phi_b | e^{-iHT} \phi_H(x_2) \phi_H(x_1) e^{-iHT} |\phi_a\rangle , \end{aligned}$$

where the completeness relation $\int \mathcal{D}\phi |\phi\rangle \langle \phi| = \mathbb{I}$ has been used, together with the correspondence between the operators in the Heisenberg and Schrödinger pictures

$$O_H(t) = e^{iHt} O_S e^{-iHt} .$$

The case $x_1^0 > x_2^0$ is described by an analogous expression with ϕ_1 and ϕ_2 exchanged. Therefore, Eq.(7.11) is equivalent to

$$\langle \phi_b | e^{-iHT} T(\phi_H(x_2) \phi_H(x_1)) e^{-iHT} |\phi_a\rangle .$$

Inserting a complete set of eigenstates of the Hamiltonian, this expression becomes

$$\sum_{m,n} e^{-i(E_n+E_m)T} \langle \phi_b | E_n \rangle \langle E_n | T(\phi_H(x_2) \phi_H(x_1)) | E_m \rangle \langle E_m | \phi_a \rangle .$$

As for relation (4.1), we are interested in evaluating this expression in the limit $T \rightarrow \infty(1-i\epsilon)$, so that only the vacuum component $|\Omega\rangle$ of the states $|\phi_a\rangle$ and $|\phi_b\rangle$ is selected. This procedure implies the assumption that $\langle \phi_a | \Omega \rangle$ and $\langle \phi_b | \Omega \rangle$ are non-vanishing. In

the limit $T \rightarrow \infty(1 - i\epsilon)$

$$e^{-iHT}|\phi_a\rangle = \sum_n e^{-iE_n T} |E_n\rangle \langle E_n | \phi_a\rangle \sim \langle \Omega | \phi_a\rangle e^{-iE_0 \cdot \infty(1-i\epsilon)} |\Omega\rangle .$$

the awkward factors cancel out if one divides (7.11) by $\int \mathcal{D}\phi \exp(i \int_{-T}^T d^4x \mathcal{L})$ (for further details see sections 4.2 and 9.2 of Peskin-Schroeder). Specifically

$$\langle \Omega | T \phi_H(x_2) \phi_H(x_1) | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \phi(x_1) \phi(x_2) e^{i \int_{-T}^T d^4x \mathcal{L}}}{\int \mathcal{D}\phi e^{i \int_{-T}^T d^4x \mathcal{L}}} .$$

The proof of the general case (7.10) is an obvious extension of the preceding derivation.

7.3 Path integral for scalar fields

At this stage, we want to consider the transition amplitude of a system driven by an external force (similar to the forced harmonic oscillator) in more general and fundamental way. It is then meaningful to require the existence of a vacuum state for the system and try to compute its corresponding transition amplitude for long enough time intervals, i.e. $t_i = -\infty$ and $t_f = +\infty$, keeping in mind the action of the external force associated to a local field, which we will denote by $J(x)$. Therefore, we will consider the simplest case, the one describing a scalar field theory with self-interaction. The expression for the transition amplitude yields

$$\langle \Omega | \Omega \rangle_J = Z[J] = N \int \mathcal{D}\phi \mathcal{D}\pi e^{i \langle \pi \dot{\phi} - \mathcal{H} + J\phi \rangle} , \quad (7.14)$$

where N is a constant, $\phi(\pi)$ denotes the product of all $d\phi_k = d\phi(x_k)$ and π the canonical momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi = \dot{\phi}$. Note that the Hamiltonian density is quadratic in π and therefore the integration over it is trivial, and we are left with an integration over ϕ . Therefore, the amplitude we intend to compute is

$$\langle \Omega | \Omega \rangle_J = Z[J] = N' \int \mathcal{D}\phi e^{i \langle \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) + J\phi \rangle} . \quad (7.15)$$

Two possible ways to treat this ill-defined integral are

- ▷ add a convergence term such that $\mathcal{L} \rightarrow \mathcal{L} + \frac{i}{2} \epsilon \phi^2$, $\epsilon > 0$,
- ▷ perform a Wick rotation, that is defining it in the Euclidean space.

Let us start by adding the convergence term $i\epsilon\phi^2/2$ to the Lagrangian density for a

scalar field

$$Z[J] = N' \int \mathcal{D}\phi \exp \left(i \left\langle \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} (m^2 - i\epsilon) \phi^2 - V(\phi) + J\phi \right\rangle \right),$$

whose expansion reads

$$Z[J] = \sum_{N=0}^{\infty} \frac{i^N}{N!} \langle G^{(N)}(1, \dots, N) J_1 \cdots J_N \rangle,$$

$$G^{(N)}(1, \dots, N) = \frac{1}{i^N} \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_N} Z[J] \Big|_{J=0}.$$

By (7.15) we see that

$$G^{(N)}(x_1, \dots, x_N) = \langle \Omega | T \phi(x_1) \cdots \phi(x_N) | \Omega \rangle.$$

For the free theory we have

$$Z_0[J] = N_0 \int \mathcal{D}\phi \exp \left(i \left\langle \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} (m^2 - i\epsilon) \phi^2 + J\phi \right\rangle \right).$$

As we will see, the generating functional $W[J]$, defined by

$$Z[J] = N e^{iW[J]},$$

will play a key role since it is the generating functional of connected Green's functions. We also set

$$Z_0[J] = Z_0[0] e^{iW_0[J]}.$$

In the following we will derive an expression for $Z[J]$ in terms of $Z_0[J]$. The procedure is similar to the one introduced in the case of the forced harmonic oscillator. Let us consider the Fourier transform

$$\tilde{J}(p) = \int \frac{d^4x}{(2\pi)^2} e^{-ipx} J(x),$$

$$J(x) = \int \frac{d^4p}{(2\pi)^2} e^{ipx} \tilde{J}(p),$$

$$\delta^{(4)}(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')}.$$

As in the case of the driven harmonic oscillator, we complete the square by considering the field shift

$$\tilde{\phi} \rightarrow \tilde{\phi}'(p) = \tilde{\phi}(p) + (p^2 - m^2 + i\epsilon)^{-1} \tilde{J}(p).$$

Since $\mathcal{D}\phi' = \mathcal{D}\phi$, we have

$$Z_0[J] = Z_0[0]e^{-\frac{i}{2}\langle J_1\Delta_{F_{12}}J_2\rangle}, \quad (7.16)$$

$\Delta_{F_{12}} \equiv \Delta_F(x_1 - x_2)$, where

$$\Delta_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon},$$

is the Feynman propagator. We then have

$$W_0[J] = -\frac{1}{2}\langle J_1\Delta_{F_{12}}J_2\rangle.$$

Note that

$$\frac{\delta Z_0[J]}{\delta J(x)} = -i\langle \Delta_F(x-y)J(y)\rangle_y Z_0[J],$$

and then

$$i\frac{\delta}{\delta J(x)} \log Z_0[J] = -\frac{\delta}{\delta J(x)} W_0[J] = \langle \Delta_F(x-y)J(y)\rangle_y.$$

In particular,

$$(\square + m^2)\Delta_F(x) = -\delta^{(4)}(x),$$

implies

$$(\square + m^2)\left(\frac{\delta}{\delta J(x)} W_0[J]\right) = J(x),$$

that is

$$\phi_{\text{cl}}^0(x) := \frac{\delta W_0[J]}{\delta J(x)} = \langle 0|\phi^0(x)|0\rangle_J,$$

satisfies the classical equation of motion of a free scalar field with external current. For the interacting case, we do the same and define

$$\phi_{\text{cl}}(x) := \frac{\delta W[J]}{\delta J(x)} = \langle \Omega|\phi(x)|\Omega\rangle_J.$$

However, as we will see, this field does not satisfy the classical equation of motion.

The Green functions of the free theory, $G_0^{(N)}$, are vanishing for N odd.⁵ Furthermore,

⁵ The reason is just the functional analogue of the relation $\int_{-a}^a dx f(x) = \int_{-a}^a dx f(-x)$, which holds for any integrable function, so that if f is odd, then $\int_{-a}^a dx f(x) = 0$.

we have

$$\begin{aligned} G_0^{(2)}(x_1, x_2) &= i\Delta_F(x - y) , \\ G_0^{(4)}(x_1, \dots, x_4) &= -(\Delta_F(x_1 - x_2)\Delta_F(x_3 - x_4) + \Delta_F(x_1 - x_3)\Delta_F(x_2 - x_4) \\ &\quad + \Delta_F(x_1 - x_4)\Delta_F(x_2 - x_3)) . \end{aligned}$$

The fact that the Green functions depend on the coordinate difference is a consequence of the translation invariance of the theory. We stress that $G_0^{(2)}$ is the building block for the higher order Green functions, that is they can always be expressed in terms of $G_0^{(2)}(x - y) = i\Delta_F(x - y)$. Furthermore, note that $G_0^{(4)}(x_1, \dots, x_4)$ is a sum of disconnected terms, that is each term in the summation is the product of functions. As we will discuss in considering the Linked Cluster Theorem, the functional $W[J]$, defined by

$$Z[J] = e^{iW[J]} ,$$

generates only the Green's functions that do not admit such a decomposition. Such functions have the same symbol of the Green's functions with the addition of the subscript c , that is $G_c^{(N)}(1, \dots, N)$. We have⁶

$$iW[J] = \sum_{N=1} \frac{i^N}{N!} \langle G_c^{(N)}(1, \dots, N) J_1 \cdots J_N \rangle .$$

The 2-point and 4-point functions have a useful Feynman pictorial representation

▷

$$G_0^{(2)}(x, y) : 1 \text{ --- } 2 \tag{7.17}$$

▷

$$G_0^{(4)}(x_1, \dots, x_4) : \begin{array}{c} 1 \text{ --- } 2 \\ 3 \text{ --- } 4 \end{array} + \begin{array}{c} 1 \text{ --- } 3 \\ 2 \text{ --- } 4 \end{array} + \begin{array}{c} 1 \text{ --- } 4 \\ 2 \text{ --- } 3 \end{array} \tag{7.18}$$

The observation made above about the translation invariance and conservation of the four momentum for the propagating particles, is one of the reasons why it is useful to work in momentum space. In order to define Green functions in momentum space, we observe that thanks to translational invariance, $G^{(N)}$ can be viewed as function of $N - 1$ variables, indeed $G^{(N)}(x_1, \dots, x_N) = G^{(N)}(x_1 - x_N, \dots, x_{N-1} - x_N, 0)$. We then set $x_N = 0$ and define

$$\tilde{G}^{(N)}(p_1, \dots, p_{N-1}) := \int d^4x_1 \dots d^4x_{N-1} e^{-i\sum_{k=1}^{N-1} p_k x_k} G^{(N)}(x_1, \dots, x_{N-1}, 0) .$$

⁶ Note that the summation starts from $N = 1$, in other words we assume $W[0] = 0$, corresponding to the normalisation $Z[0] = 1$.

Note that, always thanks to translational invariance, it does not make any difference which variable of $G^{(N)}$ is set equal to 0 before the integration. In the following, sometimes implicitly, we assume that $p_N = -\sum_{k=1}^{N-1} p_k$ and will use $\tilde{G}^{(N)}(p_1, \dots, p_N)$ to denote $\tilde{G}^{(N)}(p_1, \dots, p_{N-1})$. By making the change of variables $x_k \rightarrow x'_k = x_k - x_N$, $k = 1, \dots, N-1$, we get the following relation⁷

$$\begin{aligned} & \int d^4x_1 \dots d^4x_N e^{-i\sum_{k=1}^N p_k x_k} G^{(N)}(x_1, \dots, x_N) \\ &= \int d^4x_1 \dots d^4x_N e^{-i\sum_{k=1}^N p_k x_k} G^{(N)}(x_1, \dots, x_{N-1}, 0) \\ &= \int d^4x_N \int d^4x_1 \dots d^4x_{N-1} e^{-ip_N x_N} e^{-i\sum_{k=1}^{N-1} p_k (x_k + x_N)} G^{(N)}(x_1, \dots, x_{N-1}, 0) \\ &= \int d^4x_N e^{-i\sum_{k=1}^N p_k x_N} \int d^4x_1 \dots d^4x_{N-1} e^{-i\sum_{k=1}^{N-1} p_k x_k} G^{(N)}(x_1, \dots, x_{N-1}, 0) \\ &= (2\pi)^4 \delta^{(4)}\left(\sum_{k=1}^N p_k\right) \tilde{G}^{(N)}(p_1, \dots, p_N), \end{aligned}$$

which in most textbooks is used as the defining property of $\tilde{G}^{(N)}(p_1, \dots, p_N)$. The main benefit of the former definition is that it is a proper Fourier transform, and therefore one can use all the well-known properties, whereas in the latter this is not so evident.

Note that $\tilde{G}^{(N)}(p_1, \dots, p_N)$ is defined only for $\sum_k p_k = 0$. In the case $N = 2$ we use the notation $\tilde{G}_0^{(2)}(p) \equiv \tilde{G}_0^{(2)}(p, -p)$, so that

$$\tilde{G}_0^{(2)}(p) = \frac{i}{p^2 - m^2 + i\epsilon}.$$

Let us now consider the Euclidean formulation. We first list some formulas relating Minkowskian and Euclidean variables, denoted by a bar. We have

$$x^0 = -i\bar{x}^0.$$

One also sets $\bar{x}^k = \bar{x}_k = x^k = -x_k$, $k = 1, 2, 3$. This means that $g_{\mu\nu}$ is replaced by the Euclidean metric, that is the Kronecker $\delta_{\mu\nu}$ in \mathbb{R}^4 , $\delta_{\mu\nu} = \delta^{\mu\nu} = \delta_\mu^\nu = \delta^\mu_\nu$. Therefore,

$$\bar{x}^\mu = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) = (ix^0, x^1, x^2, x^3),$$

and

$$\bar{x}_\mu \bar{x}^\mu = \bar{x}^{0^2} + \mathbf{x} \cdot \mathbf{x} = -x^{0^2} + \mathbf{x} \cdot \mathbf{x} = -x^2.$$

⁷ Note that we are using the asymmetric normalisation of the Fourier transform, that is $\int d^4x \dots$ instead of $\int d^4x / (2\pi)^2 \dots$

Usually, to preserve the structure of the Fourier transform, the relation between the Minkowskian and Euclidean momenta differs by a minus sign with respect to the one of the space-time coordinate⁸

$$p_0 = i\bar{p}_0 .$$

Therefore, the Euclidean four-momentum reads

$$\bar{p}^\mu = (\bar{p}^0, \bar{p}^1, \bar{p}^2, \bar{p}^3) = (-ip^0, p^1, p^2, p^3) .$$

However, like in the case of spatial coordinates, even for the momentum we have

$$\bar{p}_\mu \bar{p}^\mu = \bar{p}^{0^2} + \mathbf{p} \cdot \mathbf{p} = -p^{0^2} + \mathbf{p} \cdot \mathbf{p} = -p^2 .$$

Also, note that

$$d^4x = -i d^4\bar{x} , \quad d^4p = i d^4\bar{p} .$$

Furthermore,

$$\frac{\partial}{\partial x_0} = i \frac{\partial}{\partial \bar{x}_0} , \quad \partial^2 = \frac{\partial^2}{\partial x_0^2} - \sum_{k=1}^3 \frac{\partial^2}{\partial x_k^2} = -\frac{\partial^2}{\partial \bar{x}_0^2} - \sum_{k=1}^3 \frac{\partial^2}{\partial \bar{x}_k^2} = -\bar{\partial}^2 .$$

As we said, the above transformation leaves the Fourier transform structure unchanged. More precisely, we have

$$p_\mu x^\mu = p^0 x^0 - \sum_{k=1}^3 p^k x^k = \bar{p}^0 \bar{x}^0 - \sum_{k=1}^3 \bar{p}^k \bar{x}^k \neq -\bar{p}_\mu \bar{x}^\mu . \quad (7.20)$$

⁸ Note that such a sign difference is also suggested by the fact that, how mentioned in (2.32) and (2.33), the four-momentum operator reads

$$p^\mu = i\partial^\mu := (i\partial^0, -i\nabla) . \quad (7.19)$$

Actually, the Euclidean transformation on p^0 is obtained by replacing x^0 by $-i\bar{x}^0$, so that

$$p^0 = i \frac{\partial}{\partial(-i\bar{x}^0)} = i^2 \frac{\partial}{\partial \bar{x}^0} = i\bar{p}^0 .$$

⁸ An interesting alternative is proposed in the online free version of [17], available at http://users.physik.fu-berlin.de/~kleinert/kleiner_reb6/psfiles/index.html, where in (14.202) and (14.203) at p.956 Kleinert sets

$$p_E := (p_4, \mathbf{p}) = (\omega, \mathbf{p}) , \quad x_E := (-\tau, \mathbf{x}) ,$$

in this way the Euclidean scalar product is the same as the Minkowskian one

$$p_E \cdot x_E = -\omega\tau + \mathbf{p} \cdot \mathbf{x} .$$

It is interesting to notice that also in this case one has a sort of time reversal.

Note that such a relation shows that one should be careful in considering the Euclidean version of scalar products. This must be done by replacing each four-vector component by the corresponding Euclidean version. For example, as shown by (7.20), the Euclidean version of $p_\mu x^\mu$ is not $-\bar{p}_\mu \bar{x}_\mu$.

Let us now write down the generating functional in the Euclidean space

$$Z_E[J] = N_E \int \mathcal{D}\phi \exp \left(- \left\langle \frac{1}{2} \bar{\partial}_\mu \phi \bar{\partial}_\mu \phi + \frac{1}{2} m^2 \phi^2 + V(\phi) - J\phi \right\rangle \right) . \quad (7.21)$$

The argument of the exponential in the integrand is now negative definite for positive m^2 and V when $J = 0$. One may easily check that the Euclidean Green functions $G_E^{(N)}(1, \dots, N)$ correspond to

$$\begin{aligned} G_E^{(N)}(1, \dots, N) &= Z[0]^{-1} \int \mathcal{D}\phi \phi(1) \cdots \phi(N) \exp \left(- \left\langle \frac{1}{2} \bar{\partial}_\mu \phi \bar{\partial}_\mu \phi + \frac{1}{2} m^2 \phi^2 + V(\phi) \right\rangle \right) \\ &= \langle \Omega | T \phi(1) \cdots \phi(N) | \Omega \rangle . \end{aligned} \quad (7.22)$$

To avoid possible sign errors, it is useful to keep in mind the overall $+$ sign of $\langle J\phi \rangle$ in the exponent of (7.21), so that each functional derivative of $Z_E[J]$ with respect to J has the effect of inserting ϕ to the numerator in the integrand of $Z_E[J]$. Therefore, since $G_E^{(N)}(1, \dots, N)$ corresponds to the N -point function $\langle \Omega | T \phi(1) \cdots \phi(N) | \Omega \rangle$, we have

$$G_E^{(N)}(1, \dots, N) = \frac{\delta^N Z_E[J]}{\delta J(1) \cdots \delta J(N)} \Big|_{J=0} .$$

So that we have

$$Z_E[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \langle G_E^{(N)}(1, \dots, N) J(1) \cdots J(N) \rangle . \quad (7.23)$$

As done in the Minkowski case, we introduce the generating functional for the connected Euclidean Green functions $W_E[J]$, defined by⁹

$$Z_E[J] = e^{-W_E[J]} , \quad (7.24)$$

so that

$$G_{cE}^{(N)}(1, \dots, N) = - \frac{\delta^N W_E[J]}{\delta J(1) \cdots \delta J(N)} \Big|_{J=0} .$$

Notice that the sign $-$ is due to the fact that $G_{cE}^{(N)}(1, \dots, N)$ is the connected part of

⁹ It is worth mentioning that the $-$ sign has been chosen because, as in the Minkowski case, the expression of $W_E[J]$ should correspond, in the classical limit, to the action evaluated on the classical solution.

$G_E^{(N)}(1, \dots, N)$, that should be compared with both the minus sign of $W_E[J]$ in (7.24) and with the overall + sign of $\langle J\phi \rangle$ in the exponent of (7.21).

Let us write down some useful formula in the Euclidean space. First, one may check that

$$W_{0E}[J] = -\frac{1}{2} \int d^4\bar{x} d^4\bar{y} J(\bar{x}) \Delta_{FE}(\bar{x} - \bar{y}) J(\bar{y}) ,$$

where

$$\Delta_{FE}(\bar{x}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{\exp[-i(\bar{p}_0\bar{x}_0 - \bar{p}_1\bar{x}_1 - \bar{p}_2\bar{x}_2 - \bar{p}_3\bar{x}_3)]}{\bar{p}^2 + m^2} , \quad (7.25)$$

is the Euclidean Feynman propagator. It is worth making some remarks

(i) We have the relation

$$\frac{d^4\bar{p}}{(2\pi)^4} \frac{\exp[-i(\bar{p}_0\bar{x}_0 - \mathbf{p} \cdot \mathbf{x})]}{\bar{p}^2 + m^2} = -i \frac{d^4p}{(2\pi)^4} \frac{\exp(-ip \cdot x)}{-p^2 - m^2} ,$$

where \bar{x}^0 and \bar{p}^0 are real, so that x^0 and p^0 are purely imaginary.

(ii) The Feynman propagator in the Minkowski and Euclidean spaces are different functions. In particular, Wick rotating the integration contour gives different values of the integral because the Jordan's lemma cannot be applied. In this respect, one should recall that changing contour integrals from the real to the imaginary axis, maps the argument of integration to purely imaginary values, but the equivalence of the two integral requires that the contributions one gets by closing the contours at infinities should vanish.

(iii) Note that the denominator in the integrand of the Feynman propagator is now positive definite for $m \neq 0$, so that the integrand is free of the singularities.

(iv) It is worth recalling that the Euclidean formulation is an intermediate step to perform the calculations, and then one should rotate back to the Minkowski space when computing the scattering amplitudes. We also note that

$$(-\bar{\partial}_\mu \bar{\partial}_\mu + m^2)_{\bar{x}} \Delta_{FE}(\bar{x} - \bar{y}) = \delta^{(4)}(\bar{x} - \bar{y}) ,$$

so that, in the Euclidean space, the two-point function coincides with the Feynman propagator¹⁰

$$G_{0E}^{(2)}(x - y) = \Delta_{FE}(x - y) ,$$

that, in momentum space, reads

$$\tilde{G}_{0E}^{(2)}(p) = \tilde{\Delta}_{FE}(p) = \frac{1}{p^2 + m^2} .$$

¹⁰ Recall that $G_0^{(2)}(x - y) = i\Delta_F(x - y)$.

- (v) Note that (7.20) is not $O(4)$ invariant. This rises the following question: how is it possible that the Fourier transform of an $O(4)$ invariant quantity, such as the Feynman propagator on the Euclidean, is itself $O(4)$ invariant? The answer is based on the relation

$$\int_{-a}^a dx f(x) = \int_{-a}^a dx f(-x) ,$$

showing that (7.25) is invariant if in the integrand \bar{p}_0 is replaced by $-\bar{p}_0$, i.e.

$$\Delta_{FE}(\bar{x}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{\exp[i(\bar{p}_0\bar{x}_0 + \bar{p}_1\bar{x}_1 + \bar{p}_2\bar{x}_2 + \bar{p}_3\bar{x}_3)]}{\bar{p}^2 + m^2} .$$

This means that in the case of the Feynman propagator, we can in fact safely make the replacement

$$\exp[-i(\bar{p}_0\bar{x}_0 - \mathbf{p} \cdot \mathbf{x})] \longrightarrow \exp(i\bar{p}_\mu\bar{x}_\mu) ,$$

that we will be understood in the following.

7.4 Effective action

In the previous section we have seen that

$$\phi_{\text{cl}}^0(x) := \frac{\delta W_0[J]}{\delta J(x)} = \frac{\int \mathcal{D}\phi \phi(x) e^{i(S_0 + \langle J\phi \rangle)}}{\int \mathcal{D}\phi e^{i(S_0 + \langle J\phi \rangle)}} ,$$

with S_0 the free action, satisfies the Klein-Gordon equation in the presence of an external source J .

One question that one can wonder about is what will happen if, instead of J , we consider ϕ_{cl} as an external field. In order to answer this question, we note, as a first step, that ϕ_{cl} defines the Legendre transform

$$\Gamma[\phi_{\text{cl}}] = W[J] - \int d^4x J(x)\phi_{\text{cl}}(x) , \quad (7.26)$$

that is $\phi_{\text{cl}}(x) = \delta W[J]/\delta J(x)$ is just the field that minimises $\Gamma[\phi] + \int d^4x J(x)\phi(x)$. To see this note that ϕ and J are independent, so that the field that minimises $\Gamma[\phi] + \int d^4x J(x)\phi(x)$ is the ϕ_{cl} such that

$$\left. \frac{\delta\Gamma[\phi]}{\delta\phi(x)} \right|_{\phi=\phi_{\text{cl}}} = -J(x) .$$

On the other hand, this implies

$$\frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x)} = -J(x) , \quad (7.27)$$

giving a functional relation between $\phi_{\text{cl}}(x)$ and $J(x)$. Then, taking the functional derivative of (7.26) with respect to $J(x)$ we get

$$\frac{\delta W[J]}{\delta J(x)} = \int d^4y \frac{\delta \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(y)} \frac{\delta \phi_{\text{cl}}(y)}{\delta J(x)} + \phi_{\text{cl}}(x) + \int d^4y J(y) \frac{\delta \phi_{\text{cl}}(y)}{\delta J(x)} ,$$

that, by (7.27), gives

$$\phi_{\text{cl}}(x) = \frac{\delta W[J]}{\delta J(x)} .$$

Free case. Using the equation of motion to eliminate the source J

$$\begin{aligned} \Gamma_0[\phi_{\text{cl}}^0] &= W_0[J] - \langle J \phi_{\text{cl}}^0 \rangle \\ &= -\frac{1}{2} \langle J \Delta_F J \rangle - \langle J \phi_{\text{cl}}^0 \rangle \\ &= -\frac{1}{2} \langle [(\square + m^2)\phi_{\text{cl}}^0]_1 \Delta_{F_{12}} [(\square + m^2)\phi_{\text{cl}}^0]_2 \rangle_{12} - \langle [(\square + m^2)\phi_{\text{cl}}^0] \phi_{\text{cl}}^0 \rangle \\ &= -\frac{1}{2} \langle \phi_{\text{cl}}^0 (\square + m^2) \phi_{\text{cl}}^0 \rangle , \end{aligned}$$

where we used $(\square + m^2)\Delta_F(x) = -\delta^{(4)}(x)$. This means that the effective action of the free theory coincides with the classical action S_0 ,

$$\Gamma_0[\phi_{\text{cl}}^0] = \frac{1}{2} \int d^4x (\partial_\mu \phi_{\text{cl}}^0 \partial^\mu \phi_{\text{cl}}^0 - m^2 \phi_{\text{cl}}^0{}^2) . \quad (7.28)$$

Interacting case. Let us consider the same steps for interacting theories

$$\phi_{\text{cl}}(x) \equiv -i \frac{\delta \log Z[J]}{\delta J(x)} = \frac{\delta W[J]}{\delta J(x)} = \frac{\int \mathcal{D}\phi \phi(x) e^{i(S+\langle J\phi \rangle)}}{\int \mathcal{D}\phi e^{i(S+\langle J\phi \rangle)}} . \quad (7.29)$$

Our goal is to find out which equation is satisfied by ϕ_{cl} . By

$$-i \frac{\delta}{\delta J(x)} e^{i\langle J\phi \rangle} = \phi(x) e^{i\langle J\phi \rangle} ,$$

we have

$$e^{-i\langle V(\phi) \rangle} e^{i\langle J\phi \rangle} = e^{-i\langle V(\frac{1}{i}\frac{\delta}{\delta J}) \rangle} e^{i\langle J\phi \rangle} ,$$

so that, following the Schwinger trick, we can extract the interaction away from the path integral

$$\begin{aligned} Z[J] &= \exp\{-i\langle V(-i\delta_J) \rangle\} N \int \mathcal{D}\phi \exp\left(i\left\langle \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}(m^2 - i\epsilon)\phi^2 + J\phi \right\rangle\right) \\ &= \exp\{-i\langle V(-i\delta_J) \rangle\} \frac{N}{N_0} Z_0[J] . \end{aligned} \quad (7.30)$$

Since $Z_0[J] = N_0 e^{iW_0} = N_0 e^{-\frac{i}{2}\langle J\Delta J\rangle}$, we have

$$\begin{aligned}\frac{\delta Z[J]}{\delta J(x)} &= -i \frac{N}{N_0} e^{-i\langle V(-i\delta J)\rangle} \langle \Delta_{Fx1} J_1 \rangle_1 Z_0[J] \\ &= -i e^{-i\langle V(-i\delta J)\rangle} \langle \Delta_{Fx1} J_1 \rangle_1 e^{i\langle V(-i\delta J)\rangle} Z[J],\end{aligned}\quad (7.31)$$

where in the last step we have used (7.30) to express $Z_0[J]$ in terms of $Z[J]$. Now note that, using

$$(\square + m^2)\Delta_F(y-x) = -\delta^{(4)}(y-x),$$

we have

$$(\partial_\mu \partial^\mu + m^2)\phi_{\text{cl}}(x) = Z^{-1}[J] e^{-i\langle V(-i\delta J)\rangle} J(x) e^{i\langle V(-i\delta J)\rangle} Z[J]. \quad (7.32)$$

At this point some mathematical manipulation is needed in order to go further with our quest to find the equations of motion for ϕ_{cl} .

7.5 The Schwinger-Dyson equation

Let us start by noticing that the Leibniz rule

$$\frac{d^k}{dx^k}(f(x)g(x)) = \sum_{j=0}^k \binom{k}{j} f^{(j)}(x) g^{(k-j)}(x),$$

implies the operator relation

$$\frac{d^k}{dx^k} f(x) = \sum_{j=0}^k \binom{k}{j} \left(\frac{d^j}{dx^j} f(x) \right) \frac{d^{k-j}}{dx^{k-j}},$$

that for $f(x) = x$ reduces to

$$\frac{d^k}{dx^k} x = x \frac{d^k}{dx^k} + k \frac{d^{k-1}}{dx^{k-1}},$$

that is, adding the index to x ,

$$\left[\frac{d^k}{dx_i^k}, x_j \right] = k \delta_{ij} \frac{d^{k-1}}{dx_i^{k-1}},$$

implies, in the functional case,

$$\left(-i \frac{\delta}{\delta J(y)} \right)^k J(x) = J(x) \left(-i \frac{\delta}{\delta J(y)} \right)^k - ik \delta^{(4)}(x-y) \left(-i \frac{\delta}{\delta J(y)} \right)^{k-1},$$

that is

$$\left[\left(-i \frac{\delta}{\delta J(y)} \right)^k, J(x) \right] = -ik\delta^{(4)}(x-y) \left(-i \frac{\delta}{\delta J(y)} \right)^{k-1}. \quad (7.33)$$

Expanding $V(\phi)$ in power series we then get

$$\left[V \left(-i \frac{\delta}{\delta J(y)} \right), J(x) \right] = -i\delta^{(4)}(x-y) V' \left(-i \frac{\delta}{\delta J(y)} \right). \quad (7.34)$$

Now note that this result and the identity

$$\begin{aligned} e^{-i\langle V(-i\delta_J) \rangle} J(x) e^{i\langle V(-i\delta_J) \rangle} - J(x) & \\ &= \int_0^1 d\lambda \frac{d}{d\lambda} e^{-i\lambda\langle V(-i\delta_J) \rangle} J(x) e^{i\lambda\langle V(-i\delta_J) \rangle} \\ &= \int_0^1 d\lambda e^{-i\lambda\langle V(-i\delta_J) \rangle} \left[-i \left\langle V \left(-i \frac{\delta}{\delta J} \right) \right\rangle, J(x) \right] e^{i\lambda\langle V(-i\delta_J) \rangle}, \end{aligned}$$

imply

$$\begin{aligned} e^{-i\langle V(-i\delta_J) \rangle} J(x) e^{i\langle V(-i\delta_J) \rangle} - J(x) &= - \int_0^1 d\lambda V' \left(-i \frac{\delta}{\delta J(x)} \right) \\ &= -V' \left(-i \frac{\delta}{\delta J(x)} \right), \end{aligned} \quad (7.35)$$

so that, by (7.32), we get the Schwinger-Dyson equation

$$(\partial_\mu \partial^\mu + m^2) \phi_{\text{cl}}(x) = J(x) - Z^{-1}[J] V' \left(-i \frac{\delta}{\delta J(x)} \right) Z[J]. \quad (7.36)$$

This equation can be interpreted as a quantum deformation of the classical Klein-Gordon equation in the presence of an external source. We will show that this equation can be equally well derived assuming that the integral of a total functional derivative vanishes

$$\int \mathcal{D}\phi \frac{\delta F[\phi]}{\delta \phi} = 0.$$

It is clear that the validity of such a claim depends on the structure of the “space of the ϕ ’s”, on which the integration is carried out, and also on the properties of the functional $F[\phi]$. We assume such good properties, so that

$$\int \mathcal{D}\phi \frac{\delta}{\delta \phi(x)} e^{i(S+\langle J\phi \rangle)} = i \int \mathcal{D}\phi \left(\frac{\delta S}{\delta \phi(x)} + J(x) \right) e^{i(S+\langle J\phi \rangle)} = 0, \quad (7.37)$$

which is equivalent to

$$\langle \Omega | \frac{\delta S}{\delta \phi(x)} + J(x) | \Omega \rangle_J = 0. \quad (7.38)$$

This means that the expectation value of the equations of motion, which at the functional level are identical to the classical ones, is zero. This statement can be interpreted as a field theoretic version of Ehrenfest's theorem.

The equivalence between (7.36) and (7.37) follows by

$$\frac{\delta S}{\delta \phi(x)} = -\partial_\mu \partial^\mu \phi(x) - m^2 \phi(x) - V'(\phi(x)) ,$$

so that (7.37) becomes

$$\int \mathcal{D}\phi (\partial_\mu \partial^\mu + m^2) \phi(x) e^{i(S+(J\phi))} - J(x) Z[J] + V' \left(-i \frac{\delta}{\delta J(x)} \right) Z[J] = 0 ,$$

that, dividing by $Z[J]$, coincides with (7.36).

Observe that the functional derivatives with respect to J , computed at $J = 0$, correspond to relations between ϕ correlators. To prove this, let us develop an additional way of obtaining the Schwinger-Dyson equation, which naturally extends to the case of many scalar fields ϕ_a , $a = 1, \dots, N$. The first step consists in recognising that a change of notation has no effect

$$\int \prod_1^N \mathcal{D}\phi_a(x) F[\{\phi\}] = \int \prod_1^N \mathcal{D}\phi'_a(x) F[\{\phi'\}] ,$$

where $\{\phi\} := \phi_1, \dots, \phi_N$. With this in mind, define

$$\phi'_a(x) = \phi_a(x) + \delta\phi_a(x) .$$

This is just a translation in the space of functions by a constant function $\delta\phi_a$. To see this remember that $\mathcal{D}\phi_a(x)$ can be interpreted as the generalisation of $\prod_i dq^i(t)$, where \mathbf{x} is the continuum extension of the index i . As for the case of $q^i(t)$, the integration is carried out for every instant t . Thus,

$$\mathcal{D}\phi'_a(x) = \mathcal{D}\phi_a(x) .$$

It follows that

$$\int \prod_1^N \mathcal{D}\phi_a(x) F[\{\phi\}] = \int \prod_1^N \mathcal{D}\phi_a(x) F[\{\phi'\}] = \int \prod_1^N \mathcal{D}\phi_a(x) F[\{\phi + \delta\phi\}] ,$$

so that, by

$$F[\{\phi + \delta\phi\}] = F[\{\phi\}] + \int d^4x \sum_{a=1}^N \frac{\delta F[\{\phi\}]}{\delta \phi_a(x)} \delta\phi_a(x) ,$$

one gets

$$\int \prod_1^N \mathcal{D}\phi_a(x) \int d^4x \left(\frac{\delta S}{\delta \phi_a(x)} + J_a(x) \right) \delta \phi_a(x) e^{i(S + \langle J_a \phi_a \rangle)} = 0 .$$

The n -th order derivatives with respect to J_a , computed at $J_a = 0$, give

$$\begin{aligned} \int \prod_1^N \mathcal{D}\phi_a(x) \int d^4x \left(i \frac{\delta S}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \right. \\ \left. + \sum_{k=1}^n \phi_{a_1}(x_1) \cdots \delta_{aa_k} \delta^{(4)}(x - x_k) \cdots \phi_{a_n}(x_n) \right) \delta \phi_a(x) e^{iS} = 0 . \end{aligned} \quad (7.39)$$

Since $\delta \phi_a$ is arbitrary, the previous relation implies that the expression integrated over x vanishes everywhere, that is

$$\begin{aligned} i \langle \Omega | T \frac{\delta S}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) | \Omega \rangle \\ + \langle \Omega | T \sum_{k=1}^n \phi_{a_1}(x_1) \cdots \delta_{aa_k} \delta^{(4)}(x - x_k) \cdots \phi_{a_n}(x_n) | \Omega \rangle = 0 . \end{aligned} \quad (7.40)$$

Excluding points at which contact terms show up, this implies

$$\langle \Omega | T \frac{\delta S}{\delta \phi_a(x)} \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) | \Omega \rangle = 0 , \quad \text{for } x \neq x_1, \dots, x_n .$$

For further information see section 9.6 of [11] and the discussion at [this].

As an example, we consider the Schwinger-Dyson equation (7.36) in the case $V = \frac{\lambda}{4!} \phi^4$. By

$$\begin{aligned} \frac{1}{Z[J]} V' \left(-i \frac{\delta}{\delta J(x)} \right) Z[J] &= \frac{\lambda}{3!} (-i)^3 \frac{1}{Z[J]} \frac{\delta^3}{\delta J^3(x)} Z[J] \\ &= \frac{\lambda}{3!} \left(\phi_{\text{cl}}^3(x) - \frac{\delta^2 \phi_{\text{cl}}(x)}{\delta J^2(x)} - 3i \phi_{\text{cl}}(x) \frac{\delta \phi_{\text{cl}}(x)}{\delta J(x)} \right) , \end{aligned} \quad (7.41)$$

we have

$$(\square_x + m^2) \phi_{\text{cl}}(x) = J(x) - \frac{\lambda}{3!} \phi_{\text{cl}}^3(x) + \frac{\lambda}{3!} \frac{\delta^2 \phi_{\text{cl}}(x)}{\delta J^2(x)} + i \frac{\lambda}{4} \frac{\delta \phi_{\text{cl}}^2(x)}{\delta J(x)} . \quad (7.42)$$

Note that the last two terms in the right-hand side are quantum corrections. This result

can be used to investigate the structure of the effective action

$$\Gamma[\phi_{\text{cl}}] = \int d^4x \left(-V^{\text{eff}}(\phi_{\text{cl}}) + \frac{1}{2}F(\phi_{\text{cl}})\partial_\mu\phi_{\text{cl}}\partial^\mu\phi_{\text{cl}} + \text{higher order derivatives} \right) . \quad (7.43)$$

In particular, note that the equation of motion (7.42) must be equivalent to the one given in (7.27), that is

$$\frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x)} = -J(x) . \quad (7.44)$$

The difference is that the functional derivative of $\Gamma[\phi_{\text{cl}}]$ is expressed only in terms of $\phi_{\text{cl}}(x)$ and its derivatives with respect to y , whereas (7.42) also contains functional derivatives of $\phi_{\text{cl}}(x)$ with respect to J . Nevertheless, there is a useful relation. Namely, by (7.42) and (7.44) we have

$$\frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x)} = -(\square_x + m^2)\phi_{\text{cl}}(x) - \frac{\lambda}{3!}\phi_{\text{cl}}^3(x) + \frac{\lambda}{3!}\frac{\delta^2\phi_{\text{cl}}(x)}{\delta J^2(x)} + i\frac{\lambda}{4}\frac{\delta\phi_{\text{cl}}^2(x)}{\delta J(x)} . \quad (7.45)$$

Since the last two terms are quantum corrections, it follows that the classical part of the effective action coincides with the classical action, that is

$$V^{\text{eff}}(\phi_{\text{cl}}) = \frac{m^2}{2}\phi_{\text{cl}}^2 + \frac{\lambda}{4!}\phi_{\text{cl}}^4 + \mathcal{O}(\hbar) , \\ F(\phi_{\text{cl}}) = 1 + \mathcal{O}(\hbar) .$$

Alternatively, one can consider the expansion, holding for any potential density,¹¹

$$\Gamma[\phi_{\text{cl}}] = \sum_{N=1} \frac{1}{N!} \langle \Gamma^{(N)}(1, \dots, N) \phi_{\text{cl}}(1) \cdots \phi_{\text{cl}}(N) \rangle_{1, \dots, N} ,$$

where

$$\Gamma^{(N)}(1, \dots, N) = \frac{\delta^N \Gamma[\phi]}{\delta\phi(1) \cdots \delta\phi(N)} \Big|_{\phi=0} . \quad (7.46)$$

It is also useful to introduce the Fourier transform of $\Gamma^{(N)}(1, \dots, N)$, that, as in the case of the Green's functions, is defined up to a global δ -function to take into account the momentum conservation

$$(2\pi)^4 \delta^{(4)} \left(\sum_{k=1}^N p_k \right) \tilde{\Gamma}^{(N)}(p_1, \dots, p_N) = \int d^4x_1 \cdots d^4x_N e^{-i\sum_{k=1}^N p_k x_k} \Gamma^{(N)}(x_1, \dots, x_N) .$$

As we will see the $\tilde{\Gamma}^{(N)}(p_1, \dots, p_N)$'s functions and, of course, the $\Gamma^{(N)}(x_1, \dots, x_N)$'s,

¹¹ Note that by (7.43) we have $\Gamma[0] = 0$, so that the summation starts from $N = 1$. This is a consistent with the fact that the effective action is the Legendre transform of $W[J]$, whose expansion in power series of J starts from $N = 1$.

have several important properties and can be considered the building blocks of the perturbative expansion.

Let us list some definitions concerning the $\tilde{\Gamma}^{(N)}(p_1, \dots, p_N)$'s functions that also anticipate some of their properties.

- (i) A connected Feynman diagram that can be disconnected by cutting an internal line is said *one-particle reducible* (1PR). The internal lines having such a property are said *cutlines*.
- (ii) A connected Feynman diagram that cannot be disconnected by cutting an internal line is said *one-particle irreducible* (1PI).
- (iii) A Feynman diagram with the external exact propagators removed is said to be amputated or truncated.¹²
- (iv) As we will see in proving the Jona-Lasinio theorem, the $\tilde{\Gamma}^{(N)}$'s functions correspond to amputated 1PI Feynman diagrams. For $N \geq 3$, $\tilde{\Gamma}^{(N)}$ is said *proper vertex function*.

We conclude this section by observing that the generating functional admits other representations. Let us define

$$\phi_c(x) := \int d^D y J(y) \Delta_F(y - x) ,$$

with Δ_F^{-1} the inverse of the Feynman propagator, that is

$$\delta^{(d)}(x - y) = \int d^D z \Delta_F^{-1}(x - z) \Delta_F(z - y) ,$$

$$\Delta_F^{-1}(x - y) = (-\partial_x^2 + m^2) \delta^{(D)}(x - y) = \int \frac{d^D p}{(2\pi)^D} (p^2 + m^2 + i\epsilon) e^{ip(x-y)} .$$

It turns out that [29]

$$Z[J] = \frac{N}{N_0} \exp \left\langle \frac{1}{2} \phi_c \Delta_F^{-1} \phi_c \right\rangle \exp \left\langle \frac{1}{2} \frac{\delta}{\delta \phi_c} \Delta_F \frac{\delta}{\delta \phi_c} \right\rangle \exp \left(- \int d^D x V(\phi_c(x)) \right) ,$$

whose derivation is strictly related to the Wick's theorem and to the removal of normal ordering singularities introduced in [30].

¹²It is worth stressing that the removal of the exact external propagators corresponds to remove the corresponding contributions in perturbation theory. The important point is that the external propagators are replaced by the corresponding legs of the vertexes. We also note that, sometimes in literature, by 1PI diagrams it is meant the ones that, in addition, are amputated.

Another useful representation of $Z[J]$ is [29]

$$Z[J] = \frac{N}{N_0} \exp \langle -U_0[\phi_c] \rangle \exp \left(- \int d^D x V(\mathcal{D}_{\phi_c}^-) \right) \cdot 1 ,$$

where

$$U_0[\phi_c] := -\frac{1}{2} \langle \phi_c \Delta_F^{-1} \phi_c \rangle ,$$

and

$$\mathcal{D}_{\phi}^{\pm}(x) := \mp \int d^D y \Delta_F(y-x) \frac{\delta}{\delta \phi(y)} + \phi(x) .$$

7.6 $W[J]$ as generating functional of connected Green functions¹³

We saw that the Green functions¹⁴

$$G^{(N)}(x_1, \dots, x_N) = \langle \Omega | T \phi(x_1) \dots \phi(x_N) | \Omega \rangle ,$$

correspond to a sum of terms. It happens that the dependence on x_1, \dots, x_N of such terms can be expressed in a factorised form, that is as a product of functions, each one depending on a subset of x_1, \dots, x_N . Green functions that do not admit such a factorisation are called connected Green functions, and denoted by $G_c^{(N)}(x_1, \dots, x_N)$. In this section we prove the so-called Linked Cluster Theorem, stating that $W[J]$ is the generating functional of connected Green functions, that is

$$i \frac{\delta^N W[J]}{\delta J(x_1) \dots \delta J(x_N)} = i^N G_c^{(N)}(x_1, \dots, x_N) . \quad (7.47)$$

The N -point Green function is given by a summation of terms, each one corresponding to all possible nonequivalent products of connected Green functions. These correspond to $G_c^{(N)}(x_1, \dots, x_N)$, plus the summation of all the disconnected contributions factorised on a certain number of copies of the connected contributions. More precisely, we have

$$G^{(N)}(x_1, \dots, x_N) = \sum_{\{\sigma_k\}_N} \sum_{\mathcal{P}} \mathcal{P} \left(\underbrace{[G_c^{(1)}(\cdot) \dots G_c^{(1)}(\cdot)]}_{\sigma_1} \underbrace{[G_c^{(2)}(\cdot) \dots G_c^{(2)}(\cdot)]}_{\sigma_2} \dots \right) , \quad (7.48)$$

where $\{\sigma_k\}_N$ is the set of all the possible natural numbers σ_k such that $\sum_{k=1}^N k \sigma_k = N$, while \mathcal{P} are the permutations of the coordinates x_1, \dots, x_N corresponding to nonequivalent configurations of products of connected Green functions. There are two kinds of coordinate permutations that leave the product of connected Green's functions invariant.

¹³ Umberto Natale

¹⁴ We assume the normalisation $\langle \Omega | \Omega \rangle = 1$, otherwise the right-hand side should be divided by $\langle \Omega | \Omega \rangle$.

The first one consists of $\sigma_k!$ permutations changing only the ordering of the factors in the product: an example is $x_1x_2x_3x_4 \rightarrow x_3x_4x_1x_2$, that transforms $G_c^{(2)}(x_1, x_2)G_c^{(2)}(x_3, x_4)$ in the equivalent $G_c^{(2)}(x_3, x_4)G_c^{(2)}(x_1, x_2)$. The second kind is due to the fact that all the $G_c^{(k)}$'s are completely symmetric, that is their value is invariant under the $k!$ permutations of their k arguments. It follows that the total number, $\#\mathcal{P}$, of nonequivalent permutations of the coordinates is

$$\frac{N!}{(\sigma_1! \cdots \sigma_N!)(1!)^{\sigma_1} \cdots (N!)^{\sigma_N}} .$$

Now note that inserting (7.48) in the expression for the generating functional

$$Z[J] = e^{iW[J]} = \sum_{N=0}^{\infty} \frac{i^N}{N!} \langle G^{(N)}(x_1, \dots, x_N) J(x_1) \cdots J(x_N) \rangle , \quad (7.49)$$

one sees that, due to the integration over x_1, \dots, x_N , all the permutations \mathcal{P} in (7.48) give the same contribution to (7.49). Therefore, after substituting the $G^{(N)}$'s in (7.49) with the right-hand side of (7.48), the summation over \mathcal{P} can be replaced by a single representative multiplied by a weight, corresponding to the number of permutations of x_1, \dots, x_N that give inequivalent products of connected Green functions.

The above analysis implies that the expression one obtains by substituting (7.48) in (7.49) can be rewritten in the form

$$Z[J] = \sum_{N=0}^{\infty} \sum_{\{\sigma_k\}_N} i^N \frac{(\int d^4x G_c^{(1)}(x) J(x))^{\sigma_1}}{\sigma_1!(1!)^{\sigma_1}} \frac{(\int d^4x \int d^4y G_c^{(2)}(x, y) J(x) J(y))^{\sigma_2}}{\sigma_2!(2!)^{\sigma_2}} \cdots . \quad (7.50)$$

Using the condition $\sum_{k=1}^N k\sigma_k = N$ to write i^N as a product of factors $i^{k\sigma_k}$, one for each integral, and observing that

$$\sum_{N=0}^{\infty} \sum_{\{\sigma_k\}_N} = \sum_{\sigma_1=0}^{\infty} \sum_{\sigma_2=0}^{\infty} \cdots ,$$

one sees that (7.50) is equivalent to

$$Z[J] = \sum_{\sigma_1=0}^{\infty} \frac{1}{\sigma_1!} \left[\frac{i^1}{1!} \langle G_c^{(1)}(x) J(x) \rangle \right]^{\sigma_1} \sum_{\sigma_2=0}^{\infty} \frac{1}{\sigma_2!} \left[\frac{i^2}{2!} \langle G_c^{(2)}(x, y) J(x) J(y) \rangle \right]^{\sigma_2} \cdots .$$

Each one of the above series is the expansion of an exponential, thus

$$\begin{aligned} Z[J] &= \exp \left\{ \frac{i}{1!} \langle G_c^{(1)}(x) J(x) \rangle \right\} \exp \left\{ \frac{i^2}{2!} \langle G_c^{(2)}(x, y) J(x) J(y) \rangle \right\} \cdots \\ &= \exp \left\{ \sum_{N=1}^{\infty} \frac{i^N}{N!} \langle G_c^{(N)}(x_1, \dots, x_N) J(x_1) \cdots J(x_N) \rangle \right\}, \end{aligned}$$

that is

$$iW[J] = \sum_{N=1}^{\infty} \frac{i^N}{N!} \langle G_c^{(N)}(x_1, \dots, x_N) J(x_1) \cdots J(x_N) \rangle,$$

which is equivalent to (7.47).

7.7 A note on the connected Green functions

In D dimension the generating functional of a scalar theory is

$$Z[J] = e^{iW[J]} = N \int \mathcal{D}\phi \exp \left[i \int d^D x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) + J\phi \right) \right]. \quad (7.51)$$

Rewrite (7.47) in the form¹⁵

$$\frac{1}{i^{N-1}} \frac{\delta^N W[J]}{\delta J(x_1) \cdots \delta J(x_N)} \Big|_{J=0} = \langle \Omega | T \phi(x_1) \cdots \phi(x_N) | \Omega \rangle_c. \quad (7.52)$$

If an arbitrary function f is added to the field ϕ in the external source term $\langle J\phi \rangle$ of (7.51), that is

$$J\phi \longrightarrow J(\phi + f), \quad (7.53)$$

then

$$\frac{1}{i^N} \frac{\delta^N Z[J]}{\delta J(x_1) \cdots \delta J(x_N)} = \frac{\langle \Omega | T(\phi(x_1) + f(x_1)) \cdots (\phi(x_N) + f(x_N)) | \Omega \rangle_J}{\langle \Omega | \Omega \rangle_J}. \quad (7.54)$$

Similarly, the right-hand side of (7.52) is substituted by the corresponding correlator of $\phi + f$. On the other hand, note that the transformation (7.53) is equivalent to the substitution

$$W[J] \longrightarrow W[J] + \int d^D x J(x) f(x),$$

thus, for $N \geq 2$,

$$\frac{\delta^N (W[J] + \int d^D x J(x) f(x))}{\delta J(x_1) \cdots \delta J(x_N)} = \frac{\delta^N W[J]}{\delta J(x_1) \cdots \delta J(x_N)}. \quad (7.55)$$

¹⁵ The factor $(-1)^N$ in (3.2.15) of Ramond book must be substituted by i^N .

This implies that, for $N \geq 2$, the connected Green functions of ϕ and $\phi + f$ coincide

$$\langle \Omega | T \phi(x_1) \dots \phi(x_N) | \Omega \rangle_c = \langle \Omega | T (\phi(x_1) + f(x_1)) \dots (\phi(x_N) + f(x_N)) | \Omega \rangle_c, \quad (7.56)$$

even when $\langle \Omega | \phi(x) | \Omega \rangle$ depends on x .

Consider further the case in which the vacuum expectation value of $\phi(x)$ is different from zero

$$v(x) := \langle \Omega | \phi(x) | \Omega \rangle \neq 0.$$

This means that the action of $\phi(x)$ on the vacuum contains also the vacuum itself, that is

$$\phi(x) | \Omega \rangle = v(x) | \Omega \rangle + \dots$$

On the other hand, as discussed in the case of the Källén-Lehmann representation, $\phi(x) | \Omega \rangle$ should contain only a one-particle state, so that $\langle \Omega | \phi(x) | \Omega \rangle$ should vanish. This can be achieved by considering $\eta(x)$ as the elementary field, by setting

$$\phi(x) = \eta(x) + v(x),$$

so that

$$\langle \Omega | \eta(x) | \Omega \rangle = 0.$$

Note that, written in terms of η the Lagrangian density in (7.51) is¹⁶

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 (\phi)^2 - V(\phi) \\ &= \frac{1}{2} \partial_\mu (\eta + v) \partial^\mu (\eta + v) - \frac{1}{2} m^2 (\eta + v)^2 - V(\eta + v). \end{aligned} \quad (7.57)$$

Since η is now seen as the new field, it follows that the coupling with $J(x)$ should be $\langle J \eta \rangle$. If the vacuum is translation invariant, i.e. $P_\mu | \Omega \rangle = 0$, then v is a constant and (7.57) reduces to

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 (\phi)^2 - V(\phi) \\ &= \frac{1}{2} \partial_\mu \eta \partial^\mu \eta - \frac{1}{2} m^2 (\eta + v)^2 - V(\eta + v). \end{aligned} \quad (7.58)$$

The above shift of $\phi(x)$ is one of the steps, in the Standard Model Lagrangian, of the Higgs mechanism, giving mass to gauge fields.

¹⁶ Lagrangian densities depending on c -number fields, such as $v(x)$, are not frequently investigated in literature.

7.8 Wick rotation

Let us consider the Wightman function¹⁷

$$W^{(2)}(x - y) = \langle \Omega | \phi(x) \phi(y) | \Omega \rangle .$$

We introduce the analytic function S of $z = x^4 + ix^0$, restricted to the right half-plane $x^4 > 0$, such that

$$W^{(2)}(x^0, \mathbf{x}) = \lim_{x^4 \downarrow 0} S(\mathbf{x}, x^4 + ix^0) .$$

The pointwise limit does not exist, whereas the limit exists when $S(\mathbf{x}, x^4 + ix^0)$ is seen as a distribution. This is one of the reasons why in the axiomatic formulation particular attention is paid to the fact that we are treating with distributions rather than fields. For example, the same free fields are singular quantities. For this reason we use the “smeared fields”

$$\phi(f) = \int d^4x f(x) \phi(x) ,$$

where $f(x)$ is a test function, usually belonging to the Schwarz space. Unlike $W(x^0, \mathbf{x})$, $S(\mathbf{x}, x^4 + ix^0)$ is analytic: all the points in the semiaxis $z = x^4 > 0$ are in its analyticity domain. So one can calculate S on this domain. For this reason we set $x^0 = 0$ and define

$$S(x) := S(\mathbf{x}, x^4) .$$

In the case of non-interacting theory, the Wightman function is

$$W^{(2)}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega_{\mathbf{p}}} e^{i(\mathbf{p}\mathbf{x} - \omega_{\mathbf{p}}x^0)} ,$$

$\omega_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. Therefore, we have

$$S(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega_{\mathbf{p}}} e^{i\mathbf{p}\mathbf{x} - \omega_{\mathbf{p}}x^4} ,$$

which holds only for $x^4 > 0$. Notice that

$$\frac{1}{2\omega_{\mathbf{p}}} e^{-\omega_{\mathbf{p}}x^4} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp^4 \frac{e^{ip^4x^4}}{p^2 + m^2} , \quad x^4 > 0 ,$$

¹⁷The following is taken from [31] which provides an excellent analysis of the analytic continuation. In particular, see the diagram on p. 220. A rigorous and clear book where the analytic continuation is discussed is [32]. In particular, see from p. 640 to p. 647. In the article [33] there are three ways for the calculation of the Feynman propagator. Explicit expressions of various propagators can be found at <https://en.wikipedia.org/wiki/Propagator>.

with $p = (p^1, p^2, p^3, p^4)$ and $p^2 = \sum_1^4 (p^k)^2$, from which it follows

$$S(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ipx}}{p^2 + m^2} = \begin{cases} (2\pi)^{-2} m |x|^{-1} K_1(m|x|), & \text{if } m > 0, \\ (2\pi)^{-2} |x|^{-2}, & \text{if } m = 0, \end{cases}$$

where $px = \sum_1^4 p^k x^k$, $|x| = \sqrt{x^2}$ and K_1 is the modified Bessel function. There is a significant similarity between the Schwinger function, which is the two point function in the Euclidean space, and the Feynman propagator

$$\Delta_F(x) = \lim_{\epsilon \downarrow 0} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m^2 + i\epsilon}.$$

In fact, the squared Minkowskian momentum p^2 becomes, under the Wick rotation, $-p^2$. More precisely, setting

$$p^0 + ip^4 = re^{i\alpha}, \quad (p^0)^2 + i0^+ = [p^0(1 + i0^+)]^2 = (p^0 e^{i0^+})^2,$$

we can see that the Wick rotation corresponds to the variation of the angle from 0^+ to $\pi/2$, so that

$$(p^0 e^{i0^+})^2 \rightarrow (p^0 e^{i\pi/2})^2 = -(p^0)^2.$$

Therefore, we have that the Fourier transforms $\Delta_F(p)$ and $-S(p) = -(p^2 + m^2)^{-1}$ are connected through analytic continuation with respect to the complex variable $w = p^0 + ip^4$. Since in the free case all n -points functions are built in terms of the two-point function, it follows that, at least in that case, there is a strict relation between Wightman functions and Schwinger functions.

The Wightman and Schwinger functions in coordinate space are connected through analytic continuation, but only for $x^4 > 0$, that is

$$\begin{aligned} W^{(2)}(x) &= \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\omega_{\mathbf{p}}} e^{i(\mathbf{p}\mathbf{x} - \omega_{\mathbf{p}}x^0)} \\ &= \lim_{x^4 \downarrow 0} S(\mathbf{x}, x^4 + ix^0) \\ &= \lim_{x^4 \downarrow 0} \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\omega_{\mathbf{p}}} e^{i\mathbf{p}\mathbf{x} - \omega_{\mathbf{p}}(x^4 + ix^0)}. \end{aligned} \quad (7.59)$$

Notice that the definition of $S(x)$, differently from the definition of $S(\mathbf{x}, x^4 + ix^0)$, can be extended to the whole axis x^4 . The only singularity is at the origin where $S(x)$ goes like $1/x^2$, singularity that is still integrable.

Sometimes in the literature are neglected some singularities when considering analytic continuations. Let us consider the forced harmonic oscillator. We saw in (6.70) that this provides the first example of generating functional for a one-dimensional quantum

field theory

$$Z[F] = \exp \left[-\frac{i}{2} \langle F(t_1) D(t_1 - t_2) F(t_2) \rangle \right] ,$$

where

$$D(t) = \lim_{\epsilon \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \frac{e^{-itE}}{E^2 - \omega^2 + i\epsilon} = \frac{1}{2i\omega} (\theta(t)e^{-i\omega t} + \theta(-t)e^{i\omega t}) . \quad (7.60)$$

It is sometimes stated that, even in this case, it is possible to change the integration contour from the real axis to the imaginary axis of the complex plane E . This would follow from the Cauchy theorem in that, by rotating of an angle $\pi/2$ the contour of integration counterclockwise, one does not encounter singularities. On the other hand, to rotate the contour of integration it is necessary to close the contours indefinitely such that there are no singularities and the Jordan lemma is applicable. Thus, a rotated integration contour gives the same value of the integral. In the case in question, it would be necessary to close with a quarter-circumference both in the first and third quadrants. But this is not possible because e^{-itE} , whose asymptotic behaviour depends also on t , cannot go to zero because of $\text{Im}E > 0$ and of $\text{Im}E < 0$. We conclude noticing that the fact that the rotation of the contour of integration on the complex plane of the time variable has the opposite sense with respect to the rotation of the contour of integration on the complex plane of the energy, can be also deduced from the heuristic relation¹⁸

$$p_0 \sim i\hbar \frac{\partial}{\partial x_0} = i\hbar \frac{\partial}{\partial \bar{x}_0} \sim i\bar{p}_0 .$$

7.9 $\tilde{\Gamma}_E^{(2)}(p) \tilde{G}_{cE}^{(2)}(p) = 1$

In this section we show that the two-point 1PI function

$$\Gamma_E^{(2)}(\bar{x}_1, \bar{x}_2) = \left. \frac{\delta^2 \Gamma_E[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(\bar{x}_1) \delta \phi_{\text{cl}}(\bar{x}_2)} \right|_{\phi_{\text{cl}}=0} ,$$

is the inverse of the connected two-point function

$$G_{cE}^{(2)}(\bar{x}_1, \bar{x}_2) = - \left. \frac{\delta^2 W_E[J]}{\delta J(\bar{x}_1) \delta J(\bar{x}_2)} \right|_{J=0} .$$

¹⁸ In quantum mechanics

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} ,$$

corresponds to the expression of the momentum in coordinate space, where $\hat{x} = x$. However, t , differently from x , is a parameter. In quantum field theory all the four coordinates are parameters.

The effective Euclidean action is

$$\Gamma_E[\phi_{\text{cl}}] = W_E[J] - \int d^4\bar{x} J(\bar{x}) \frac{\delta W_E[J]}{\delta J(\bar{x})}, \quad (7.61)$$

where

$$\phi_{\text{cl}}(\bar{x}) := -\frac{\delta W_E[J]}{\delta J(\bar{x})}. \quad (7.62)$$

In the following we show that the choice of the sign of $\Gamma_E[\phi_{\text{cl}}]$ in (7.61) is the right one. This also offers the opportunity to comment on some signs in the text of Ramond. In this regard, notice that, consistently with (3.4.7) and (3.4.32) in the Ramond book, $J(\bar{x})$ in Eq.(3.4.31) must be substituted with $-J(\bar{x})$. Apparently, even the effective action in Eq.(3.4.33) should have the opposite sign. However, (3.4.33) follows from (3.4.31) using (3.3.9)

$$J(x) = -\frac{\delta\Gamma[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(x)}. \quad (7.63)$$

On the other hand, this relationship is valid in the Minkowskian. As in the case of the definition of ϕ_{cl} , which changes sign on the Euclidean ($\phi_{\text{cl}} = \delta_J W[J]$ in the Minkowskian and $\phi_{\text{cl}} = -\delta_J W_E[J]$ in the Euclidean), consistently with $\phi_{\text{cl}} = \langle\Omega|\phi|\Omega\rangle_J$, one expects that in the Euclidean

$$J(\bar{x}) = \frac{\delta\Gamma_E[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(\bar{x})}. \quad (7.64)$$

To see that (7.64) is the right choice notice that in the classical limit the effective action must correspond to the classical action calculated on the classical solution, both in the Minkowskian and in the Euclidean. From this it follows that the right choice is indeed (7.64). In fact, the (7.64), and not the (3.3.9) in the Ramond book, together with (3.4.31) with the opposite sign, imply (3.4.33).

It is straightforward to verify that (7.61) implies (7.64).

$$\frac{\delta\Gamma_E[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(\bar{x})} = \int d^4\bar{z} \frac{\delta W_E[J]}{\delta J(\bar{z})} \frac{\delta J(\bar{z})}{\delta\phi_{\text{cl}}(\bar{x})} + J(\bar{x}) - \int d^4\bar{z} \frac{\delta W_E[J]}{\delta J(\bar{z})} \frac{\delta J(\bar{z})}{\delta\phi_{\text{cl}}(\bar{x})} = J(\bar{x}).$$

By deriving (7.62) with respect to $\phi_{\text{cl}}(\bar{y})$ one has

$$\delta^{(4)}(\bar{x} - \bar{y}) = - \int d^4\bar{z} \frac{\delta^2 W_E[J]}{\delta J(\bar{x}) \delta J(\bar{z})} \frac{\delta^2 \Gamma_E[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(\bar{z}) \delta\phi_{\text{cl}}(\bar{y})}. \quad (7.65)$$

On the other hand, the left-hand side shows that this expression is independent of J and of ϕ_{cl} . By computing it for $J = 0$ one has

$$\int d^4\bar{z} G_{cE}^{(2)}(\bar{x}, \bar{z}) \left. \frac{\delta^2 \Gamma_E[\phi_{\text{cl}}]}{\delta\phi_{\text{cl}}(\bar{z}) \delta\phi_{\text{cl}}(\bar{y})} \right|_{J=0} = \delta^{(4)}(\bar{x} - \bar{y}).$$

Notice that for $J = 0$ one has $\phi_{\text{cl}}[J = 0] = \langle \Omega | \phi | \Omega \rangle_{J=0}$ that, in most of the treated cases, is vanishing. When $\phi_{\text{cl}}[J = 0] = 0$ one has the following relation

$$\int d^4\bar{z} G_{cE}^{(2)}(\bar{x}, \bar{z}) \Gamma_E^{(2)}(\bar{z}, \bar{y}) = \delta^{(4)}(\bar{x} - \bar{y}) ,$$

which is the Fourier transform of

$$\tilde{\Gamma}_E^{(2)}(p) \tilde{G}_{cE}^{(2)}(p) = 1 .$$

The analogous relation in Minkowski space reads

$$\tilde{\Gamma}^{(2)}(p) \tilde{G}_c^{(2)}(p) = i ,$$

which is a consequence of (7.63), $\phi_{\text{cl}} = \delta W / \delta J$ and

$$G_c^{(2)}(x, y) = -i \frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} , \quad \Gamma^{(2)}(x, y) = \frac{\delta^2 \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x) \delta \phi_{\text{cl}}(y)} .$$

Chapter 8

Perturbation Theory

In this chapter we introduce the main techniques to investigate the physical effects of interactions for small perturbation of a free quantum field theory. This leads to a power expansion in the coupling strength. We start by considering such an expansion in the classical approximation, given in terms of Feynman diagrams in the tree approximation. Although the analysis is focused on the scalar theory with potential density ϕ_4^4 , the methods and many of the results extend to other theories.

We will then investigate the quantum corrections by studying the effective action. To this end, we will introduce some mathematical tools, such as the ζ -function method to compute and regularise functional determinants. The analysis will show some interesting quantum phenomena, such as the quantum corrections to the effective potential density and the scaling properties of determinants.

Another topic concerns the Feynman rules to construct the loop expansion. We will then investigate the ultraviolet divergences due to the integration over the loop momenta, by giving a classification of renormalisable theories. Finally, we will prove that the effective action is the generating functional of the amputated one-particle irreducible functions, which are the building blocks of perturbation theory.

8.1 Saddle point approximation¹

The saddle point approximation is a useful approximation method to evaluate integrals of the form

$$I = \int dx e^{-a(x)} .$$

If the exponent $a(x)$ has a sharp minimum (say at x_0), then the idea is that the greatest and dominant contribution to the integral comes from the region where this sharp

¹ Luca Teodori

minimum lies. Therefore, we expand $a(x)$ near x_0 up to the second order (the first order is null since we are dealing with a minimum)

$$a(x) \simeq a(x_0) + \frac{1}{2}a''(x_0)(x - x_0)^2 ,$$

where $a''(x_0) > 0$ since we are at a minimum. We then have

$$I \simeq e^{-a(x_0)} \int dx e^{-\frac{a''(x_0)}{2}(x-x_0)^2} = e^{-a(x_0)} \sqrt{\frac{\pi}{a''(x_0)}} ,$$

where in the last step we recognised the gaussian integral.

Now we apply these ideas to the path integral. We will make this discussion in the Euclidean, that is, we will use

$$Z_E[J] = N \int \mathcal{D}\phi e^{-S_E[\phi, J]} ,$$

where

$$S_E[\phi, J] = \int d^4\bar{x} \left(\frac{1}{2} \bar{\partial}_\mu \phi \bar{\partial}_\mu \phi + \frac{1}{2} m^2 \phi^2 + V(\phi) - J\phi \right) .$$

To make the saddle point approximation we must consider the expansion near the minimum ϕ_0 . To find it we evaluate²

$$\left. \frac{\delta S_E}{\delta \phi} \right|_{\phi=\phi_0} = -\square \phi_0 + m^2 \phi_0 + V'(\phi_0) - J = 0 , \quad (8.2)$$

² We made the integration by parts $\int d^4\bar{x} \bar{\partial}_\mu \phi \bar{\partial}_\mu \phi = -\int d^4\bar{x} \phi \square \phi$, where, as usual, we dropped the boundary terms. To evaluate its functional derivative with respect to ϕ , one may first consider

$$\int d^4\bar{x} \phi(\bar{x}) \square_{\bar{x}} \phi(\bar{x}) = \int d^4\bar{x} d^4\bar{z} \phi(\bar{x}) \square_{\bar{x}} \delta^{(4)}(\bar{x} - \bar{z}) \phi(\bar{z}) ,$$

so that in the integrand appears only ϕ and not its derivatives. In this way one gets

$$\begin{aligned} \frac{\delta}{\delta \phi(\bar{y})} \int d^4\bar{x} d^4\bar{z} \phi(\bar{x}) \square_{\bar{x}} \delta^{(4)}(\bar{x} - \bar{z}) \phi(\bar{z}) \\ = \int d^4\bar{x} d^4\bar{z} \delta^{(4)}(\bar{y} - \bar{x}) \square_{\bar{x}} \delta^{(4)}(\bar{x} - \bar{z}) \phi(\bar{z}) + \int d^4\bar{x} d^4\bar{z} \phi(\bar{x}) \square_{\bar{x}} \delta^{(4)}(\bar{x} - \bar{z}) \delta^{(4)}(\bar{z} - \bar{y}) \\ = 2 \square_{\bar{y}} \phi(\bar{y}) , \end{aligned} \quad (8.1)$$

which is equivalent to

$$\frac{\delta}{\delta \phi(\bar{y})} \int d^4\bar{x} \phi(\bar{x}) \square_{\bar{x}} \phi(\bar{x}) = \int d^4\bar{x} \delta^{(4)}(\bar{y} - \bar{x}) \square_{\bar{x}} \phi(\bar{x}) + \int d^4\bar{x} \phi(\bar{x}) \square_{\bar{x}} \delta^{(4)}(\bar{y} - \bar{x}) = 2 \square_{\bar{y}} \phi(\bar{y}) .$$

As already clear by its definition, this shows that the functional derivative filters the standard derivative.

that is

$$(-\bar{\partial}_\mu \bar{\partial}_\mu + m^2)\phi_0 = J - V'(\phi_0) . \quad (8.3)$$

Let us consider the expansion of the action near³ ϕ_0

$$S_E[\phi, J] = S_E[\phi_0] + \frac{1}{2} \left\langle \frac{\delta S_E}{\delta \phi_1 \delta \phi_2} \Big|_{\phi=\phi_0} (\phi - \phi_0)_1 (\phi - \phi_0)_2 \right\rangle_{12} ,$$

To compute the right-hand side, note that

$$\frac{\delta}{\delta \phi(\bar{x}_2)} \square_{\bar{x}_1} \phi(\bar{x}_1) = \square_{\bar{x}_1} \frac{\delta \phi(\bar{x}_2)}{\delta \phi(\bar{x}_1)} = \square_{\bar{x}_1} \delta^{(4)}(\bar{x}_1 - \bar{x}_2) ,$$

so that

$$\frac{\delta^2 S_E}{\delta \phi_1 \delta \phi_2} \Big|_{\phi_i=\phi_0(\bar{x}_i)} = (-\bar{\partial}_\mu \bar{\partial}_\mu + m^2 + V''(\phi_0))_{\bar{x}_1} \delta^{(4)}(\bar{x}_1 - \bar{x}_2) , \quad (8.4)$$

$i = 1, 2$. We then have⁴

$$Z_E[J] \simeq e^{-S_E[\phi_0, J]} \int \mathcal{D}\phi \exp \left(-\frac{1}{2} \left\langle \frac{\delta^2 S_E}{\delta \phi_1 \delta \phi_2} \Big|_{\phi=\phi_0} (\phi - \phi_0)_1 (\phi - \phi_0)_2 \right\rangle_{12} \right) . \quad (8.5)$$

To evaluate this, we must first see how to compute Gaussian integrals of the form

$$G(A) = \int_{\mathbb{R}^n} \prod_{k=1}^n dx_k e^{-\mathbf{x}^T A \mathbf{x}} ,$$

where A is a positive-definite real symmetric matrix. We can diagonalise A with a certain $R \in SO(n)$,

$$A = R^T D R , \quad D = \text{diag}(d_1, \dots, d_n) .$$

Now note that by $\det R = 1$, it follows that the Jacobian of the transformation

$$x_k \longrightarrow y_k := R_{kj} x_j ,$$

is 1, so that

$$G(A) = \int_{\mathbb{R}^n} \prod_{k=1}^n dy_k e^{-\mathbf{y}^T D \mathbf{y}} = \int_{\mathbb{R}^n} \prod_{k=1}^n dy_k e^{-y_k^2 d_k} = \pi^{n/2} \left(\prod_{k=1}^n d_k \right)^{-\frac{1}{2}} = \pi^{n/2} (\det A)^{-\frac{1}{2}} .$$

For a positive definite Hermitian matrix C , the steps are similar but the integration is

³ The notation used here is $\phi_i \equiv \phi(\bar{x}_i)$ and $\langle \dots \rangle_{12}$ means integration over x_1 and x_2 .

⁴ Notice that this result is exact for a quadratic Lagrangian density.

over a $2n$ dimensional real space, so that we have

$$\int \prod_{k=1}^n dz_k dz_k^* e^{-\mathbf{z}^\dagger C \mathbf{z}} = \pi^n (\det C)^{-1} .$$

Let us go back to the Gaussian integral $G(A)$. If there are m zero modes, i.e. eigenvalues of A that are zero, so that A is positive semidefinite,⁵ we can restrict $G(A)$ to $n - m$ non-zero eigenvalues,

$$G(A) \rightarrow G_{\text{rest}}(A) \approx \int dy_1 \dots dy_{n-m} e^{-\mathbf{x}^T(y) A \mathbf{x}(y)} , \quad (8.6)$$

then one inserts a Dirac delta function instead of the variables y_{n-m+1}, \dots, y_n that are contributing to the zero mode in the integral. Therefore, one can modify a little bit (8.6) as

$$\begin{aligned} G_{\text{rest}}(A) &= \int dy_1 \dots dy_{n-m} dy_{n-m+1} \dots dy_n \delta(y_{n-m+1}) \dots \delta(y_n) e^{-\mathbf{x}^T(y) A \mathbf{x}(y)} \\ &= \int \left(\prod_{k=1}^n dx_k \right) \det \left| \frac{\partial y}{\partial x} \right|_{k=n-m+1}^n \prod_{k=n-m+1}^n \delta(y_k) e^{-\mathbf{x}^T A \mathbf{x}} . \end{aligned} \quad (8.7)$$

The problem here consists in the choice of the “smart” y .

The above formalism extends to the case where the indices are continuous ones. In particular, (8.5) is equivalent to

$$Z_E[J] \simeq e^{-S_E[\phi_0, J]} \det [(-\square + m^2 + V''(\phi_0))_{\bar{x}} \delta^{(4)}(\bar{x} - \bar{y})]^{-1/2} , \quad (8.8)$$

so that, recalling that

$$\det M = \exp(\text{Tr} \log M) ,$$

we have

$$Z_E[J] \simeq \exp \left\{ -S_E[\phi_0, J] - \frac{1}{2} \text{Tr} \log [(-\square + m^2 + V''(\phi_0))_{\bar{x}} \delta^{(4)}(\bar{x} - \bar{y})] \right\} . \quad (8.9)$$

We now have an approximate expression for $Z_E[J]$ which is given in terms of J and of the solution ϕ_0 of the classical equation of motion. The strategy is to first solve this equation in order to express ϕ_0 in terms of J and then replace it in (8.9). In the following we solve this problem of the classical theory expressing, perturbatively in λ , the classical action $S_E[\phi_0, J]$ in terms of J .

Let us then consider the classical equation of motion in the case of $V(\phi) = \lambda \phi^4/4!$, so

⁵ This can happen for example in the discussion of Faddeev-Popov ghosts in covariant gauges.

that (8.3) becomes

$$(-\bar{\partial}_\mu \bar{\partial}_\mu + m^2)\phi_0 = J - \frac{\lambda}{3!}\phi_0^3, \quad (8.10)$$

and by exploiting it one can expand ϕ_0 in powers of λ

$$\phi_0 = \sum_{k=0}^{\infty} \lambda^k \phi^{[k]}, \quad \phi^{[k]} = \frac{1}{k!} \left. \frac{d^k \phi_0}{d\lambda^k} \right|_{\lambda=0},$$

in this way equation (8.10) becomes

$$(-\bar{\partial}_\mu \bar{\partial}_\mu + m^2)(\phi^{[0]} + \lambda\phi^{[1]} + \lambda^2\phi^{[2]} + \dots) = J - \frac{\lambda}{3!}(\phi^{[0]} + \lambda\phi^{[1]} + \lambda^2\phi^{[2]} + \dots)^3. \quad (8.11)$$

Set

$$\hat{O} \equiv (-\bar{\partial}_\mu \bar{\partial}_\mu + m^2).$$

At order zero, equation (8.11) implies⁶

$$\hat{O}\phi^{[0]} = J, \quad \phi^{[0]}(\bar{x}) = \langle G_{\bar{x}\bar{y}} J_{\bar{y}} \rangle_{\bar{y}}.$$

At first order in λ we have

$$\hat{O}\phi^{[1]} = -\frac{1}{3!}(\phi^{[0]})^3.$$

To find $\phi^{[1]}$, note that

$$(\phi^{[0]}(\bar{x}))^3 = (\langle G_{\bar{x}\bar{a}} J_{\bar{a}} \rangle_{\bar{a}})^3 = \langle G_{\bar{x}\bar{a}} J_{\bar{a}} \rangle_{\bar{a}} \langle G_{\bar{x}\bar{b}} J_{\bar{b}} \rangle_{\bar{b}} \langle G_{\bar{x}\bar{c}} J_{\bar{c}} \rangle_{\bar{c}},$$

so that

$$\phi^{[1]}(\bar{x}) = -\frac{1}{3!} \langle G_{\bar{x}\bar{y}} G_{\bar{y}\bar{a}} G_{\bar{y}\bar{b}} G_{\bar{y}\bar{c}} J_{\bar{a}} J_{\bar{b}} J_{\bar{c}} \rangle_{\bar{a}\bar{b}\bar{c}\bar{y}}.$$

It is clear that any $\phi^{[k]}$ can be obtained by such a recursive method.

To find $S_E[\phi_0, J]$, note that, integrating by parts the kinetic term, yields

$$S_E[\phi_0, J] = \int d^4\bar{x} \left(-\frac{1}{2}\phi_0 \bar{\partial}_\mu \bar{\partial}_\mu \phi_0 + \frac{1}{2}m^2\phi_0^2 + V(\phi_0) - J\phi_0 \right).$$

Replacing $(-\bar{\partial}_\mu \bar{\partial}_\mu + m^2)\phi_0$ by the right-hand side of (8.10), yields

$$S_E[\phi_0, J] = \int d^4\bar{x} \left(-\frac{1}{2}J\phi_0 - \frac{\lambda}{24}\phi_0^4 \right). \quad (8.12)$$

Finally, replacing ϕ_0 by $\phi^{[0]} + \lambda\phi^{[1]}$, we get, up to order λ^2 , $S_E[\phi_0, J]$ expressed only in

⁶ In the following we will often indicate the dependence on the variables by subscripts, for example $G_{\bar{x}\bar{y}} \equiv G(\bar{x} - \bar{y})$.

terms of J

$$\begin{aligned}
S_E[\phi_0[J], J] = & -\frac{1}{2} \langle J_{\bar{x}} G_{\bar{x}\bar{y}} J_{\bar{y}} \rangle_{\bar{x}\bar{y}} + \frac{\lambda}{4!} \langle G_{\bar{x}\bar{y}} G_{\bar{y}\bar{a}} G_{\bar{y}\bar{b}} G_{\bar{y}\bar{c}} J_{\bar{x}} J_{\bar{a}} J_{\bar{b}} J_{\bar{c}} \rangle_{\bar{a}\bar{b}\bar{c}\bar{x}\bar{y}} \\
& - \frac{\lambda^2}{3 \cdot 4!} \langle G_{\bar{x}\bar{a}} G_{\bar{x}\bar{b}} G_{\bar{x}\bar{c}} G_{\bar{x}\bar{y}} G_{\bar{y}\bar{d}} G_{\bar{y}\bar{e}} G_{\bar{y}\bar{f}} J_{\bar{a}} J_{\bar{b}} J_{\bar{c}} J_{\bar{d}} J_{\bar{e}} J_{\bar{f}} \rangle_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}\bar{x}\bar{y}} + \mathcal{O}(\lambda^3) .
\end{aligned} \tag{8.13}$$

Since at the classical level we have

$$W_E[J] = S_E[\phi_0[J], J] ,$$

it follows that the connected Green functions at order \hbar^0 are

$$G_{cE}^{(2k+2)}(1, \dots, 2k+2) = - \left. \frac{\delta^{2k+2} S_E[\phi_0[J], J]}{\delta J_1 \dots \delta J_{2k+2}} \right|_{J=0} . \tag{8.14}$$

It is easy to see that $\phi^{[k]}$ is always expressed in terms of the product of $2k+1$ factors J . Since $\phi^{[k]}$ is the contribution at order λ^k to ϕ_0 , it follows by (8.12) that

$$S_E[\phi_0[J], J] = \int d^4\bar{x} \left(-\frac{1}{2} J \phi_0[J] - \frac{\lambda}{24} \phi_0^4[J] \right) = \sum_{k=1} \lambda^k F_k[J] , \tag{8.15}$$

with $F_k[J]$ a homogeneous functional of J of degree $2k+2$, that is

$$F_k[\alpha J] = \alpha^{2k+2} F_k[J] .$$

It then follows that, at order λ^k , $S_E[\phi_0[J], J]$ contributes only to $G_E^{(2k+2)}$. This is a particular property due to the classical approximation. One may also check that in terms of the 2-point function, the action has the structure

$$S_E[\phi_0[J], J] \simeq \sum_{k=1}^{\infty} \lambda^k \langle \underbrace{G \dots G}_{3k+1 \text{ terms}} \underbrace{J \dots J}_{2k+2 \text{ terms}} \rangle .$$

In order to compute the connected Green functions in the saddle point approximation at the classical level, i.e. without quantum corrections, we only need (8.13) and (8.14).

Let us write down the explicit expressions for some connected Green functions in configuration space. In the case of the 2- and 4-point functions, we have

$$G_{cE}^{(2)}(\bar{x}_1, \bar{x}_2) = G(\bar{x}_2 - \bar{x}_1) = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{\exp(i\bar{p}(\bar{x}_2 - \bar{x}_1))}{\bar{p}^2 + m^2} ,$$

Now since the internal lines do not have free ends, each one is attached to two vertex, whereas the external lines have a free end so they are attached to one vertex only. Therefore, we have

$$NV_N = 2I + E . \tag{8.18}$$

We saw that, at order \hbar^0 for ϕ_4^4 theory, the contribution at the order λ^k in the expansion is due only to the $G^{(2k+2)}$ connected Green functions. Therefore, to this order $E = 2k + 2$ and since order λ^k means that the corresponding diagram has k vertexes, by (8.18), we have

$$4V_4 - 2I = E \implies 4k - 2I = 2k + 2 \implies I = k - 1 ,$$

that implies

$$I = V_4 - 1 .$$

This last equation is valid only at the order \hbar^0 , i.e. for diagrams at the tree level (so it does not hold for diagrams with loops) and that, in addition, are connected. An example for which the relation $I = V_4 - 1$ does not hold is in the case of two vertexes with a loop and four external lines. In this case we have $V_4 = 2$, $I = 2$, $E = 4$ and $L = 1$, with L the number of loops. Another example is given by three disconnected vertexes. In a vertex the legs are glued by an internal line in pair. The other two vertexes having all legs attached to external lines. In this case we have $V_4 = 3$, $I = 2$, $E = 8$, $L = 2$.

8.3 $\Gamma[\varphi]$ at the order \hbar

Let us consider the effective action in the Minkowskian. We saw that at the order \hbar^0 this coincides with the classical action. In the following we determine the contribute of order \hbar to Γ . Note that the equation of motion satisfied by ϕ_{cl} , i.e. the Schwinger-Dyson equation, fixes the relation with the source J . In particular, one may consider J as a functional of ϕ_{cl} , that is we can consider ϕ_{cl} as independent variable, so that we set

$$\varphi := \phi_{\text{cl}} ,$$

with the warning that J is a functional of φ . Notice that

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi]\right) = Z[J] \exp\left(-\frac{i}{\hbar}\langle J\varphi\rangle\right) = \int \mathcal{D}\tilde{\phi} \exp\left[\frac{i}{\hbar}(S[\tilde{\phi}] + \langle J(\tilde{\phi} - \varphi)\rangle)\right] .$$

Let us define a new field ϕ setting

$$\tilde{\phi} = \phi + \varphi .$$

Since $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$, one has

$$\exp\left(\frac{i}{\hbar}\Gamma[\varphi]\right) = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\left(S[\phi + \varphi] - \left\langle \frac{\delta\Gamma[\varphi]}{\delta\varphi}\phi \right\rangle\right)\right]. \quad (8.19)$$

If the potential density is a polynomial of degree n , then

$$S[\phi + \varphi] = S[\varphi] + \sum_{k=1}^n \frac{1}{k!} S_k[\varphi] \phi^k,$$

where

$$S_k[\varphi] \phi^k := \left\langle \frac{\delta^k S[\varphi]}{\delta\varphi(x_1) \dots \delta\varphi(x_k)} \phi(x_1) \dots \phi(x_k) \right\rangle.$$

Let us consider the power series expansion⁸

$$\Gamma[\varphi] = \sum_{k=0}^{\infty} \Gamma_k[\varphi] \hbar^k.$$

Next, recall that

$$\Gamma_0[\varphi] = S[\varphi],$$

so that, moving the term $S[\varphi]$ on the left-hand side of (8.19) and noticing that

$$\sum_{k=1}^n \frac{1}{k!} S_k[\varphi] \phi^k - \left\langle \frac{\delta\Gamma[\varphi]}{\delta\varphi} \phi \right\rangle = \sum_{k=2}^n \frac{1}{k!} S_k[\varphi] \phi^k - \left\langle \left(\hbar \frac{\delta\Gamma_1[\varphi]}{\delta\varphi} + \mathcal{O}(\hbar^2) \right) \phi \right\rangle,$$

we have

$$\exp[i(\Gamma_1[\varphi] + \mathcal{O}(\hbar))] = \int \mathcal{D}\phi \exp\left\{ \frac{i}{\hbar} \left[\sum_{k=2}^n \frac{1}{k!} S_k[\varphi] \phi^k - \left\langle \left(\hbar \frac{\delta\Gamma_1[\varphi]}{\delta\varphi} + \mathcal{O}(\hbar^2) \right) \phi \right\rangle \right] \right\}. \quad (8.20)$$

Before proceeding we consider the dimensional analysis of the scalar theories, in D dimension, with potential density

$$V(\phi) = \frac{\lambda}{n!} \phi^n,$$

in the presence of an external source J . The convention for the metric in Minkowski space is $(+, -, \dots, -)$. Moreover, $x^\mu = (x^0, x^1, \dots, x^{D-1})$, $x^0 = ct$, $\partial_\mu = (\partial_0, \nabla)$, where $\nabla = \partial_i = \partial/\partial x^i = -\partial^i = \partial/\partial x_i$, $i = 1, \dots, D-1$, is the $(D-1)$ gradient, whereas the

⁸ As we will see, the meaning of this expansion must be specified, aspect that requires a dimensional analysis.

D momentum operator is $p^\mu = i\hbar\partial^\mu = (i\hbar\partial^0, -i\hbar\nabla)$. The equation of motion is

$$(\hbar^2\partial_\mu\partial^\mu + m^2c^2)\phi = J - \frac{\lambda}{(n-1)!}\phi^{n-1}. \quad (8.21)$$

In the following we will consider the most general form

$$\left(\hbar^\alpha\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^{2-\alpha}}\right)\phi = J - \frac{\lambda}{(n-1)!}\phi^{n-1}, \quad (8.22)$$

$\forall\alpha \in \mathbb{R}$, which differs with respect to (8.21) by a rescaling of the right-hand side. Eq.(8.22) implies

$$[J] = L^{-2}[\hbar]^\alpha[\phi], \quad [\lambda][\phi]^{n-1} = [J], \quad [\phi]^{n-2} = L^{-2}[\lambda]^{-1}[\hbar]^\alpha. \quad (8.23)$$

The Lagrangian density associated to (8.22) is

$$\mathcal{L} = \frac{\hbar^\alpha}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\frac{m^2c^2}{\hbar^{2-\alpha}}\phi^2 - \frac{\lambda}{n!}\phi^n + J\phi. \quad (8.24)$$

A crucial aspect of the path integral formulation is the presence of the term \hbar that divides the action. We must require that $S = \int d^Dx \mathcal{L}$ has indeed the dimension of an action. However, note that the equations of motion remain unchanged if one multiplies \mathcal{L} by an arbitrary constant, that is

$$S = K \int d^Dx \mathcal{L}.$$

By Eq.(8.24) it follows that the condition $[K \int d^Dx \mathcal{L}] = [\hbar]$, implies

$$[K] = L^{2-D}[\hbar]^{1-\alpha}[\phi]^{-2}. \quad (8.25)$$

Together with (8.23), this relation implies

$$[K] = L^{\frac{n(2-D)+2D}{n-2}}[\hbar]^{\frac{n(1-\alpha)-2}{n-2}}[\lambda]^{\frac{2}{n-2}}. \quad (8.26)$$

If we require K to be dimensionless, then

$$[\lambda] = L^{\frac{n}{2}(D-2)-D}[\hbar]^{\frac{n}{2}(\alpha-1)+1}, \quad [\phi] = L^{(2-D)/2}[\hbar]^{(1-\alpha)/2}, \quad [J] = L^{-(D+2)/2}[\hbar]^{(1+\alpha)/2}. \quad (8.27)$$

By reabsorbing the dimensions in $[\hbar]$ of λ , ϕ and J

$$\lambda \rightarrow \hbar^{\frac{n}{2}(\alpha-1)+1}\lambda, \quad \phi \rightarrow \hbar^{(1-\alpha)/2}\phi, \quad J \rightarrow \hbar^{(1+\alpha)/2}J, \quad (8.28)$$

the Lagrangian density becomes

$$\mathcal{L} = \frac{\hbar}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \frac{m^2 c^2}{\hbar} \phi^2 - \hbar \frac{\lambda}{n!} \phi^n + \hbar J \phi . \quad (8.29)$$

Note that such an expression of \mathcal{L} immediately follows also by (8.24), by requiring

$$[\int d^D x \mathcal{L}] = [\hbar] ,$$

and, in addition, that the dimensions of ϕ and J correspond to powers of L only.

Let us go back to Eq.(8.27). Note that for $D = 4$ and $n = 4$ it gives

$$[\lambda] = [\hbar]^{2\alpha-1} , \quad [\phi] = L^{-1} [\hbar]^{(1-\alpha)/2} , \quad [J] = L^{-3} [\hbar]^{(\alpha+1)/2} , \quad (8.30)$$

that for $\alpha = 1$ reads

$$[\lambda] = [\hbar] , \quad [\phi] = L^{-1} , \quad [J] = L^{-3} [\hbar] , \quad (8.31)$$

whereas, for $\alpha = 0$,

$$[\lambda] = [\hbar]^{-1} , \quad [\phi] = L^{-1} [\hbar]^{1/2} , \quad [J] = L^{-3} [\hbar]^{1/2} . \quad (8.32)$$

Since usually in the Lagrangian density one sets $\hbar = 1$, there are no obvious reasons to make a choice of the value of α . However, the canonical way to treat the contributions in powers of \hbar to the effective action is equivalent, even if not always explicitly declared, to choose $\alpha = 0$. For example, on p. 288 of the Itzykson and Zuber's book [7], it is considered, for $n = 4$, the choice (8.24) with $\alpha = 0$. Such a choice suggests to rescale ϕ and J by a factor $\hbar^{1/2}$, that is to substitute in the Lagrangian density the field ϕ with $\hbar^{1/2} \phi$ and J with $\hbar^{1/2} J$. Therefore, after this rescaling Eq.(8.24) becomes, for $\alpha = 0$ and $n = 4$,

$$\mathcal{L} = \frac{\hbar}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \frac{m^2 c^2}{\hbar} \phi^2 - \hbar^2 \frac{\lambda}{4!} \phi^4 + \hbar J \phi , \quad (8.33)$$

where

$$[\lambda] = [\hbar]^{-1} , \quad [\phi] = L^{-1} , \quad [J] = L^{-3} . \quad (8.34)$$

In this respect, it is worth stressing that the fact that the coupling constant has the dimension of \hbar^{-1} does not imply that it must have a term \hbar^{-1} in it.

Let us go back to Γ_1 , that denotes the contribution to Γ at order \hbar . Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \frac{m^2 c^2}{\hbar^2} \phi^2 - \frac{\lambda}{4!} \phi^4 + J \phi , \quad (8.35)$$

which is (8.24) with $\alpha = 0$. In this case the dimensions of λ , ϕ and J , are the ones reported in Eq.(8.32). We have seen that, except the term $\langle J\phi \rangle$, the action corresponding to (8.35) disappears from (8.20). At this point one chooses to rescale the “quantum fluctuation” ϕ , removing its dimensionality in \hbar . In this regard, note that the rescaling of J is fixed by the equation of motion $J = -\delta\Gamma[\varphi]/\delta\varphi$. It follows that the dependence of J on \hbar is completely fixed by the functional structure of Γ , and therefore by the series expansion of Γ in powers of \hbar .

Doing the rescaling $\phi \rightarrow \hbar^{1/2}\phi$, Eq.(8.20) becomes

$$\exp[i(\Gamma_1[\varphi] + \mathcal{O}(\hbar))] = \int \mathcal{D}\phi \exp \left\{ i \left[\sum_{k=2}^n \frac{\hbar^{\frac{k}{2}-1}}{k!} S_k[\varphi] \phi^k - \left\langle \left(\hbar^{1/2} \frac{\delta\Gamma_1[\varphi]}{\delta\varphi} + \mathcal{O}(\hbar^{3/2}) \right) \phi \right\rangle \right] \right\}. \quad (8.36)$$

Note that the exponent in the integrand has a quadratic term given by $iS_2[\varphi]\phi^2/2$. For our purpose it is useful to keep this term in the exponent and expand the exponential of the remaining polynomial into a power series in ϕ . Such an expansion has the form

$$\int \mathcal{D}\phi \exp \left(\frac{i}{2} S_2[\varphi] \phi^2 \right) \left[1 + \hbar^{1/2} \left(\frac{1}{3!} S_3[\varphi] \phi^3 - \left\langle \frac{\delta\Gamma_1[\varphi]}{\delta\varphi} \phi \right\rangle \right) + \mathcal{O}(\hbar) \right]$$

Now recall that the odd part of any functional $G[\phi]$ does not contribute to the integral, that is

$$\int \mathcal{D}\phi G[\phi] = \frac{1}{2} \int \mathcal{D}\phi (G[\phi] + G[-\phi]) .$$

Therefore, we have⁹

$$\exp[i(\Gamma_1[\varphi] + \mathcal{O}(\hbar))] = \int \mathcal{D}\phi \exp \left(\frac{i}{2} S_2[\varphi] \phi^2 \right) (1 + \mathcal{O}(\hbar)) , \quad (8.37)$$

that is

$$\Gamma_1[\varphi] = -i \log \int \mathcal{D}\phi \exp \left(\frac{i}{2} S_2[\varphi] \phi^2 \right) . \quad (8.38)$$

This is the contribution to order \hbar to the effective action.¹⁰ Such a path integral is just the Minkowskian analogue of the one computed in (8.9) for the \hbar contribution in the

⁹ Note that in the case where φ is replaced by a constant, the term $iS_2[\varphi]\phi^2/2$ would coincide with the action S_0 of a free particle with square mass $m^2 + V''(\text{constant})$. One would then have, according to Wick’s theorem,

$$\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_{2k+1}) e^{iS_0} = 0 ,$$

whereas

$$\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_{2k}) e^{iS_0} = \frac{i^n}{2^n n!} \sum_{perm.} \Delta_F(x_{P_1} - x_{P_2}) \cdots \Delta_F(x_{P_{2n-1}} - x_{P_{2n}}) .$$

¹⁰ Recall that the \hbar factor multiplying $i\Gamma_1[\varphi]$ simplified by the \hbar^{-1} multiplying $i\Gamma[\varphi]$.

saddle point approximation. We then have

$$\Gamma_1[\varphi] = \frac{1}{2} \text{Tr} \log [(-\partial_\mu \partial^\mu + m^2 + V''(\varphi))_x \delta^{(4)}(x - y)] . \quad (8.39)$$

As shown by the previous dimensional analysis, the fields and parameters in the Lagrangian density depend on \hbar . Such a dependence has not been considered in deriving the expression of $\Gamma_1[\varphi]$. This is of interest when considering perturbation theory¹¹ and, in particular, in the renormalisation group equation that depends on a mass scale.

Let us further investigate such a point by anticipating the explicit expression of the effective potential density at order \hbar . As we will see, it turns out that, in the Euclidean space,

$$V^e(\varphi) = \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{\hbar}{64\pi^2} \left(m^2 + \frac{\lambda}{2} \varphi^2 \right)^2 \left(-\frac{3}{2} + \log \frac{m^2 + \lambda \varphi^2 / 2}{\mu^2} \right) + \mathcal{O}(\hbar^2) , \quad (8.40)$$

where μ is a mass scale we will introduce for dimensional reasons in solving a differential equation. As an example of the role of \hbar in (8.40), we consider the case $\alpha = 0$, that is Eq.(8.35). To take into account the \hbar^2 term that divides m^2 , we rewrite (8.40) with the substitution¹²

$$m^2 \longrightarrow \frac{m^2}{\hbar^2} , \quad \mu^2 \longrightarrow \frac{\mu^2}{\hbar^2} ,$$

that is

$$V^e(\varphi) = \frac{1}{2} \frac{m^2}{\hbar^2} \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{\hbar}{64\pi^2} \left(\frac{m^2}{\hbar^2} + \frac{\lambda}{2} \varphi^2 \right)^2 \left(-\frac{3}{2} + \log \frac{m^2/\hbar^2 + \lambda \varphi^2 / 2}{\mu^2/\hbar^2} \right) + \dots . \quad (8.41)$$

Note that by (8.30) with $\alpha = 0$, we have that the dimension of φ is $[\hbar]^{1/2} L^{-1}$, and the one of λ is $[\hbar]^{-1}$. Since $[m] = [\hbar] L^{-1}$, one can check that (8.41) is dimensionally correct.

Such a result shows that the \hbar expansion is subtle. As a matter of fact, the analysis of the contribution up to order \hbar to the effective potential density, and therefore even to $\Gamma[\varphi]$, has been done in the following way. Consider the Lagrangian density in Minkowski space (8.35) with $c = 1$ and set

$$\nu_m := m/\hbar ,$$

as parameter independent of \hbar . We then have

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \nu_m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + J \phi .$$

¹¹ As already mentioned, the loop expansion corresponds to an expansion in \hbar .

¹² Here we use $c = 1$.

Then rescale the μ parameter in (8.41) by setting

$$\nu_\mu := \mu/\hbar ,$$

that now should be seen as a parameter independent of \hbar . Then, expanding the effective action in power series of \hbar , one gets

$$\Gamma[\varphi; \nu_m, \nu_\mu] = \sum_{k=0}^{\infty} \hbar^k \Gamma_k[\varphi; \nu_m, \nu_\mu] .$$

Since $\Gamma[\varphi] = \Gamma[\varphi; \nu_m, \nu_\mu]$, one has

$$\Gamma_k[\varphi] = \Gamma_k[\varphi; \nu_m, \nu_\mu] .$$

8.4 ζ -function evaluation of determinants¹³

We want to go beyond the order \hbar^0 . Let us then consider again (8.8). In order to evaluate such a determinant, we consider the case of a positive definite operator A such that¹⁴

$$Af_n = a_n f_n , \quad a_n > 0 , \quad \forall n ,$$

where the a_n 's are the positive definite eigenvalues and the f_n 's the corresponding eigenfunctions. A frequent problem in quantum field theory is that the relevant determinants diverge. A powerful method to regularise such determinants is to use the so-called ζ -function regularisation. The starting point is to consider the $\zeta_A(s)$ -function, defined by

$$\zeta_A(s) = \sum_{n=1}^{\infty} \frac{1}{a_n^s} ,$$

for the values of $\text{Re}(s)$ for which it converges, and by analytic continuation elsewhere.

A famous example of such a regularisation by analytic continuation is the Riemann ζ -function, corresponding to $a_n = n$, that is

$$\sum_{n=1}^{\infty} \frac{1}{n^s} .$$

For $s = -1$ it corresponds to $\sum_{n=1}^{\infty} n = \infty$. However, by analytic continuation we can substitute $\sum_{n=1}^{\infty} n$ by $\zeta(-1) = -1/12$. It is remarkable that such a result has been first obtained by Euler without using complex analysis, unavailable at that time. Such a

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¹⁴ The problem of zero modes, that is zero eigenvalues that would give a vanishing determinant can be treated by a suitable application to the present method of the strategy leading to (8.7).

result is used in several research fields besides number theory. For example, it can be used to prove that the bosonic string critical dimension is 26.

Other values of the Riemann ζ -function are

$$\sum_{n=1}^{\infty} 1 \xrightarrow{\text{analytic cont.}} \zeta(0) = -\frac{1}{2},$$

and

$$\sum_{n=1}^{\infty} n^2 \xrightarrow{\text{analytic cont.}} \zeta(-2) = 0.$$

More generally

$$\sum_{n=1}^{\infty} n^s \xrightarrow{\text{analytic cont.}} \zeta(-s) = -\frac{B_{s+1}}{s+1},$$

with B_k the k th Bernoulli number.

The connection between the $\zeta_A(s)$ and determinants follows by

$$\left. \frac{d\zeta_a(s)}{ds} \right|_{s=0} = - \left. \sum_{n=1}^{\infty} \frac{\log a_n}{a_n^s} \right|_{s=0} = - \left. \sum_{n=1}^{\infty} \log a_n e^{-s \log a_n} \right|_{s=0} = - \log \prod_n a_n,$$

that is

$$\det A = e^{-\zeta'_A(0)}.$$

The key observation is that the ζ_A function can be evaluated by using the heat equation. The consequence is that the problem of finding the determinant of an operator is equivalent to solve a partial differential equation. To show this, we consider the heat function

$$G(\bar{x}, \bar{y}, \tau) = \sum_n e^{-a_n \tau} f_n(\bar{x}) \bar{f}_n(\bar{y}), \quad (8.42)$$

called in this way because it satisfies the heat equation¹⁵

$$A_{\bar{x}} G(\bar{x}, \bar{y}, \tau) = -\frac{\partial}{\partial \tau} G(\bar{x}, \bar{y}, \tau). \quad (8.43)$$

Note that by (8.42) it follows that the solution of (8.43) must satisfy the initial condition

$$G(\bar{x}, \bar{y}, 0) = \delta^{(4)}(\bar{x} - \bar{y}). \quad (8.44)$$

It is easy to check that

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} \int d^4 \bar{x} G(\bar{x}, \bar{x}, \tau), \quad (8.45)$$

¹⁵ $A_{\bar{x}}$ means that the operator acts only on the \bar{x} variable.

where

$$\Gamma(s) = \int_0^\infty dt t^{s-1} e^{-t} ,$$

is the Euler Γ -function. To prove (8.45), note that, exploiting the fact that the f_n 's are orthonormal eigenfunctions, we have

$$\sum_n \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-a_n \tau} \underbrace{\int d^4 \bar{x} |f_n(\bar{x})|^2}_{=1} = \sum_n \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{a_n^s} e^{-t} = \sum_{n=1}^\infty \frac{1}{a_n^s} = \zeta_A(s) ,$$

where in the first equality it has been used the change of variable $t = a_n \tau$.

Let us summarise the above strategy to find the determinant of an operator.

- (i) Find the solutions to the heat equation (8.43) satisfying the initial condition (8.44).
- (ii) Insert the solution $G(\bar{x}, \bar{y}, \tau)$ in (8.45).
- (iii) Use $\det A = e^{-\zeta'_A(0)}$.

Needless to say, the very hard step is the first one.

8.5 Effective potential density¹⁶

In this section we compute the effective potential density in Euclidean space at order \hbar . Let us first consider the Euclidean version of the determinant in (8.39)

$$\det [(-\bar{\partial}_\mu \bar{\partial}_\mu + m^2 + V''(\varphi))_{\bar{x}_1} \delta^{(4)}(\bar{x}_1 - \bar{x}_2)] . \quad (8.46)$$

Before using the ζ -function treatment of determinants, it is instructive to extend the treatment of Gaussian integrals to quantum field path integrals. Let us then consider

$$\int \mathcal{D}\phi \exp \left(- \int d^4 \bar{x} \phi(\bar{x}) O_{\bar{x}} \phi(\bar{x}) \right) ,$$

with $O_{\bar{x}}$ some differential operator. Let $\{\psi_n\}$ be the set of orthonormal eigenfunctions of $O_{\bar{x}}$

$$O_{\bar{x}} \psi_n(\bar{x}) = \lambda_n \psi_n(\bar{x}) ,$$

where we assume positive definiteness of the eigenvalues, that is $\lambda_n > 0, \forall n$. Considering the expansion

$$\phi = \sum_k c_k \psi_k , \quad (8.47)$$

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we have

$$\int d^4\bar{x}\phi(\bar{x})O_{\bar{x}}\phi(\bar{x}) = \sum_k c_k^2\lambda_k .$$

Let us now consider the following qualitative argument concerning the measure $\mathcal{D}\phi$. The idea is that any element in the space Φ on which is defined the measure can be expanded as in (8.47). In this way, any element $\phi \in \Phi$ corresponds to a point the $\mathcal{C} := \{c_k\}$. We then consider the map $\phi \in \Phi \rightarrow \mathcal{C}$, and set

$$\mathcal{D}\phi \cong \prod_k \frac{dc_k}{2\pi} .$$

This leads to an infinite product of Gaussian integrals, and therefore to $(\det O_{\bar{x}})^{-1/2}$. More precisely, we have

$$\begin{aligned} \int \mathcal{D}\phi \exp\left(-\int d^4\bar{x}\phi(\bar{x})O_{\bar{x}}\phi(\bar{x})\right) &= \int \prod_k \frac{dc_k}{2\pi} \exp\left(-\sum_k c_k^2\lambda_k\right) \\ &= \prod_k \int \frac{dc_k}{2\pi} \exp\left(-c_k^2\lambda_k\right) \\ &= \prod_k \frac{1}{2\sqrt{\pi}}\lambda_k^{-1/2} = N(\det O_{\bar{x}})^{-1/2} , \end{aligned}$$

where the infinite constant

$$N = \prod_k \frac{1}{2\sqrt{\pi}} ,$$

needs to be regularised. A possible method is just the use of the ζ -function we have seen in the previous section. As we said, such a receipt is also called ζ -function regularisation.

Let us comment on the role of $\delta^{(4)}(\bar{x}_1 - \bar{x}_2)$ in the operator (8.46). Note that this is analogue of the δ_{jk} in the diagonal matrix $A_{jk} = A_j\delta_{jk}$. So that the analogous of the multiplication of a vector v by A , that is $\sum_k A_{jk}v_k = A_jv_j$, is, in the case of the operator in (8.46),

$$\int d^4\bar{x}_2 \left(-\bar{\partial}_\mu\bar{\partial}_\mu + m^2 + V''(\varphi)\right)_{\bar{x}_1} \delta^{(4)}(\bar{x}_1 - \bar{x}_2)\phi(\bar{x}_2) = \left(-\bar{\partial}_\mu\bar{\partial}_\mu + m^2 + V''(\varphi)\right)_{\bar{x}_1} \phi(x_1) .$$

However, note that this does not mean that we have diagonalised the differential operator. Let us show this by considering the massless case and with $\varphi = v$, where v is a constant.¹⁷ Consider

$$O_{\bar{x}} := -\bar{\partial}_\mu\bar{\partial}_\mu + \frac{\lambda}{2}v^2 ,$$

and let $\{\psi_{\bar{p}}\}$ be the set of orthonormalised eigenfunctions of $O_{\bar{x}}$ of four momentum \bar{p} ,

¹⁷ This is what we will do later to compute V^e .

so that

$$O_{\bar{x}}\psi_{\bar{p}}(\bar{x}) = \left(\bar{p}^2 + \frac{\lambda}{2}v^2\right)\psi_{\bar{p}}(\bar{x}) ,$$

with

$$\int d^4x \bar{\psi}_{\bar{p}_2}\psi_{\bar{p}_1}(\bar{x}) = \delta^{(4)}(\bar{p}_1 - \bar{p}_2) .$$

For any function¹⁸ F of $O_{\bar{x}}$, consider

$$\int d^4\bar{x} \bar{\psi}_{\bar{p}_2}F(O_{\bar{x}})\psi_{\bar{p}_1}(\bar{x}) = F(\bar{p}_1^2 + \lambda v^2/2) \int d^4x \bar{\psi}_{\bar{p}_2}\psi_{\bar{p}_1}(\bar{x}) = \delta^{(4)}(\bar{p}_1 - \bar{p}_2)F(\bar{p}_1^2 + \lambda v^2/2) .$$

Note that \bar{p}_1 and \bar{p}_2 are the continuous indices of the matrix

$$M_{\bar{p}_1\bar{p}_2} := \delta^{(4)}(\bar{p}_1 - \bar{p}_2)F(\bar{p}_1^2 + \lambda v^2/2) ,$$

so that its trace reads $\int d^4\bar{p}M_{\bar{p}\bar{p}}$, that is

$$\text{Tr} F(O_{\bar{x}}) = \delta^{(4)}(0) \int d^4\bar{p}F(\bar{p}^2 + \lambda v^2/2) .$$

The integral representation of the δ -function shows that $\delta^{(4)}(0)$ can be interpreted as the space-time volume divided by $(2\pi)^4$. Later we will show a method to treat such an infrared singularity.

In the case F is the log function, we get an expression for the effective potential density that needs to be regularised. Instead of considering such an analysis, we now follow the related ζ -function method to compute V^e .

Let us start by considering the \hbar contribution to the effective action. We saw that

$$\Gamma_{1E}[\varphi] = -\frac{1}{2}\zeta'_{[-\bar{\partial}_\mu\bar{\partial}_\mu+m^2+\frac{\lambda}{2}\varphi^2]}(0) . \tag{8.48}$$

Following our three steps, we should now find the solution of the heat equation

$$\left(-\bar{\partial}_\mu\bar{\partial}_\mu + m^2 + \frac{\lambda}{2}\varphi^2(\bar{x})\right)G(\bar{x}, \bar{y}, \tau) = -\frac{\partial G(\bar{x}, \bar{y}, \tau)}{\partial \tau} . \tag{8.49}$$

This is very hard to be solved for an arbitrary φ , what we can do is to consider the effective action, that we saw has the form

$$\Gamma_E[\varphi] = \int d^4\bar{x} \left(V^e(\varphi) + \frac{1}{2}F(\varphi)\bar{\partial}_\mu\varphi\bar{\partial}_\mu\varphi + \text{higher order derivatives}\right) . \tag{8.50}$$

We are interested to the contribution at order \hbar to V^e and this can be obtained by using a constant field configuration, that is $\phi = v$, with v constant, so that the derivative

¹⁸ Recall that functions of operators are defined by their formal power expansion.

terms in (8.50) cancel, and the effective action reads

$$\Gamma_E[v] = V^e(v) \int d^4\bar{x} .$$

The space-time integral should be infinite, nevertheless we can perform a standard trick. Suppose our Euclidean space is S^4 rather than \mathbb{R}^4 . In this case we can consider S^4 as the surface of a five dimensional sphere, so that we obtain a finite result and avoid this infrared divergence. It follows that (8.48)

$$V^e(v) \int d^4\bar{x} = \left(\frac{1}{2}m^2v^2 + \frac{\lambda}{4!}v^4 \right) \int d^4\bar{x} - \frac{\hbar}{2} \zeta'_{[-\bar{\partial}_\mu \bar{\partial}_\mu + m^2 + \frac{\lambda}{2}v^2]}(0) + \mathcal{O}(\hbar^2) . \quad (8.51)$$

Now with a constant v we can integrate (8.49) and find

$$G(\bar{x}, \bar{y}, \tau) = \frac{\mu^2}{16\pi^2\tau^2} e^{-\mu^2(\bar{x}-\bar{y})^2/(4\tau)} e^{-(m^2 + \frac{\lambda}{2}v^2)\tau/\mu^2} ,$$

where μ is a scale that we introduced for dimensional reasons. Then, by (8.45), we get¹⁹

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} \int d^4\bar{x} \frac{\mu^4}{16\pi^2\tau^2} e^{-(m^2 + \lambda^2 v^2)\tau/\mu^2} \\ &= \frac{\mu^2}{16\pi^2} \left(\frac{m^2 + \frac{\lambda}{2}v^2}{\mu^2} \right)^{2-s} \frac{\Gamma(s-2)}{\Gamma(s)} \int d^4\bar{x} , \end{aligned}$$

so that

$$\begin{aligned} \zeta'_{[-\bar{\partial}_\mu \bar{\partial}_\mu + m^2 + \frac{\lambda}{2}v^2]}(0) &= \frac{\mu^4}{16\pi^2} \frac{d}{ds} \left[\frac{1}{(s-2)(s-1)} \left(\frac{m^2 + \lambda v^2/2}{\mu^2} \right)^{2-s} \right] \Big|_{s=0} \int d^4\bar{x} \\ &= -\frac{1}{32\pi^2} \left(m^2 + \frac{\lambda}{2}v^2 \right)^2 \left(-\frac{3}{2} + \log \frac{m^2 + \lambda v^2/2}{\mu^2} \right) \int d^4\bar{x} . \quad (8.52) \end{aligned}$$

Finally, by (8.51) we get the following expression for the Euclidean effective potential density

$$V^e(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{\hbar}{64\pi^2} \left(m^2 + \frac{\lambda}{2}\varphi^2 \right)^2 \left(-\frac{3}{2} + \log \frac{m^2 + \frac{\lambda}{2}\varphi^2}{\mu^2} \right) + \mathcal{O}(\hbar^2) . \quad (8.53)$$

However, as (8.41) shows, a complete dimensional analysis should include the Lagrangian density \mathcal{L} . We saw that this leads to the insertion of \hbar factors. There are

¹⁹The integration in τ is defined only for $s > 2$, nevertheless, the result can be extended by analytic continuation.

many possibilities, parameterised by α . Eq.(8.41) corresponds to the choice $\alpha = 0$.

Let us consider the μ -dependence of V^e . Note that until now the parameters m^2 and λ have been introduced by hand in the Lagrangian density. On the other hand, it is natural to define the mass squared as the coefficients of the ϕ^2 term in $V^e(\phi)$. Let us consider the case $m^2 = 0$. It is immediate to check that in this case we have, at order \hbar , $V^{e''}(0) = 0$. Therefore, the squared mass is zero if it is zero classically.

Let us now consider the coupling constant λ . We can define it as the coefficient of the ϕ^4 term in $V^e(\phi)$, that is

$$\lambda = \left. \frac{d^4 V^e(\phi)}{d\phi^4} \right|_{\phi=M} , \tag{8.54}$$

with M some non-zero constant. Note that $M = 0$ cannot be a choice because this leads to a logarithmic divergence. This is an infrared divergence, which is a typical property of theories which are classically massless.

Let us compute (8.54), with V^e given by (8.53) in the case $m^2 = 0$

$$\lambda = \lambda + 24A \left(B + \frac{25}{6} + \log M^2 \right) , \tag{8.55}$$

with

$$A = \frac{\hbar\lambda^2}{256\pi^2} , \quad B = -\frac{3}{2} + \log \frac{\lambda}{2\mu^2} .$$

Therefore, we have

$$\log \frac{\lambda M^2}{2\mu^2} = -\frac{8}{3} . \tag{8.56}$$

We can then express the term $\lambda/(2\mu^2)$ in (8.53) in terms of M^2 , that is,

$$-\frac{3}{2} + \log \frac{\lambda\varphi^2}{2\mu^2} = -\frac{25}{6} + \log \frac{\varphi^2}{M^2} ,$$

so that

$$V^e(\varphi) = \frac{\lambda}{4!}\varphi^4 + \frac{\hbar\lambda^2\varphi^4}{256\pi^2} \left(\log \frac{\varphi^2}{M^2} - \frac{25}{6} \right) + \mathcal{O}(\hbar^2) . \tag{8.57}$$

Such a result, due to Coleman and Weinberg,²⁰ shows a key property, related to the renormalisation group equation that will be discussed in great detail later. Namely, if we now change the scale, from M to M' , then the physical content does not change. In other words, this is just a reparametrisation of the same function, to the order in which we are working. As such, it is a change of definition, not a change of physics. To see

²⁰S. Coleman and E. Weinberg, Phys. Rev. **D7** (1973) 1888.

this, set²¹

$$\lambda' = \frac{d^4 V^e(\phi, \lambda)}{d\phi^4} \Big|_{\phi=M'} . \quad (8.58)$$

Note that now Eq.(8.55) becomes

$$\lambda' = \lambda + \frac{3\hbar\lambda^2}{32\pi^2} \left(\frac{8}{3} + \log \frac{\lambda}{2\mu^2} + \log M'^2 \right) , \quad (8.59)$$

that, by

$$\log \frac{\lambda}{2\mu^2} = -\frac{8}{3} - \log M^2 ,$$

gives

$$\lambda' = \lambda + \frac{3\hbar\lambda^2}{16\pi^2} \log \frac{M'}{M} . \quad (8.60)$$

To express λ in terms of λ' one may solve the algebraic equation using the expansion

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \dots ,$$

leading to

$$\lambda = \lambda' + \frac{3\hbar\lambda'^2}{16\pi^2} \log \frac{M}{M'} + \mathcal{O}(\lambda'^3) . \quad (8.61)$$

Replacing λ in (8.57) with the right-hand side of (8.61), yields

$$V^e(\varphi) = \frac{\lambda'}{4!} \varphi^4 + \frac{\hbar\lambda'^2 \varphi^4}{256\pi^2} \left(\log \frac{\varphi^2}{M'^2} - \frac{25}{6} \right) + \mathcal{O}(\lambda'^3) . \quad (8.62)$$

Denoting by $V^e(\lambda, M)$ and $V^e(\lambda', M')$ the effective potential densities in (8.57) and (8.62), respectively, one may check that, at the order λ^2 , we in fact have

$$V^e(\lambda', M') = V^e(\lambda, M) ,$$

that is, we have the same physics.

8.6 Scaling of determinants²²

In this section we consider the scaling properties of determinants.²³ Suppose to implement the transformation

$$A \rightarrow A' = e^{ad} A ,$$

²¹ Here we add λ in the argument of V^e , to stress that we are still using the same V^e in (8.54), and therefore with the same λ .

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²³ This can be useful also for the discussion of a conformal quantum field theory.

where d is the (natural) dimension of A and a a parameter. We have

$$\zeta_{A'}(s) = \sum_{n=1}^{\infty} \frac{1}{(e^{ad}a_n)^s} = e^{-ads} \zeta_A(s) ,$$

so that

$$\det(e^{ad}A) = e^{-\zeta_{A'}(0)} = \exp\left(-\frac{d}{ds}\bigg|_{s=0} (e^{-ads} \zeta_A(s))\right) = e^{ad\zeta_A(0)} e^{-\zeta_A'(0)} .$$

Therefore, we have

$$\zeta_{A'}'(0) = \zeta_A'(0) - ad\zeta_A(0) , \tag{8.63}$$

and

$$\det A' = e^{ad\zeta_A(0)} \det A .$$

Let us now consider the effective action at order \hbar^0 for the massless $\lambda\phi^4$ theory. We saw that at order \hbar^0 the effective action coincides with the classical action. We then have

$$\Gamma_{0E}[\varphi] = - \int d^4\bar{x} \left(\varphi \bar{\partial}_\mu \bar{\partial}_\mu \varphi - \frac{\lambda}{4!} \varphi^4 \right) .$$

Such an action is invariant under the rescaling

$$x^\mu \longrightarrow x^{\mu'} = e^a x^\mu , \quad \varphi \longrightarrow \varphi' = e^{-a} \varphi . \tag{8.64}$$

However, such a dilatation symmetry is broken by quantum effects. Namely, the effective action is not invariant under such a transformation. To see this, we consider (8.48) in the massless case

$$\Gamma_{1E}[\varphi] = -\frac{1}{2} \zeta'_{[-\bar{\partial}_\mu \bar{\partial}_\mu + \lambda\varphi^2/2]}(0) . \tag{8.65}$$

Under the rescaling (8.64) we have

$$\bar{\partial}_\mu \bar{\partial}_\mu - \frac{\lambda}{2} \varphi^2 \longrightarrow e^{-2a} \left(\bar{\partial}_\mu \bar{\partial}_\mu - \frac{\lambda}{2} \varphi^2 \right) .$$

On the other hand, by (8.63),

$$\zeta'_{[e^{-2a}(-\bar{\partial}_\mu \bar{\partial}_\mu + \lambda\varphi^2/2)]}(0) = \zeta'_{[-\bar{\partial}_\mu \bar{\partial}_\mu + \lambda\varphi^2/2]}(0) + 2a \zeta_{[-\bar{\partial}_\mu \bar{\partial}_\mu + \lambda\varphi^2/2]}(0) ,$$

that is

$$\Gamma_E[\varphi] \longrightarrow \Gamma'_E[\varphi] = \Gamma_E[\varphi] - a\hbar \zeta_{[-\bar{\partial}_\mu \bar{\partial}_\mu + \lambda\varphi^2/2]}(0) + \mathcal{O}(\hbar^2) .$$

In particular, it turns out that [34]

$$\Gamma'_E[\varphi] = \Gamma_E[\varphi] - \hbar a \frac{\lambda^2}{128\pi^2} \int d^4\bar{x} \varphi^4(\bar{x}) + \mathcal{O}(\hbar^2) .$$

This means that at the order \hbar the effect is just the following shift of the coupling constant

$$\lambda \longrightarrow \lambda' = \lambda - \frac{3\hbar a \lambda^2}{16\pi^2} . \quad (8.66)$$

This is another signal of a general property of quantum field theory: the coupling constants must be defined at some scale because even if they may be classically scale independent, this is no more true at the quantum level.

We note that the above result is consistent with the scale dependence of λ we found in the previous section. This can be explicitly seen by setting

$$a = \log \frac{M}{M'} .$$

Actually, by (8.54) and (8.58), we see that the effect of the rescaling of φ by e^{-a} is that the original M maps to $M' = e^{-a}M$. Then, by (8.60), we recover (8.66).

As we will see in the framework of the renormalisation group, the running of the coupling constant is describe in terms of the so-called β -function. Let us then consider (8.61), with λ' and M' as reference values. We then have

$$\beta(\lambda) := \frac{d\lambda}{d \log M^2} = \frac{3\hbar \lambda^2}{32\pi^2} .$$

We see then that, in the case of the theory ϕ_4^4 , the β -function is positive definite. As we will discuss later, in the case of QCD, the β -function is negative definite, meaning that at high momenta the theory is essentially a free theory. Such a phenomenon is called asymptotic freedom.

8.7 Feynman rules for the ϕ_4^4 theory²⁴

In this section we will consider the perturbative expansion of the 2- and 4-point functions, in the case of the ϕ_4^4 theory.

In the following, to simplify the notation, we will omit the corresponding subscripts and superscripts of the Euclidean formulation. The use of the Minkowskian or Euclidean formulation is clear from the context.

Let start by expressing $Z[J]$ in the form

$$Z[J] = N e^{-W_0[J]} e^{W_0[J]} e^{-\langle V(\frac{\delta}{\delta J}) \rangle} e^{-W_0[J]} ,$$

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and note that

$$W[J] = -\log N + W_0[J] - \log \left[1 + e^{W_0[J]} \left(e^{-\langle V(\frac{\delta}{\delta J})} \rangle - 1 \right) e^{-W_0[J]} \right]. \quad (8.67)$$

Such an expression leads to a convenient expansion of $Z[J]$ in terms of

$$\delta[J] := e^{W_0[J]} \left(e^{-\langle V(\frac{\delta}{\delta J})} \rangle - 1 \right) e^{-W_0[J]}. \quad (8.68)$$

Actually, by

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k, \text{ for } -1 < x \leq 1,$$

we have

$$W[J] = -\log N + W_0[J] - \delta[J] + \frac{1}{2} \delta^2[J] + \dots. \quad (8.69)$$

Expanding of $\delta[J]$ in power series of λ

$$\delta[J] = \sum_{k=1}^{\infty} \delta_k[J] \lambda^k,$$

we get

$$W[J] = -\log N + W_0[J] - \lambda \delta_1[J] - \lambda^2 (\delta_2[J] - \frac{1}{2} \delta_1^2[J]) + \dots.$$

To find the expression of the $\delta_k[J]$'s, we expand the term $\exp\left(\frac{\lambda}{4!} \langle \frac{\delta^4}{\delta J^4} \rangle\right)$ in the right-hand side of (8.68)

$$\delta[J] = e^{W_0[J]} \left(\underbrace{-\frac{\lambda}{4!} \left\langle \frac{\delta^4}{\delta J^4} \right\rangle}_{\lambda \delta_1[J]} + \underbrace{\frac{\lambda^2}{2(4!)} \left\langle \frac{\delta^4}{\delta J^4} \right\rangle \left\langle \frac{\delta^4}{\delta J^4} \right\rangle}_{\lambda^2 \delta_2[J]} + \dots \right) e^{-W_0[J]}. \quad (8.70)$$

Therefore, performing the fourth derivative and using

$$W_0[J] = -\frac{1}{2} \langle J(x) \Delta_F(x-y) J(y) \rangle_{xy},$$

we get

$$\delta_1[J] = -\frac{1}{4!} (\langle \Delta_{xa} \Delta_{xb} \Delta_{xc} \Delta_{xd} J_a J_b J_c J_d \rangle - 6 \langle \Delta_{xx} \Delta_{xa} \Delta_{xb} J_a J_b \rangle + 3 \langle \Delta_{xx}^2 \rangle),$$

and one can compute similarly any other term. An important remark is that, in the computation of the $\delta_k[J]$, one can notice that the disconnected part (i.e. the contribution that can be written as a product of functional of J , for example a possible $\delta_1^2[J]$ term in the computation of $\delta_2[J]$) drops out. This is a consequence of the linked cluster theorem that we proved, and stating that all the disconnected contributions in the ex-

pansion drop out order by order, confirming that $W[J]$ generates only and all connected Green functions.

We can thus plug the expressions of $\delta_k[J]$ in $W[J]$ and then use

$$G_c^{(N)}(x_1, \dots, x_N) = - \frac{\delta^N W[J]}{\delta J_1 \dots \delta J_N} \Big|_{J=0} .$$

Let us consider the case of the 2-point function. We have,

$$\begin{aligned} G_c^{(2)}(x_1, x_2) = & \Delta(x_2 - x_1) - \frac{\lambda}{2} \int d^4y \Delta(x_1 - y) \Delta(y - y) \Delta(y - x_2) \\ & + \frac{\lambda^2}{6} \int d^4x d^4y \Delta(x_1 - x) \Delta^3(x - y) \Delta(y - x_2) \\ & + \frac{\lambda^2}{4} \int d^4x d^4y \Delta(x_1 - x) \Delta^2(x - y) \Delta(y - y) \Delta(x - x_2) \\ & + \frac{\lambda^2}{4} \int d^4x d^4y \Delta(x_1 - x) \Delta(x - x) \Delta(x - y) \Delta(y - y) \Delta(y - x_2) + \mathcal{O}(\lambda^3) , \end{aligned}$$

and similarly other rather cumbersome expressions for the higher Green functions. In momentum space this reads

$$\begin{aligned} \tilde{G}_c^{(2)}(p, -p) = & \frac{1}{p^2 + m^2} - \frac{\lambda}{2} \frac{1}{(p^2 + m^2)^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} \\ & + \frac{\lambda^2}{6} \frac{1}{(p^2 + m^2)^2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{(2\pi)^4 \delta^{(4)}(p - q_1 - q_2 - q_3)}{(q_1^2 + m^2)(q_2^2 + m^2)(q_3^2 + m^2)} \\ & + \frac{\lambda^2}{4} \frac{1}{(p^2 + m^2)^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{(2\pi)^4 \delta^{(4)}(q_1 - q_2)}{(q_1^2 + m^2)(q_2^2 + m^2)} \\ & + \frac{\lambda^2}{4} \frac{1}{(p^2 + m^2)^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} \int \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2 + m^2)} + \mathcal{O}(\lambda^3) . \end{aligned} \tag{8.71}$$

Needless to say, we should find another way to find these expressions at any order, and here is where Feynman diagrams can greatly help in simplifying things by applying simple rules. Such rules generate all the possible contributions to the Green's function, including the disconnected ones.

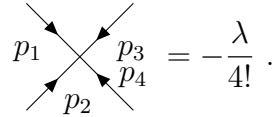
As we will see the prescription to recover the perturbative expansion is to draw all possible topologically inequivalent diagrams, by using vertexes and lines, and then identifying them with the coupling constants and propagators.

In the case of the ϕ_4^4 theory, the Feynman rules are the following

(i) Each line represents a propagator

$$\longrightarrow = \frac{1}{\bar{p}^2 + m^2} . \tag{8.72}$$

(ii) Each vertex corresponds to $-\lambda/4!$,



$$= -\frac{\lambda}{4!} .$$

(iii) Identify all the external legs and find the topological weight to each graph, i.e. the number of possible topologically equivalent graphs.

(iv) Attach

$$(2\pi)^4 \delta^{(4)} \left(\sum_{k=1}^4 p_k \right) ,$$

to each vertex and integrate over the internal momenta with

$$\int \frac{d^4 p}{(2\pi)^4} .$$

With the last rule one gets

$$(2\pi)^4 \delta^{(4)} \left(\sum_{k=1}^N p_k \right) \tilde{G}^{(N)}(p_1, \dots, p_N) ,$$

so that, to get the N -point function, one should remove the overall $(2\pi)^4 \delta^{(4)} \left(\sum_{k=1}^N p_k \right)$ term. This is what has been done in (8.71). However, note that in (8.71) there remain two more δ 's, each one multiplied by $(2\pi)^4$. The effect of each one of such δ 's is to eliminate an integration, together with a factor $(2\pi)^4$, and simultaneously reducing the number of momenta. However, it is faster to do such a simplification by writing at each vertex only the independent momenta. Therefore, it is worth replacing the fourth Feynman's rule by

(iv) Use the momentum conservation at each vertex to eliminate the redundant momenta, and then integrate over each unfixed momenta p with

$$\int \frac{d^4 p}{(2\pi)^4} .$$

In this way one directly gets $\tilde{G}^{(N)}(p_1, \dots, p_N)$ and avoids to perform the trivial integra-

tions due to the δ -functions.

Using the Feynman rules we can diagrammatically express (8.71) as

$$\begin{aligned}
 \text{---} \xrightarrow{p} \text{---} \text{---} \text{---} \xrightarrow{p} \text{---} &= \text{---} \xrightarrow{p} \text{---} + \text{---} \xrightarrow{p} \text{---} \text{---} \xrightarrow{p} \text{---} + \text{---} \xrightarrow{p} \text{---} \text{---} \text{---} \xrightarrow{p} \text{---} \\
 &+ \text{---} \xrightarrow{p} \text{---} \text{---} \text{---} \xrightarrow{p} \text{---} + \text{---} \xrightarrow{p} \text{---} \text{---} \text{---} \text{---} \xrightarrow{p} \text{---} .
 \end{aligned} \tag{8.73}$$

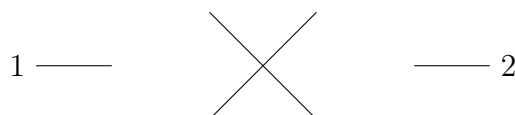
Now we see how to recover the analytical expressions for the Green function thanks to these rules. Take the so-called tadpole (the second diagram in the right-hand side of (8.73)): it has one vertex and three propagators, and one of these three corresponds to an internal momenta. Using then the rules 1, 2 and 4, we can write

$$\text{---} \xrightarrow{p} \text{---} \text{---} \text{---} \xrightarrow{p'} \text{---} = T = -\frac{M\lambda}{4!} \frac{1}{p^2 + m^2} \frac{1}{p'^2 + m^2} \int \frac{d^4q}{(2\pi)^4} \frac{(2\pi)^4 \delta^{(4)}(p - p' + q - q)}{q^2 + m^2}, \tag{8.74}$$

where M is the topological factor. Note that the right-hand side of (8.74) is a contribution to the two-point function times $(2\pi)^4 \delta(p - p')$. As we said, the above expression could be directly obtained using as forth Feynman's rule the faster alternative of writing directly the independent momenta. In other words, one may directly write

$$T = -\frac{M\lambda}{4!} \frac{1}{(p^2 + m^2)^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2}.$$

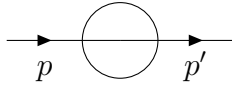
To find the value of M one decomposes the diagram in the following way



i.e on the two propagators and the vertex; then one can see that there are 4 ways to attach one “vertex leg” to the propagator 1 and then 3 ways to attach one of the remaining three vertex legs to the propagator 2; then to close the loop there remains only one choice, so $M = 4 \times 3 \times 1 = 12$.

For the setting sun (the third diagram in the left-hand side of (8.73)), we need two

external legs, two vertices and two loops; again using Feynman rules we can write

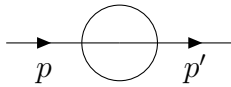


$$= \frac{\lambda^2}{(4!)^2} \frac{M}{(p^2 + m^2)(p'^2 + m^2)} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \frac{(2\pi)^8 \delta^{(4)}(p - Q) \delta^{(4)}(Q - p')}{(q_1^2 + m^2)(q_2^2 + m^2)(q_3^2 + m^2)},$$

where $Q = q_1 + q_2 + q_3$ is the sum of the right going internal propagators momenta. Note that

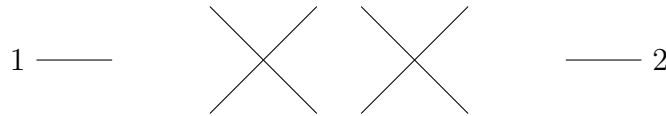
$$\delta^{(4)}(p - Q) \delta^{(4)}(Q - p') = \delta^{(4)}(p - Q) \delta^{(4)}(p' - p).$$

In this respect, as in the case of the tadpole, recognising the independent momenta directly at the vertexes avoids both the use of the δ 's, and therefore a superfluous integration, and the overall $(2\pi)^4 \delta(p' - p)$. In particular, a direct application of the simplified Feynman's rules leads to



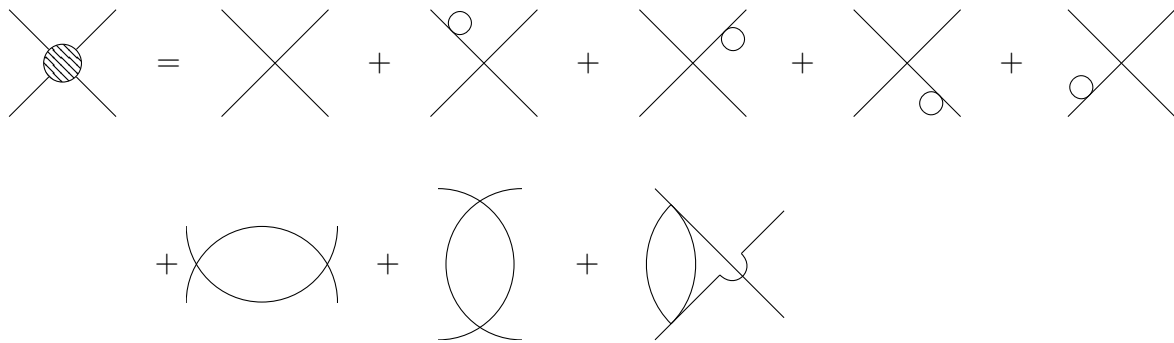
$$= \frac{\lambda^2}{(4!)^2} \frac{M}{(p^2 + m^2)^2} \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{1}{(q_1^2 + m^2)(q_2^2 + m^2)[(p - q_1 - q_2)^2 + m^2]}.$$

To determine M , again divide the diagram in the two external legs and the two vertices



now there are 4 ways to attach one vertex leg of the first vertex to the propagator 1, 4 ways to attach one vertex leg of the second vertex to the propagator 2, then 3 ways to attach one of the remaining three vertex leg on the first vertex to the second vertex and other 2 ways to attach the third vertex leg to the second vertex; then there remains only one choice, so $M = 4 \times 4 \times 3 \times 2 \times 1 = 96$.

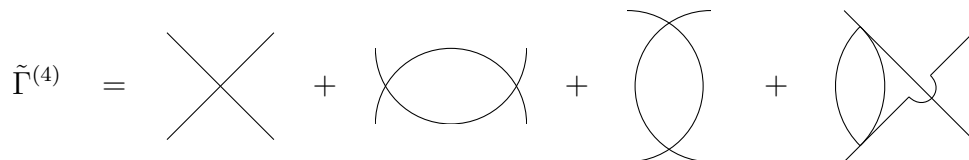
Now we use a diagrammatic representation to write the 4 point Green function:



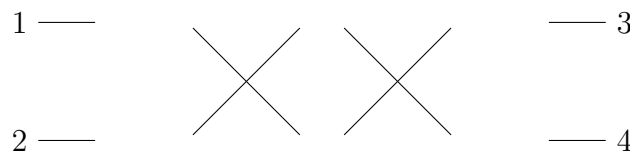
In general we have for $\tilde{G}^{(N)}(p_1, \dots, p_N)$ the multiplicative factor

$$\prod_{k=1}^N \frac{1}{p_k^2 + m^2},$$

corresponding to the external free propagators. It is simpler to restrict the analysis to the diagrams which are 1PI and that, in addition, are also amputated.²⁵ As we will see, these are the diagrams generated by the effective action. Therefore, $\tilde{\Gamma}^{(4)}$ contains only the following diagrams



and these must be computed *without considering the contributions due to the external propagators*. As an example let us see the Feynman rules for the so-called fish diagram (the second one in the right-hand side of the previous equation); first of all let us try to find out the topological factor, again with the usual decomposition in vertices and legs



now there are four ways to attach one vertex leg of the first vertex to the propagator 1, three ways to attach one of the remaining three vertex legs of the first vertex to

²⁵ Recall that these are the connected Feynman diagrams that cannot be disconnected by cutting only one internal line and with the external legs removed. Also recall that, in a generic theory, $\tilde{\Gamma}^{(N)}$, with $N \geq 3$, is said proper vertex function.

the propagator 2, then again the same reasoning applies for the second vertex and propagator 3 and 4, then two ways to attach one of the remaining two vertex leg on the first vertex to the second vertex; then there remains only one choice, so that

$$M = 4 \times 4 \times 3 \times 3 \times 2 \times 1 = 288 .$$

We then get

$$\begin{array}{c}
 p_2 \swarrow \quad \searrow p_3 \\
 \text{---} \text{---} \text{---} \\
 p_1 \swarrow \quad \searrow p_4
 \end{array}
 = \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2)[(q - p_1 - p_2)^2 + m^2]} \sim \log q .$$

Note that here we reported the contribution to the proper vertex function $\Gamma^{(4)}$, so that the external free propagators have been omitted.

To resume, $W[J]$ is the generating functional of the connected diagrams, whereas Γ is the generating functional of the 1PI amputated (or truncated) diagrams. As we will soon see, the $\tilde{\Gamma}^{(N)}$'s can be considered as building blocks of quantum field theory.

8.8 A combinatorial proof of the Feynman rules²⁶

In this section, we give a combinatorial proof of the validity of Feynman rules in position space for the ϕ_4^4 theory. The same methods can be applied in a scalar theory with generic potential $V(\phi)$.

We begin with the generating functional for a scalar theory with potential $V(\phi)$ in Minkowski space

$$Z[J] = \int \mathcal{D}\phi \exp \left[\frac{i}{\hbar} (S + \langle J\phi \rangle) \right] = \int \mathcal{D}\phi \exp \left[\frac{i}{\hbar} \left(\frac{1}{2} \langle \phi_1 \Delta_{12}^{-1} \phi_2 \rangle_{12} - V(\phi) + \langle J\phi \rangle \right) \right] ,$$

where $\Delta^{-1}(x, y)$ is the free propagator. Using the Schwinger trick one gets

$$Z[J] = \exp \left[-\frac{i}{\hbar} \left\langle V \left(-i\hbar \frac{\delta}{\delta J} \right) \right\rangle \right] Z_0[J] = \exp \left[-\frac{i}{\hbar} \left\langle V \left(-i\hbar \frac{\delta}{\delta J} \right) \right\rangle \right] \exp \left[-\frac{i}{2\hbar} \langle J_1 \Delta_{12} J_2 \rangle_{12} \right] ,$$

(we have assumed $Z_0[0] = 1$) now we expand both exponentials in series

$$Z[J] = \sum_{v=0}^{\infty} \frac{1}{v!} \left(-\frac{i}{\hbar} \left\langle V \left(-i\hbar \frac{\delta}{\delta J} \right) \right\rangle \right)^v \sum_{p=0}^{\infty} \frac{1}{p!} \left(-\frac{i}{2\hbar} \langle J_1 \Delta_{12} J_2 \rangle_{12} \right)^p .$$

²⁶ Elia de Sabbata

The N -point Green function $G^{(N)}(x_1, \dots, x_N)$ is given by

$$G^{(N)}(x_1, \dots, x_N) = \frac{1}{i^N} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} Z[J] \Big|_{J=0},$$

if we insert the expansion of $Z[J]$ in the previous equation, the generic term of the double sum reads

$$\frac{1}{v! p!} \frac{1}{i^N} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} \left(-\frac{i}{\hbar} \left\langle V \left(-i\hbar \frac{\delta}{\delta J} \right) \right\rangle \right)^v \left(-\frac{i}{2\hbar} \langle J_1 \Delta_{12} J_2 \rangle_{12} \right)^p \Big|_{J=0}.$$

Now we use $V(\phi) = \frac{\lambda}{4!} \phi^4$

$$\frac{(-i)^{p+N}}{v! p!} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} \left(\left\langle -\frac{i \lambda \hbar^4}{\hbar 4!} \frac{\delta^4}{\delta J^4} \right\rangle \right)^v \left(\frac{1}{2\hbar} \langle J_1 \Delta_{12} J_2 \rangle_{12} \right)^p \Big|_{J=0},$$

we label the v variables of the $4v$ functional derivatives in the middle by y_1, \dots, y_v , we carry out their integrations and we collect \hbar factors

$$\frac{(-i)^{p+N}}{v! p!} \hbar^{3v-p} \left(\frac{-i\lambda}{4!} \right)^v \int d^4 y_1 \cdots d^4 y_v \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} \frac{\delta^4}{\delta J(y_1)^4} \cdots \frac{\delta^4}{\delta J(y_v)^4} \left(\frac{\langle J_1 \Delta_{12} J_2 \rangle_{12}}{2} \right)^p \Big|_{J=0}.$$

Note that, since at the end of the calculation J is set to be equal to 0, in order for the term to be non-vanishing the $4v + N$ functional derivatives must cancel exactly the $2p$ external currents. This means the only non-vanishing terms are those for which $4v + N = 2p$ (so N must be even). Using the latter relation we have $(-i)^N = (-i)^{2p-4v} = (-1)^p$, thus it follows $(-i)^{p+N} = i^p$. We can bring this factor inside the integral, in the term which is elevated to the p

$$\frac{1}{v! p!} \hbar^{3v-p} \left(\frac{-i\lambda}{4!} \right)^v \int d^4 y_1 \cdots d^4 y_v \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_N)} \frac{\delta^4}{\delta J(y_1)^4} \cdots \frac{\delta^4}{\delta J(y_v)^4} \left(\frac{i \langle J_1 \Delta_{12} J_2 \rangle_{12}}{2} \right)^p \Big|_{J=0}.$$

Thanks to the Leibniz rule, the problem of finding the generic term of the N -point Green function is then reduced to sum all the possible ways in which the $4v + N$ functional derivatives act on $\left(\frac{i \langle J_1 \Delta_{12} J_2 \rangle_{12}}{2} \right)^p$ and then multiply by the right combinatorial factor.

Moreover, for each of those ways, there are $v! p!$ equivalent ways obtained by permuting y_1, \dots, y_v and the p equal terms $\frac{i \langle J_1 \Delta_{12} J_2 \rangle_{12}}{2}$.

Thus if we remove the $v! p!$ in the denominator of the previous equation we can consider y_1, \dots, y_v and the p terms $\frac{i \langle J_1 \Delta_{12} J_2 \rangle_{12}}{2}$ as ordered.

We further note that when two functional derivatives act on $\frac{i \langle J_1 \Delta_{12} J_2 \rangle_{12}}{2}$, the result is

$$\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} \frac{i \langle J_1 \Delta_{12} J_2 \rangle_{12}}{2} = i \Delta(x, y).$$

Therefore, to calculate the generic term of the N -point Green function we proceed as

follows:

- (i) We draw N arrows and we write near them x_1, \dots, x_N . Each arrow represents the action of a functional derivative $\frac{\delta}{\delta J(x_k)}$, $k = 1, \dots, n$.
- (ii) We draw v 4-vertices with 4 arrows each and we write near them $-\frac{i\lambda}{4!} \int d^4 y_k$, $k = 1, \dots, v$. Each arrow represents the action of a functional derivative $\frac{\delta}{\delta J(y_k)}$, the variable y_k being the same as the integration variable of the vertex.
- (iii) We connect pairs of arrows with lines. Each line represents a factor $\frac{i\langle J_1 \Delta_{12} J_2 \rangle_{12}}{2}$ which has to be derived by the arrows. The result of the derivation is simply $i\Delta(x, y)$, where the variables are those of the arrows (since $\Delta(x, y)$ is symmetric, we do not care about the order).
- (iv) We multiply each of the inequivalent diagrams obtained using the previous tree rules by a symmetry factor given by the possible different ways to draw an equivalent diagram (this symmetry factor comes from the fact that the 4 arrows of each vertex are not ordered, while as said before we have to consider the v vertices and the N external arrows as ordered). We then sum the contributions of all the possible inequivalent diagrams.
- (v) Finally we multiply the result by \hbar^{3v-p} . We set $I := p - N$ (the number of lines which are connected only to vertices, i.e. the internal lines) we have $3v - p = 4v - v - I - N = 2(I + N) - N - v - I - N = I - v$, which is equal to $L + 1$, where L is the number of loops of the diagram. Hence we can as well multiply the result by \hbar^{L+1} .

Notice that the previous rules, which basically are diagrammatic rules for computing the action of some derivatives on a certain product according to the Leibniz rule, are exactly the Feynman rules in the position space.

To get the full N -point Green function we observe that, since N is fixed, given v then p is fully determined by $4v + n = 2p$, hence we just need to sum all the diagrams with all possible number of vertices. Obviously, by rule (ii), if a diagram has v vertices then the contribution it gives is proportional to λ^v .

8.9 Divergences of Feynman diagrams²⁷

In doing the dimensional analysis of the terms in the Lagrangian density, we saw that a natural choice is (8.24) with $\alpha = 0$. In this case, as reported in (8.32), the dimension of ϕ is

$$[\phi] = L^{-1}[\hbar]^{1/2} .$$

²⁷ Luca Teodori

In the following we will connect the loop expansion of Green functions with the expansion in powers of \hbar . The same argument holds for the expansion of the generating functional $Z[J]$ and $W[J]$. Even in this case, the expansion in powers of \hbar should ignore the factor \hbar^2 in the Lagrangian density due to the mass term, that is $m^2 c^2 / \hbar^2 \phi^2 / 2$. As previously showed, we should set $\nu_m = mc/\hbar$ and then consider the loop expansion with ν_m treated as independent parameter, showing its correspondence with the \hbar expansion, and finally substitute ν_m by mc/\hbar . So we take for granted this passage, leaving out the c/\hbar term in mc/\hbar .

Let us consider the Minkowskian free propagator

$$\langle 0|T\phi(x)\phi(y)|0\rangle = i\hbar \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon}, \quad (8.75)$$

which is just the free propagator one would get using the Lagrangian density (8.24) by taking into account the insertion of \hbar in the right places. It is useful to make some observation, that also helps in checking the signs.

- (i) Note that the integration variable in (8.75) is $k = p/\hbar$. In this respect, recall that in the path integral representation of $Z[J]$ one should add the term \hbar^{-1} that multiplies both the action and the external source. In addition, \hbar^{-1} factors arise when considering the Fourier transform,²⁸ so that d^4p should be replaced by $d^4p/\hbar^4 = d^4k$. This means that

$$Z_0[J] = Z_0[0] \exp\left(-\frac{1}{2\hbar^2} \langle J(x)\langle 0|T\phi(x)\phi(y)|0\rangle J(y)\rangle\right). \quad (8.76)$$

As a check, note that since the dimension of J is $[J] = L^{-3}[\hbar]^{1/2}$ (see (8.32)), it follows that the exponent in the right-hand side of (8.76) is in fact dimensionless.

Also note that

$$\left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(y)} = \langle 0|T\phi(x)\phi(y)|0\rangle,$$

where the factor $(\hbar/i)^2$ is due to the fact that in the generating functional the source appears in the form $\exp(i\langle J\phi\rangle/\hbar)$. It follows that to get $\phi(x)$ in the numerator of the path integral, one should take the functional derivative with respect to $iJ(x)/\hbar$.

- (ii) The appearance of k^2 in the denominator of the integrand in (8.75) is a consequence of the absence of the \hbar^2 factor in the kinetic term of (8.24).
- (iii) As we said, it is understood that m^2 should be replaced by $\nu_m^2 = m^2 c^2 / \hbar^2$. This explains the dimensional discrepancy between k^2 and m^2 in (8.75).
- (iv) The factor \hbar multiplying the integral in (8.75) can be also obtained by observing

²⁸ Recall that, as done to derive (7.16), a way to get the expression of the Feynman propagator includes the Fourier transform of both $J(x)$ and $\phi(x)$.

that the dimension of the first member is $L^{-2}[\hbar]$.

Let us now determine, in a scalar theory with a potential density $\lambda\phi^N/N!$, the relation between

- L : number of loops ,
- I : number of internal propagators (internal lines) ,
- E : number of external propagators (external lines) ,
- V_N : number of vertexes with N legs .

Note that the number of loops is the number of independent internal momenta, unfixed by momentum conservation. On the other hand, the number of internal momenta is just the number of internal lines. However, as we now show, such I -momenta satisfy $V_N - 1$ conditions, so that the independent momenta are $L = I - (V_N - 1)$.

Let us first observe that there are V_N conditions concerning the momentum conservation, implemented by the distribution

$$-i \frac{\lambda}{N!} (2\pi)^4 \delta^{(4)} \left(\sum_{k=1}^N p_k \right) ,$$

associated to each vertex. However, these V_N conditions are redundant. In fact, Feynman's prescription requires the integration over all independent internal momenta. After such integrations there is still one δ -function assuring the conservation of the external momentum, that is

$$(2\pi)^4 \delta^{(4)} \left(\sum_{k=1}^E q_k \right) .$$

Since such a condition concerns the external momenta, it is clear that it has not direct influence on the number of independent internal momenta. It follows that the number of internal independent momenta, i.e. the number of loops, is

$$L = I - (V_N - 1) . \tag{8.77}$$

At each vertex is associated a factor \hbar^{-1} , coming from the term $\exp(-i \langle V(\phi) \rangle / \hbar)$ in the path integral. Moreover, to each line, i.e. to each free propagator, it corresponds, according to (8.75), a \hbar factor. Therefore, each diagram has a factor

$$\hbar^{I+E-V_N} = \hbar^{E+L-1} . \tag{8.78}$$

Since the expansion of each $\tilde{G}^{(E)}$ the value of E is fixed, it follows that for every Green

function the expansion in loops corresponds to the expansion in powers of \hbar , that is

$$\tilde{G}^{(E)} = \hbar^{E-1} \sum_{L=0}^{\infty} c_L \hbar^L .$$

As we have seen, the fish diagram has a divergence growing logarithmically with the momentum. In general, a divergence which arises when the momentum goes to infinity is called ultraviolet divergence. The ultraviolet divergences in Feynman diagrams are associated to the presence of loops.

Let us consider the criteria to distinguish the divergent diagrams from the finite ones. To this end note that in a Feynman diagram there are L integrations over loop momenta ℓ_k . In d -dimension each $d^d \ell_k$ contributes by a factor d to the total dimension in momenta of a given Feynman's diagram. Since there are L integrations, it follows that the total dimension in momenta due to such integrations is dL . On the other hand, each internal line corresponds to a factor $i/(q^2 - m^2 + i\epsilon)$, where q is a linear combination external momenta p_k and loop momenta ℓ_k . Each internal line gives a contribution -2 to the total dimension in momenta. This suggests introducing the *superficial degree of (ultraviolet) divergence*²⁹

$$D := dL - 2I . \quad (8.79)$$

On general grounds (but this is not always valid) we can say that a diagram converges if $D < 0$.

We now extend the analysis of the topological relations between E , I and V_N , discussed at the end of section 8.1, to the case with loops. It is clear that, as in the \hbar^0 approximation, even in this case we have

$$NV_N = 2I + E . \quad (8.80)$$

Therefore, by (8.77), (8.79) and (8.80) we get

$$D = d - \frac{1}{2}(d-2)E + V_N \left(\frac{N-2}{2}d - N \right) . \quad (8.81)$$

A useful analysis is to relate D to the dimension of the coupling constant in a Lagrangian density. Let us consider the general scalar theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \sum_{N=3}^{\infty} \frac{\lambda_N}{N!} \phi^N . \quad (8.82)$$

²⁹ Lowering the number of dimensions can make the integral convergent. Such an observation will be exploited in considering the dimensional regularisation method in the next chapter.

The diagram expansion for the E -point proper vertex function, $\tilde{\Gamma}^{(E)}$, starts with the vertex with $N = E$ legs³⁰

$$\tilde{\Gamma}^{(E)}(p_1, \dots, p_E) = -\frac{\lambda_E}{E!} + \dots . \tag{8.83}$$

Set

$$(X) = \text{momentum dimension of } X .$$

By (8.83) we have

$$(\tilde{\Gamma}^{(E)}) = (\lambda_E) .$$

On the other hand, since all the contributions to $\tilde{\Gamma}^{(E)}$ must have the same dimension, we have

$$(\tilde{\Gamma}^{(E)}) = dL - 2I + \sum_{N=3}^{\infty} V_N(\lambda_N) ,$$

that, together with $D = dL - 2I$ and $(\tilde{\Gamma}^{(E)}) = (\lambda_E)$, gives

$$D = (\lambda_E) - \sum_{N=3}^{\infty} V_N(\lambda_N) . \tag{8.84}$$

This shows that if λ_N has negative momentum dimension, then D grows with the addition of such vertex. On the other hand, since $(\partial_\mu \phi \partial^\mu \phi) = d$, we have

$$(\phi) = \frac{d-2}{2}$$

so that

$$(\lambda_N) = d - N \frac{d-2}{2} ,$$

that is

$$(\lambda_N) < 0 \text{ for } N > \frac{2d}{d-2} . \tag{8.85}$$

Let us go back to (8.81). In the case $d = 4$ it reads

$$D(d = 4) = 4 - E + (N - 4)V_N .$$

In the case of ϕ_4^4 , that is $N = 4$, we have

$$D(\phi_4^4) = 4 - E .$$

³⁰ Note that, at the tree level, the Green function $\tilde{G}^{(E)}$ may have other contributions coming by gluing legs of different vertexes $-\lambda_N/N!$. For example, besides a terms proportional to λ_6 , $\tilde{G}^{(6)}$ has tree level contributions containing the product of more λ_N 's, such as $\lambda_3\lambda_4$, whose vertexes are connected by internal lines. On the other hand, since our analysis concerns the tree level, such internal lines are not part of loops, so that, all but the contribution λ_6 only contribute to 1PR Green's functions and not to $\tilde{\Gamma}^{(6)}$.

As we will see, such a property of the ϕ_4^4 theory is crucial for its renormalisability. In particular, note that the 2- and 4-point interactions³¹ are already present in the initial Lagrangian density \mathcal{L} . On the other hand, a key point of renormalisation concerns the addition to \mathcal{L} of a Lagrangian density $\mathcal{L}_{count.}$, containing counterterms, which has the same functional structure of \mathcal{L} . The fact that D is non-negative only for $E = 2$ and $E = 4$, means that the ϕ_4^4 theory satisfies such a necessary condition for its renormalisability.

It should be also stressed that the superficial, or apparent, degree of divergence, only provides a dimensional analysis of the total dimension in momentum of a given diagram. As such it cannot be considered as a warranty of convergence or divergence of a Feynman's diagram. For example, in the case of the vertex with the four external momenta, the superficial degree of divergence is zero, so that, apparently, it would be logarithmic divergent. It is also true the opposite, namely the fact that $D = 4 - E$ cannot be considered a proof that the Green functions $\tilde{G}^{(2k+2)}$ are convergent for $k \geq 3$. It is easy to see that this cannot be the case. The reason is that one may construct a Feynman diagram starting from $\tilde{G}^{(2)}$ and/or $\tilde{G}^{(4)}$ and then attach to their legs other Feynman's diagrams to get a Green function $\tilde{G}^{(2k+2)}$ with $k \geq 3$. A simple example is the 2-point function with each one of the two legs attached to tree level 4-point function, resulting in a divergent 6-point function. Nevertheless, despite such remarks, one may easily verify by a topological analysis that all the possible divergences of $\tilde{G}^{(2k+2)}$ with $k \geq 3$ are always due to sub-diagrams which are 2- and/or 4-point functions.

We have seen that in the ϕ_4^4 theory all the ultraviolet divergences are due to the 2- and 4-point functions. Therefore, taking care of such divergent Green functions will guarantee that even all other Green functions will be finite. For such a reason $\tilde{G}^{(2)}$ and $\tilde{G}^{(4)}$ are called *primitively* divergent graphs. More generally, we have

A Feynman diagram is called primitively divergent if it becomes convergent by cutting any internal line, so that such a line transforms in two external lines.

It is clear by its definition that a primitively divergent graph has the following properties.

- (i) It does not contain any divergent subgraph. The reason is that any subgraph could be separated by cutting lines. this would mean that one could cut at least one line of the original graph without making it convergent.
- (ii) It must be a proper graph, that is, it cannot be disconnected in two sub-graphs

³¹ As we will see, the propagator also follows by considering the kinetic term, that provides the p^2 term, together with 2-leg vertex due to the term $m^2\phi^2/2$. This is not surprising, since ϕ^2 is a self-interaction and can be seen as the functional analogue of the harmonic oscillator potential.

by cutting an internal line. The reason is that divergences come from loop integrations, so that to get a convergent diagram one should cut at least a line of a divergent loop. On the other hand, if the diagram would disconnect by cutting an internal line, then such a line cannot be part of a loop.

As we will see the fact that the primitively divergent diagrams of the ϕ_4^4 theory are only $\tilde{G}_c^{(2)}$ ($D = 2$) and $\tilde{G}_c^{(4)}$ ($D = 0$), allows a systematic strategy for removing the ultraviolet divergences. A theory admitting the removal of ultraviolet divergences is called *renormalisable*.

Quantum field theories can be classified according to their kind of divergences. In particular, we have the following standard definitions.

- (i) Super-Renormalisable: the theory has a finite number of superficially divergent Feynman diagrams and divergences do not occur at all orders in perturbation theory.
- (ii) Renormalisable: the theory has a finite number of superficially divergent Feynman diagrams, but divergences occur at all orders in perturbation theory.
- (iii) Non-renormalisable: all Feynman diagrams are divergent at a sufficiently high order in perturbation theory. According to (8.84) we have that if the theory has any λ_N with negative mass dimension, then it is non-renormalisable. Nevertheless, it is important to note that even if the number of counterterms to subtract the divergences is needed, a non-renormalisable theory may provide useful predictions at energy below some ultraviolet cutoff. In this respect, it is worth mentioning that in recent years non-renormalisable theories have been widely investigated.

We now consider (8.81) and (8.85) to comment on some of the scalar theories admitting a finite number of primitively divergent graphs.

- (i) In the case $d = 4$ we have

$$D = 4 - E + (N - 4)V_N ,$$

so that for $N > 4$, D grows with the number of vertexes V_N . This means that the $\lambda\phi^N/N!$ theories with $N \geq 5$ have infinitely many primitively divergent diagrams. This also follows by (8.85) that, for $d = 4$, reads

$$(\lambda_N) < 0 \text{ for } N > 4 ,$$

implying infinitely many counterterms.

- (ii) In the case $d = 2$ we have

$$D = 2 - 2V_N ,$$

which is independent of N and E . In particular, the convergence improves with the number of vertexes. Divergences are due only to the graphs with $V_N = 0$ and $V_N = 1$ vertexes. Such theories are super-renormalisable. Since divergences arise only from loop integration, and since the maximum number of vertexes a divergent diagram may contain is one, it follows that the only possible divergences are due to a diagram in which two legs of the *same* vertex are glued by a propagator. Such a kind of divergences are removable by normal ordering the interaction term in the Lagrangian density

$$\phi^N \longrightarrow : \phi^N : ,$$

so that, renormalisation is essentially trivial in these cases.

- (iii) The highest dimension for which there exists a renormalisable scalar theory³² is $d = 6$ where the $\lambda\phi^3/6!$ theory has a dimensionless λ . In this case

$$D = 6 - 2E ,$$

so that the primitively divergent graphs are the 1-, 2- and 3-point functions, having $D = 4$, $D = 2$ and $D = 0$, respectively. Although such a theory has a potential density unbounded below, so that it seems unphysical, it possesses several interesting properties. In particular, it provides a scalar model with the property of asymptotic freedom, which is a phenomenon that usually appears in some gauge theories.

- (iv) It turns out that theories with spin 0 and spin 1/2 whose couplings have non-negative momentum dimension are renormalisable. Four-dimensional theories with a field of spin 1 are renormalisable if and only such a field is associated with a gauge symmetry. Theories with spin greater than one are not renormalisable for $d \geq 4$.

We conclude this section by stating the Weinberg theorem, that says that a sufficient, but not necessary, condition for a Feynman diagram to be convergent is that $D < 0$ is satisfied also by all its subdiagrams.

8.10 LSZ reduction formula and truncated functions

As it is known, in the computation of the scattering amplitudes one assumes that both at the far away past and at the far away future the interacting field ϕ corresponds to a free field ϕ_{in} and ϕ_{out}

$$\lim_{x_0 \rightarrow -\infty} \phi(x) = Z^{1/2} \phi_{\text{in}}(x) , \quad \lim_{x_0 \rightarrow +\infty} \phi(x) = Z^{1/2} \phi_{\text{out}}(x) ,$$

³²Note that for $d \geq 7$ all the scalar theories have infinitely many primitively divergent graphs.

where Z is the normalisation factor in

$$\langle 0|\phi(x)|1\rangle = Z^{\frac{1}{2}}\langle 0|\phi_{\text{in}}(x)|1\rangle = Z^{\frac{1}{2}}\langle 0|\phi_{\text{out}}(x)|1\rangle .$$

Let us denote by $|a, \text{in}\rangle$ ($|a, \text{out}\rangle$) a state that at the time $t = -\infty$ ($t = +\infty$) is in the state $|a\rangle$. Let us consider the unitary operator S defined by the relation

$$\langle f, \text{out}|i, \text{in}\rangle = \langle f, \text{in}|S|i, \text{in}\rangle ,$$

where $|i\rangle$ ($|f\rangle$) is the initial (final) state. One has

$$\phi_{\text{in}}(x) = S\phi_{\text{out}}(x)S^{-1} ,$$

$$|i, \text{in}\rangle = S|i, \text{out}\rangle ,$$

$$\langle f, \text{in}|S|i, \text{in}\rangle = \langle f, \text{out}|S|i, \text{out}\rangle .$$

Now consider the reduction formula, already introduced during the lectures, derived by Lehmann, Symanzik and Zimmermann, connects on-shell transition amplitudes with Green functions (in the following we will omit the shift $m^2 \rightarrow m^2 - i\epsilon$)

$$\begin{aligned} \langle p_1, \dots, p_n, \text{out}|q_1, \dots, q_m, \text{in}\rangle &= \langle p_1, \dots, p_n, \text{in}|S|q_1, \dots, q_m, \text{in}\rangle \\ &= \text{disconnected terms} \\ &+ (iZ^{-1/2})^{n+m} \int d^4y_1 \dots d^4x_m \exp \left[i \left(\sum_1^n p_j y_j - \sum_1^m q_k x_k \right) \right] \\ &\times (\square_{y_1} + m^2) \dots (\square_{x_m} + m^2) \langle \Omega|T\phi(y_1) \dots \phi(x_m)|\Omega\rangle . \end{aligned} \quad (8.86)$$

The disconnected terms do not contribute in the case that none of the initial momenta coincides with one of the final momenta. This means that the disconnected part contains terms in which at least one particle is not affected by the collision. Notice that as we are dealing with on-shell transitions, in the previous formula we have $p_j^2 = m^2$, $j = 1, \dots, n$ e $q_j^2 = m^2$, $j = 1, \dots, m$. In particular, by expressing the correlators in terms of the Green functions in momentum space, the connected part of the transition amplitude is

$$\begin{aligned} \langle p_1, \dots, p_n, \text{out}|q_1, \dots, q_m, \text{in}\rangle_c &= (2\pi)^4 \delta^{(4)} \left(\sum_1^n p_j - \sum_1^m q_k \right) (-i)^{m+n} Z^{-(m+n)/2} \\ &\times \lim_{p_j^2, q_k^2 \rightarrow m^2} \prod_{j=1}^n (p_j^2 - m^2) \prod_{k=1}^m (q_k^2 - m^2) \tilde{G}^{(m+n)}(p_1, \dots, p_n, -q_1, \dots, -q_m) . \end{aligned}$$

This shows that the S matrix is proportional to the product of the residues of the Green function³³ $\tilde{G}^{(m+n)}(p_1, \dots, p_n, -q_1, \dots, -q_m)$. Notice that since for $p^2 \sim m^2$ the inverse of the exact propagator goes like

$$1/\tilde{G}^{(2)}(p) \sim (iZ)^{-1}(p^2 - m^2) ,$$

we can replace $p^2 - m^2$ by $iZ/\tilde{G}^{(2)}(p)$, so that

$$\begin{aligned} \langle p_1, \dots, p_n, \text{out} | q_1, \dots, q_m, \text{in} \rangle_c &= (2\pi)^4 \delta^{(4)} \left(\sum_1^n p_j - \sum_1^m q_k \right) (-1)^{m+n} Z^{(m+n)/2} \\ &\times \tilde{G}_{\text{trunc}}^{(m+n)}(p_1, \dots, p_n, -q_1, \dots, -q_m) \Big|_{p_j^2 = q_k^2 = m^2} , \end{aligned} \quad (8.87)$$

where

$$\tilde{G}_{\text{trunc}}^{(N)}(p_1, \dots, p_N) = \tilde{G}^{(N)}(p_1, \dots, p_N) / \prod_{k=1}^N \tilde{G}^{(2)}(p_k) , \quad N > 2 ,$$

is the truncated Green function, i.e. with the exact external propagators removed. Feynman diagrams in the diagrammatic expansion of Green truncated functions are also called truncated diagrams.

It is interesting to note the similarity between $\tilde{G}_{\text{trunc}}^{(N)}(p_1, \dots, p_N)$ and the 1PI truncated functions

$$\tilde{\Gamma}^{(N)}(p_1, \dots, p_N) = \left[\tilde{G}_c^{(N)}(p_1, \dots, p_N) \prod_{k=1}^N (\tilde{G}_c^{(2)}(p_k))^{-1} \right] \Big|_{\text{1PI}} , \quad (8.88)$$

where the subscript 1PI denotes the selection of the 1PI graphs. To prove such a relation, note that

$$\tilde{G}_c^{(N)}(p_1, \dots, p_N) = \tilde{\Gamma}^{(N)}(p_1, \dots, p_N) \prod_{k=1}^N \tilde{G}_c^{(2)}(p_k) + \dots ,$$

where the remanent part contains the terms $\tilde{\Gamma}^{(N-k)}$, $k \in [1, N-3]$. Eq.(8.88) then follows by observing that none of such terms can correspond to a 1PI graph, because

³³ From what has been said it follows that in some ways the physically relevant quantity of Green's functions is constituted by the product of its residues. This aspect is of considerable importance because it implies that by performing a diffeomorphism of the fields the amplitudes remain unchanged. In this respect note that a field redefinition may imply a change of the transition amplitude between the vacuum and the one-particle states, $\langle 1|\phi|\Omega \rangle$, which can be done by a wave function renormalisation of the Lehmann-Symanzik-Zimmermann formula to give the same results.

The fact that the S -matrix is independent of such field redefinitions is not entirely surprising, since fields do not appear directly on the cross sections. See the discussion on page 68 in [35], and p. 447-448 of [7]. We also suggest the comments at <http://physics.stackexchange.com/questions/69828/equivalence-theorem-of-the-s-matrix>.

this would be a doubling of a graph already included in $\tilde{\Gamma}^{(N)}(p_1, \dots, p_N) \prod_{k=1}^N \tilde{G}_c^{(2)}(p_k)$. We also stress that the Green functions involved in the LSZ formula are the finite ones. Furthermore, we note that in considering the LSZ formula it is assumed that $\langle \Omega | \phi(x) | \Omega \rangle = 0$, so that³⁴

$$\tilde{G}^{(2)}(p) = \tilde{G}_c^{(2)}(p) .$$

The 1PI functions are the fundamental building blocks of the perturbative formulation. The reason is that the integration on internal momenta of a given diagram can be performed independently in each sub-diagram corresponding to a perturbative contribution to the proper vertex function. In fact, if by cutting the internal line the diagram disconnects in two parts, this means that such a line corresponds to a factor $(p^2 - m^2)^{-1}$, where p_μ is not a loop momenta. It follows that such a line cannot generate singularities.

This property of the proper vertex functions is of particular relevance in the renormalisation procedure since, to make finite a divergent diagram, it is necessary and sufficient to make finite all the 1PI sub-diagrams.

We conclude by observing that since a term in the expansion of Green functions with L loops has, by (8.78), an overall factor \hbar^{E+L-1} , it follows that the corresponding factor for $\tilde{\Gamma}^{(N)}$ is just \hbar^{L-1} .

8.11 Comments on the proper vertex functions³⁵

In this section we make some comments concerning the relations between the connected Green's functions and the proper vertex functions. Before starting let us remember the definitions of the Green's functions. In Minkowski space we have

$$G^{(N)}(x_1, \dots, x_N) = \frac{1}{i^N Z[0]} \frac{\delta^N Z[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0} ,$$

$$G_c^{(N)}(x_1, \dots, x_N) = \frac{1}{i^{N-1}} \frac{\delta^N W[J]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0} ,$$

while in the Euclidean space

$$G_E^{(N)}(\bar{x}_1, \dots, \bar{x}_N) = \frac{1}{Z_E[0]} \frac{\delta^N Z_E[J]}{\delta J(\bar{x}_1) \dots \delta J(\bar{x}_N)} \Big|_{J=0} ,$$

$$G_{cE}^{(N)}(\bar{x}_1, \dots, \bar{x}_N) = - \frac{\delta^N (\log Z_E[J])}{\delta J(\bar{x}_1) \dots \delta J(\bar{x}_N)} \Big|_{J=0} .$$

³⁴ In section 6 of the Itzykson-Zuber book there are some typos exchanging the connected with the total Green's functions.

³⁵ Matteo Turco

In theories with even potential density, that is such that $V(-\phi) = V(\phi)$, all the $2k + 1$ -point functions vanish. It is clear that since the only possible non-connected components of the 2-point function is given by the product of two 1-point functions, it follows that in the case of even potential densities the two-point Green's function and the connected two-point Green's function coincide. Equivalently,

$$\begin{aligned} G_c^{(2)}(x, y) &= -\frac{\delta^2(\log Z[J])}{\delta J(x)\delta J(y)} \Big|_{J=0} \\ &= -\frac{\delta}{\delta J(x)} \left(\frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(y)} \right) \Big|_{J=0} \\ &= \frac{1}{Z^2[0]} \frac{\delta Z[J]}{\delta J(x)} \Big|_{J=0} \frac{\delta Z[J]}{\delta J(y)} \Big|_{J=0} - \frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} \Big|_{J=0} \\ &= -\frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} \Big|_{J=0} = G^{(2)}(x, y) . \end{aligned}$$

In momentum space, the exact propagator of the ϕ_4^4 theory is

$$\tilde{G}^{(2)}(p) = \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \bigcirc \text{---} + \text{---} \bigcirc \bigcirc \text{---} + \dots$$

Let us introduce, in Minkowski space, the *proper self-energy* $\Sigma(p)$. It is defined to be $i\tilde{G}^{(2)}(p)$ with the external legs amputated, and from which are excluded the 1-particle reducible graphs. This means that $\Sigma(p)/i$ corresponds to the following diagrammatic expansion

$$\frac{\Sigma(p)}{i} = \text{---} \bigcirc \text{---} = \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \bigcirc \text{---} + \dots$$

We can expand diagrammatically $\tilde{G}^{(2)}(p)$ using the usual line (corresponding to the free propagator) and the self-energy as 2-legs vertex

that, as we said, is (8.88) with $N = 2$.

8.12 Jona-Lasinio theorem: $\Gamma[\phi]$ as generating functional of $\Gamma^{(N)}$

In the following we demonstrate an important result, due to Giovanni Jona-Lasinio, i.e. that the effective action

$$\Gamma[\phi_{\text{cl}}] = W[J] - \int d^D x J(x)\phi_{\text{cl}}(x) , \quad (8.92)$$

is the generating functional of the³⁸ $\Gamma^{(N)}$'s. The demonstration, considered in the Minkowskian, fits the one shown in section 21 of Srednicki's text and contains some additional comments.

Let us start by considering some simple aspects concerning $\Gamma[\phi_{\text{cl}}]$. First of all we observe that in the free case

$$W_0[J] = -\frac{1}{2} \langle J \Delta_F J \rangle ,$$

and

$$\phi_{\text{cl}}(x) := \frac{\delta W_0[J]}{\delta J(x)} = - \int d^D y J(y) \Delta_F(y-x) .$$

The corresponding effective action reads

$$\Gamma_0[\phi_{\text{cl}}] = W_0[J] - \int d^D x J(x)\phi_{\text{cl}}(x) = \frac{1}{2} \langle \phi_{\text{cl}} \Delta_F^{-1} \phi_{\text{cl}} \rangle = -W_0[J] , \quad (8.93)$$

where $\Delta_F^{-1}(y-x)$ denotes the inverse Feynman propagator, defined by

$$\int d^D z \Delta_F^{-1}(x-z) \Delta_F(z-y) = \delta^{(D)}(x-y) ,$$

so that

$$\Delta_F^{-1}(y-x) = \int \frac{d^D p}{(2\pi)^D} (p^2 - m^2 + i\epsilon) e^{ip(y-x)} .$$

Notice that this expression is equivalent to

$$\Delta_F^{-1}(y-x) = (-\partial_\mu \partial^\mu - m^2 + i\epsilon) \delta^{(D)}(y-x) .$$

As known, the inverse of the Feynman propagator also appears in the expression of

³⁸ Recall that for $N \geq 3$ such functions are called proper vertex functions.

$S_0[\phi]$, corresponding to the free part of the action of the scalar theory

$$\begin{aligned}
S_0[\phi] &= \frac{1}{2} \int d^D x [\partial_\mu \phi(x) \partial^\mu \phi(x) - (m^2 - i\epsilon) \phi^2(x)] \\
&= \frac{1}{2} \int d^D x \int d^D y \phi(y) (-\partial_\mu \partial^\mu - m^2 + i\epsilon) \delta^{(4)}(y-x) \phi(x) \\
&= \frac{1}{2} \int d^D x \int d^D y \phi(y) \Delta_F^{-1}(y-x) \phi(x) .
\end{aligned} \tag{8.94}$$

Recall that in the classical approximation the effective and classical actions coincide, that is

$$\Gamma_0[\phi] = S[\phi] .$$

In particular, in the free case

$$\Gamma[\phi] = \Gamma_0[\phi] = S_0[\phi] .$$

The fact that $\Gamma_0[\phi]$ matches the classical action suggests investigating which type of generating functional would we get if $S[\phi]$ is replaced by $\Gamma[\phi]$. As we will see in a moment, this new generating functional is used to show that $\Gamma[\phi]$ is the generating functional of the 1PI amputated functions. In this respect, note that until now we just claimed that the $\Gamma^{(N)}$'s are the 1PI amputated functions, but we did not give any proof that they have such a property.

The idea of the proof is to first construct a generating functional $\Gamma'[\phi]$ for the N -point functions which are really the amputated 1PI functions, let us denote such functions by $\Gamma^{(N)'$. Our aim is to show that

$$\Gamma'[\phi] = \Gamma[\phi] ,$$

which, in turn, would imply

$$\Gamma^{(N)'} = \Gamma^{(N)} .$$

Let us consider the exact connected 2-point Green function in Minkowski space

$$G_c^{(2)}(y-x) = i \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip(y-x)}}{p^2 - m^2 - \Sigma(p) + i\epsilon} .$$

Note that in the free case, where the self-energy $\Sigma(p)$ is null, $\hat{\Delta}_c(y-x) := -iG_c^{(2)}(y-x)$

reduces to the Feynman propagator. Let us define the functional

$$\begin{aligned} \Gamma'[\phi] &= \frac{1}{2} \int d^D x \int d^D y \phi(y) \hat{\Delta}_c^{-1}(y-x) \phi(x) \\ &+ \sum_{N=3}^{\infty} \frac{1}{N!} \int d^D x_1 \dots \int d^D x_N \Gamma^{(N)'}(x_1, \dots, x_N) \phi(x_1) \dots \phi(x_N), \end{aligned} \quad (8.95)$$

where

$$\hat{\Delta}_c^{-1}(y-x) = \int \frac{d^D p}{(2\pi)^D} (p^2 - m^2 - \Sigma(p) + i\epsilon) e^{ip(y-x)}.$$

The key point is to consider a new generating functional with action $\Gamma'[\phi]$

$$Z_{\Gamma'}[J] = \exp(iW_{\Gamma'}[J]) = \int \mathcal{D}\phi \exp \left[i(\Gamma'[\phi] + \int d^D x J(x)\phi(x)) \right].$$

Let us note that

- (i) the perturbative expansion of the generating functional $W_{\Gamma'}[J]$ corresponds to the infinite sum of all connected Feynman diagrams with source.
- (ii) The propagator is now the exact one of the original theory $\hat{\Delta}_c(y-x)$. For every N , $\Gamma^{(N)'}$ corresponds to a vertex that contributes with the factor $-\frac{i}{N!}\Gamma^{(N)'}$.
- (iii) In a diagram that contributes to $W_{\Gamma'}[J]$, the propagator ends can be connected to vertices and/or to the J source.
- (iv) An analysis of the diagrammatic expansion shows that, using the exact propagator and the vertex functions, the original generating functional $W[J]$ corresponds to the tree contribution to $W_{\Gamma'}[J]$.
- (v) The effective action does not describe a local quantum field theory.³⁹
- (vi) It is useful to bear in mind that if the term ϕ^N is absent in the Lagrangian density, then, at the tree level, $\Gamma^{(N)}$ is null. Thus, for example, in the ϕ_4^4 theory, the $\Gamma^{(N)}$'s, $N > 4$, are null at the tree level.

In the following we determine the tree contribution to $W_{\Gamma'}[J]$. To this end, we introduce a dimensionless parameter, \hbar' , in the following way

$$Z_{\Gamma', \hbar'}[J] = \exp(iW_{\Gamma', \hbar'}[J]) = \int \mathcal{D}\phi \exp \left[\frac{i}{\hbar'} \left(\Gamma'[\phi] + \int d^D x J(x)\phi(x) \right) \right].$$

Using Schwinger's formalism, we can express $Z_{\Gamma', \hbar'}[J]$ in the form⁴⁰ (the normalisation

³⁹ Note that this is not in contradiction with the fact that the tree level reproduces the generating functional of a local quantum field theory.

⁴⁰ Observe that the $-i\hbar'$ multiplying the functional derivative in the vertex functions is necessary to

constant is omitted)

$$Z_{\Gamma',\hbar'}[J] = \exp\left(\frac{i}{\hbar'} \sum_{N=3}^{\infty} \left\langle \Gamma^{(N)'}[-i\hbar'\delta_J] \right\rangle\right) \exp(iW_{0,\Gamma',\hbar'}[J]) , \quad (8.96)$$

where

$$W_{0,\Gamma',\hbar'}[J] = -\frac{1}{2} \left\langle \frac{J(x)}{\hbar'} \hbar' \hat{\Delta}_c(x-y) \frac{J(y)}{\hbar'} \right\rangle , \quad (8.97)$$

and

$$\left\langle \Gamma^{(N)'}[\phi] \right\rangle := \left\langle \Gamma^{(N)'}(x_1, \dots, x_N) \phi(x_1) \cdots \phi(x_N) \right\rangle .$$

Note that in $W_{0,\Gamma',\hbar'}[J]$

- (i) J is divided by \hbar' ,
- (ii) the exact propagator is multiplied by \hbar' .

The perturbative series is obtained by power expanding both exponentials in (8.96)

$$Z_{\Gamma',\hbar'}[J] = \sum_{V=0}^{\infty} \frac{1}{V!} \left(\frac{i}{\hbar'} \sum_{N=3}^{\infty} \left\langle \Gamma^{(N)'}[-i\hbar'\delta_J] \right\rangle \right)^V \sum_{P=0}^{\infty} \frac{1}{P!} \left(-\frac{1}{2} \left\langle \frac{J(x)}{\hbar'} \hbar' \hat{\Delta}_c(x-y) \frac{J(y)}{\hbar'} \right\rangle \right)^P .$$

Analysing the action of the functional derivatives in such an expansion, one may easily check that each diagram, connected or disconnected, satisfies the following properties

- (i) each vertex and each external source J contributes with a factor \hbar'^{-1} ,
- (ii) each propagator contributes with a factor \hbar' .
- (iii) If P is the number of propagators in a given diagram, E the number of sources⁴¹ and V the number of vertices, then such a diagram has a factor \hbar'^{P-E-V} .
- (iv) If the sources are removed, then E matches the number of external legs of a given diagram.
- (v) Since the number of loops L in a diagram with sources is the same as that of a diagram with the sources removed, we can conclude that, in the case of connected diagrams, the relation

$$P - E - V = L - 1 ,$$

already derived in the case of connected diagrams with external legs,⁴² is still valid.

cancel the i/\hbar' factor in $i\langle J\phi\rangle/\hbar'$, so that its effect is to replace $-i\hbar'\delta_J$ by ϕ .

⁴¹ Note that the source arises in the form $iJ(x)/\hbar$ integrated together with the variable x of a propagator.

⁴² In (8.78) we derived the relation $I - V_N = L - 1$. On the other hand, the number of internal lines (propagators) is equal to the total number of propagators minus the number of the external propagators.

Consider the expansion

$$W_{\Gamma',\hbar'}[J] = \sum_{L=0}^{\infty} \hbar'^{L-1} W_{\Gamma',L}[J] .$$

Now recall that the generating functional $W[J]$ corresponds to the tree contribution to $W_{\Gamma',\hbar'}[J]$, that is

$$W[J] = W_{\Gamma',L=0}[J] . \quad (8.98)$$

On the other hand, the above analysis showed that the tree approximation is just the one of order \hbar'^{-1} , that is

$$Z_{\Gamma',\hbar'}[J] = \exp \left[\frac{i}{\hbar'} \left(\Gamma'[\phi_J] + \int d^D x J(x) \phi_J(x) \right) + \mathcal{O}(\hbar'^0) \right] , \quad (8.99)$$

where ϕ_J is the solution to the equation of motion

$$\frac{\delta \Gamma'[\phi]}{\delta \phi(x)} = -J(x) . \quad (8.100)$$

Comparing (8.98) with (8.99) one has

$$W[J] = \Gamma'[\phi_J] + \int d^D x J(x) \phi_J(x) , \quad (8.101)$$

implying that

$$\phi_{\text{cl}}(x) := \frac{\delta W[J]}{\delta J(x)} = \int d^D y \frac{\delta \Gamma'[\phi_J]}{\delta \phi_J(y)} \frac{\delta \phi_J(y)}{\delta J(x)} + \phi_J(x) + \int d^D x J(y) \frac{\delta \phi_J(y)}{\delta J(x)} .$$

Since ϕ_J is solution of the equation of motion (8.100), this relation reduces to

$$\phi_{\text{cl}} = \phi_J .$$

By (8.101) it follows that Γ' evaluated at $\phi = \phi_{\text{cl}}$ is the Legendre transform of $W[J]$, i.e.

$$\Gamma'[\phi_{\text{cl}}] = W[J] - \int d^D x J(x) \phi_{\text{cl}}(x) ,$$

and by (8.92) it follows that

$$\Gamma'[\phi] = \Gamma[\phi] ,$$

which also implies that, as promised, $\Gamma[\phi]$ is the generating functional of the amputated 1PI N -point functions, that is

$$\Gamma^{(N)} = \Gamma^{(N)'} .$$

Chapter 9

Renormalisation

We have seen that integration over internal loops in Feynman diagrams gives divergent results. The approach to solve this problem is to proceed, order by order in the perturbative expansion, and show that, at each order, the quantities of physical interest (masses, coupling constants, Green's functions) can be renormalised to finite values.

A key step in renormalisation is to regularise the divergent quantities to get finite quantities. One then add, at each order of the perturbative expansion, counterterms to the Lagrangian densities in such a way that after that one gets finite Green's functions even when the regulator is removed.

There are several regularisation methods. A fundamental regularisation concerns the formulation on the lattice, which has been discussed in Chapter 1. Another example concerns a cutoff Λ on the momenta. For example, using the polar coordinates, one may check that with such a cutoff, the loop integral of the tadpole diagram reads

$$\int_{\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m^2} = \frac{m^2}{16\pi^2} \left(\frac{\Lambda^2}{m^2} - \log \frac{\Lambda^2}{m^2} \right) + \mathcal{O}[(\Lambda^{-1})^0] .$$

A problem with such a regularisation method is that it explicitly breaks translational invariance. As we will see, this is in fact useful when combining several propagators into a single one using the Feynman parameters. Another problem with the momentum cutoff is that it is difficult to maintain gauge invariance.

Another regularisation method is the one introduced by Pauli and Villars. It consists in subtracting off the same loop integral with a much larger mass

$$\frac{1}{p^2 + m^2} \longrightarrow \frac{1}{p^2 + m^2} - \frac{1}{p^2 + M^2} = \frac{M^2 - m^2}{(p^2 + m^2)(p^2 + M^2)} .$$

Such a regularisation maintains translational invariance. It also possible to maintain

gauge invariance by considering the new propagator as the one associated to a scalar field but with the fermionic statistics. However, this does not satisfy the spin-statistics theorem and then violates causality and/or positivity of the energy.

In our investigation we will use the dimensional regularisation. It has two main properties are

- (i) preserves translational invariance,
- (ii) preserves gauge invariance.

The main drawback is that one must consider space-time in non-integer dimensions. In this sense, apparently, dimensional regularisation seems only a formal powerful tool to manipulate divergences. Furthermore, as we will see, dimensional regularisation concerns the regularisation of Feynman diagrams, so that it is just a perturbative regularisation scheme.

Another problem with dimensional regularisation is that it is difficult to extend the gamma's matrices to non-integer dimensions. Particularly interesting is the case of γ_5 .

9.1 Dimensional regularisation of Feynman integrals

We start by evaluating, in the Euclidean space, divergent Feynman integrals, whose general form is

$$I_4(k) = \int_{-\infty}^{+\infty} d^4l F(l, k) , \quad (9.1)$$

where, for large l , F behaves either as l^{-2} or l^{-4} .

The basic idea behind the technique of dimensional regularisation is that by lowering the number of dimensions over which one integrates, the divergences trivially disappear. Let us introduce the function

$$I(\omega, k) = \int d^{2\omega}l F(l, k) , \quad (9.2)$$

regarded as a function of a (complex) variable ω .

Dimensional regularisation is based on the use of analytic continuation of a complex function. A nice example is the analytic continuation of the Γ -function.

For a complex number z with $\text{Re } z > 0$, the Euler representation of the Γ -function is

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} . \quad (9.3)$$

The integral diverges when $\text{Re } z < 0$, because as t approaches zero, the integrand behaves as $dt/t^{1+|\text{Re } z|}$, which leads to an infinity. Starting from (9.3) we can split up

the troublesome integration limit

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\alpha} dt t^{n+z-1} + \int_{\alpha}^{\infty} dt t^{z-1} e^{-t}, \quad (9.4)$$

where α is totally arbitrary and we used the Taylor expansion for the exponential, valid for any t . The second integral is well-defined even when $\operatorname{Re} z \leq 0$ as long as $\alpha > 0$. Moreover, if $\operatorname{Re} z > 0$ the first integral can be rewritten in the following form

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\alpha^{z+n}}{z+n} + \int_{\alpha}^{\infty} dt t^{z-1} e^{-t}. \quad (9.5)$$

Note that (9.5) makes sense even for $\operatorname{Re} z < 0$, except when z is a negative integer or zero, and it defines a holomorphic function with simple poles. Furthermore, (9.5) generalises (9.4) in the sense that they are equivalent whenever $z \in \{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$ (which is obviously a set that contains an accumulation point), therefore (9.5) must be the unique analytic continuation of (9.4). Finally, (9.5) does not depend on the arbitrary coefficient α (this is also a consequence of the unicity of the analytic continuation), indeed

$$\frac{d\Gamma(z)}{d\alpha} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha^{z+n-1} - \alpha^{z-1} e^{-\alpha} = \alpha^{z-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \alpha^n - \alpha^{z-1} e^{-\alpha} = 0.$$

For $\alpha = 1$, (9.5) is the Weierstrass representation of the Γ -function.

Still, to isolate the singularities we did introduce an arbitrary scale in the process, although the final result is independent of it. We want to find the equivalent of the Weierstrass' representation for the (9.2).

Our procedure is as follows

- (i) establish a finite domain of convergence for the loop integral in the ω plane. For divergent integrals, it will typically lie to the left of the $\omega = 2$ line.
- (ii) construct a new function which overlaps with the loop integral in its domain of convergence, but is defined in a larger domain which encloses the point $\omega = 2$.
- (iii) take the limit $\omega \rightarrow 2$.

We now follow 't Hooft and Veltman to show how this is done in the case of the integral

$$I(\omega) := \int d^{2\omega} \ell \frac{1}{l^2 + m^2}. \quad (9.6)$$

Let us split up the domain of integration as

$$d^{2\omega} l \rightarrow d^4 l d^{2\omega-4} l.$$

Next in the $2\omega - 4$ space, introduce polar coordinate,¹ and set

$$L^2 = \ell_\mu \ell_\mu .$$

The integral (9.6) now reads

$$I(\omega) = \int d^4l \int d\Omega_{2\omega-4} \int_0^\infty dL L^{2\omega-5} \frac{1}{L^2 + l^2 + m^2} . \quad (9.7)$$

Doing the integration over the angles (see the Appendix), we get

$$I(\omega) = \frac{2\pi^{\omega-2}}{\Gamma(\omega-2)} \int d^4l \int_0^\infty dL L^{2\omega-5} \frac{1}{L^2 + l^2 + m^2} . \quad (9.8)$$

To study the convergence of $I(\omega)$ with respect to ω we set

$$L = x\sqrt{l^2 + m^2} , \quad (9.9)$$

so that

$$I(\omega) = \frac{2\pi^{\omega-2}}{\Gamma(\omega-2)} \int d^4l (l^2 + m^2)^{\omega-3} \int_0^\infty dx x^{2\omega-5} \frac{1}{x^2 + 1} . \quad (9.10)$$

It is now easy to see the divergences in the integral, in fact the integration over x converges in the line $2 < \omega < 3$, and over l for $\omega < 1$.²

One may check that the IR divergence is an artifact of the splitting of the measure. Consider the identity

$$L^{2\omega-6} = \frac{1}{\omega-2} \frac{d}{dL^2} (L^2)^{\omega-2} , \quad (9.11)$$

so that

$$dL L^{2\omega-5} = \frac{dL^2}{2(\omega-2)} \frac{d}{dL^2} (L^2)^{\omega-2} .$$

Then, integrating (9.8) by parts over L^2 , and throwing away the surface term,³

$$I(\omega) = \frac{\pi^{\omega-2}}{\Gamma(\omega-1)} \int d^4l \int_0^\infty dL^2 (L^2)^{\omega-2} \left(-\frac{d}{dL^2} \right) \frac{1}{L^2 + l^2 + m^2} , \quad (9.12)$$

where we used $\Gamma(\omega-1) = (\omega-2)\Gamma(\omega-2)$. Using again (9.9) one may check how by (9.11) we have moved the region of the IR divergence of a unit and now the integral is IR divergent for $\omega \leq 1$. However, the region of UV divergence is still $\omega \geq 1$.

Therefore, neither (9.12) has an overlapping region of convergence. However, we have

¹ See the Appendix.

² Note that the IR divergence is due only to the term $x^{2\omega-5}$.

³ The surface term is singular at the extremum $L^2 = 0$. A more detailed analysis shows that canceling such a term balances the fictitious IR singularity introduced by the splitting of the measure.

learned how to shift the IR problem. We then reconsider the same trick, and obtain

$$I(\omega) = \frac{\pi^{\omega-2}}{\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 (L^2)^{\omega-1} \left(-\frac{d}{dL^2} \right)^2 \frac{1}{L^2 + l^2 + m^2}, \quad (9.13)$$

an expression which is well-defined for $0 < \omega < 1$. Note that we had to move the IR convergence region two units to obtain a non-zero region of convergence. Had the loop been logarithmically divergent, one such step would have sufficed. Having obtained an expression for $I(\omega)$ convergent in a finite domain, we reached the first task in the list above. Now, we should continue by finding a region of definiteness for $I(\omega)$ in (9.13) that includes the physical point $\omega = 2$ as an isolated singularity.

The above procedure did not remove the UV divergence, to get the continuation up to the physical point $\omega = 2$ we must in some way to fix the position of the IR divergence (determined by the power in L at the numerator) and increase the power of the denominator, so that moving the region where the integral is UV divergent to the right of $\omega = 1$. We will repeat such a procedure until we get a domain that includes $\omega = 2$.

In place of (4.3.11) in Ramond's book, we use the analogous identity

$$1 = \frac{1}{5} \left(2 \frac{\partial L^2}{\partial L^2} + \frac{\partial l^\mu}{\partial l^\mu} - 1 \right). \quad (9.14)$$

Inserting such an identity in (9.13), gives

$$I(\omega) = \frac{2\pi^{\omega-2}}{\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 \frac{1}{5} \left(2 \frac{\partial L^2}{\partial L^2} + \frac{\partial l^\mu}{\partial l^\mu} - 1 \right) \frac{(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^3},$$

that, after integrating by parts, corresponds to

$$\begin{aligned} I(\omega) &= -\frac{2\pi^{\omega-2}}{5\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 \left(2L^2 \frac{\partial}{\partial L^2} + l^\mu \frac{\partial}{\partial l^\mu} + 1 \right) \frac{(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^3} \\ &= -\frac{2\pi^{\omega-2}}{5\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 \left[\frac{-6l_\mu l^\mu (L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^4} + \frac{2L^2(\omega-1)(L^2)^{\omega-2}}{(L^2 + l^2 + m^2)^3} + \right. \\ &\quad \left. - \frac{6L^2(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^4} + \frac{(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^3} \right]. \end{aligned}$$

Noticing the relation

$$I(\omega) = \frac{2\pi^{\omega-2}}{5\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 \left[6(l^2 + L^2 + m^2 - m^2) \frac{(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^4} \right] - \frac{1}{5} (2\omega - 1) I(\omega),$$

we get

$$I(\omega) = -\frac{2\omega - 1 - 6}{5}I(\omega) - 6m^2 \frac{2\pi^{\omega-2}}{5\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 \frac{(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^4},$$

that is

$$I(\omega) = -\frac{3m^2}{\omega - 1} \frac{2\pi^{\omega-2}}{\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 \frac{(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^4}, \quad (9.15)$$

which is (4.3.13) of Ramond book. The result is interesting because what was a divergence extended to the whole plane for $\omega \geq 1$, now is reduced to a simple pole at $\omega = 1$. We have therefore analytically continued our function, in fact now the integral is UV finite in the region $\omega < 2$. Using again the identity (9.14) by inserting it in (9.15), we get

$$I(\omega) = \frac{4!m^4}{(\omega - 1)(\omega - 2)} \frac{\pi^{\omega-2}}{\Gamma(\omega)} \int d^4l \int_0^\infty dL^2 \frac{(L^2)^{\omega-1}}{(L^2 + l^2 + m^2)^5}. \quad (9.16)$$

This is the desired result. The key point is that there is a common region which is both IR and UV finite, namely

$$0 < \omega < 3,$$

except for the ultraviolet divergence at

$$\omega = 2.$$

As we will see, we now have an expression that can be treated by analytical methods.

Let us summarise. We first define a finite integral in the ω plane to be what we mean by (9.2): in this case it is the expression given by (9.13), and constitutes our starting point. Then if the region of convergence does not include $\omega = 2$, we continue analytically by iterating the tricky insertion of 1 in the integrand, as done above.

It would be nice to show that for a convergent integral, the procedure that leads to (9.13) indeed gives the right answer. Take as an example the convergent integral⁴

$$J(\omega) = \int d^{2\omega}l \frac{1}{(l^2 + m^2)^6}.$$

It is easy to see that the same procedure leads to the expression⁵

$$J(\omega) = \frac{\pi^{\omega-2}}{\Gamma(\omega - 1)} \int d^4l \int_0^\infty dL^2 (L^2)^{\omega-2} \left(-\frac{d}{dL^2} \right) \frac{1}{(L^2 + l^2 + m^2)^6},$$

⁴ Convergent in 4 dimension.

⁵ The only change is the power in the denominator in the initial function F .

which is finite at $\omega = 2$. We find

$$J(2) = \int d^4l \frac{-1}{(L^2 + l^2 + m^2)^6} \Big|_0^\infty = \int d^4l \frac{1}{(l^2 + m^2)^6},$$

as desired. Therefore, the procedure is entirely consistent.

If we were to blindly plug in (A.11), we would obtain

$$I(\omega) = \int d^{2\omega}l \frac{1}{l^2 + m^2} = \frac{\pi^\omega \Gamma(1 - \omega)}{(m^2)^{1-\omega}}. \quad (9.17)$$

We now show how to go from (9.16) to (9.17).

Setting $L = \sqrt{x(l^2 + m^2)}$ in (9.16) yields

$$I(\omega) = C \int d^4l (l^2 + m^2)^{\omega-5} \int_0^\infty dx \frac{x^{\omega-1}}{(x+1)^5},$$

where

$$C := \frac{4!m^4}{(\omega-1)(\omega-2)\Gamma(\omega)} \pi^{\omega-2}.$$

Using the definition of the Beta function we get

$$I(\omega) = C \frac{\Gamma(\omega)\Gamma(5-\omega)}{\Gamma(5)} \int d^4l (l^2 + m^2)^{\omega-5}.$$

Let us split the measure in the “time” and “space” components

$$\int d^4l (l^2 + m^2)^{\omega-5} = \int d^3l \int_{-\infty}^\infty dl_0 (l_0^2 + \mathbf{l}^2 + m^2)^{\omega-5}.$$

Next, setting $l_0 = \tilde{l}_0 \sqrt{\mathbf{l}^2 + m^2}$

$$\begin{aligned} \int d^3\mathbf{l} \int_{-\infty}^\infty dl_0 (l_0^2 + \mathbf{l}^2 + m^2)^{\omega-5} &= \int d^3\mathbf{l} (\mathbf{l}^2 + m^2)^{\omega-\frac{9}{2}} \int_{-\infty}^\infty d\tilde{l}_0 (\tilde{l}_0^2 + 1)^{\omega-5} \\ &= \int d^3l (\mathbf{l}^2 + m^2)^{\omega-\frac{9}{2}} \int_0^\infty dx \frac{x^{-\frac{1}{2}}}{(x+1)^{5-\omega}}, \end{aligned}$$

where in the second equality we used the fact the integrand is an even function of \tilde{l}_0 . The second integral in the right-hand side can be expressed in terms of the Beta function. For the spatial integral, we note that the function depends only on the radius,

so passing in polar coordinates we obtain

$$\begin{aligned} \int d^3\mathbf{l} (\mathbf{l}^2 + m^2)^{\omega - \frac{9}{2}} &= 4\pi (m^2)^{\omega - \frac{9}{2}} \int_0^\infty dr r^2 \left(\frac{r^2}{m^2} + 1 \right)^{\omega - \frac{9}{2}} \\ &= 2\pi m^{2\omega - 6} \int_0^\infty dx \frac{x^{\frac{1}{2}}}{(x + 1)^{\frac{9}{2} - \omega}}, \end{aligned}$$

that can be easily computed. Putting together all the pieces leads to

$$I(\omega) = \frac{4!m^4}{(\omega - 1)(\omega - 2)\Gamma(\omega)} \frac{\pi^{\omega - 2}\Gamma(\omega)\Gamma(5 - \omega)}{\Gamma(5)} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{9}{2} - \omega)}{\Gamma(5 - \omega)} \frac{\Gamma(\frac{3}{2})\Gamma(3 - \omega)}{\Gamma(\frac{9}{2} - \omega)} 2\pi m^{2\omega - 6}.$$

Making the various simplifications and remembering the values for the Γ -function we get

$$I(\omega) = \frac{\Gamma(3 - \omega)}{(\omega - 1)(\omega - 2)} \pi^\omega m^{2\omega - 2},$$

which is equivalent to (9.17).

Set

$$\epsilon = 2 - \omega,$$

and consider the following expansion⁶ of $\Gamma(-n + \epsilon)$ for $n = 0, 1, 2, \dots$ around $\epsilon = 0$

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n + 1) + \frac{\epsilon}{2} \left(\frac{\pi^2}{3} + \psi^2(n + 1) - \psi'(n + 1) \right) + \mathcal{O}(\epsilon^2) \right], \quad (9.18)$$

where

$$\psi(n + 1) = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \gamma_E, \quad \psi(s) = \frac{d \log(\Gamma(s))}{ds}, \quad (9.19)$$

furthermore

$$\psi'(n + 1) = \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2}, \quad \psi'(1) = \frac{\pi^2}{6}, \quad (9.20)$$

and γ_E being the Euler-Mascheroni constant

$$\psi(1) = -\gamma_E = -0.5772\dots \quad (9.21)$$

The result is

$$\lim_{\omega \rightarrow 2} \int d^2\omega l \frac{1}{l^2 + m^2} = -\pi^2 m^2 \left(\frac{1}{2 - \omega} + \psi(2) \right) + \mathcal{O}(2 - \omega). \quad (9.22)$$

In appendix A we investigate some of the above formulas.

⁶ See the Appendix for a derivation of such an expansion.

9.2 Evaluation of Feynman integrals

We have seen in the previous section how to calculate integrals in arbitrary dimensions and thanks to this we are able to calculate the Feynman diagrams for the $\lambda\phi^4$ theory. In 2ω dimensions in general the coupling constant λ is no longer dimensionless. We find it convenient to redefine it in terms of a dimensionless coupling constant by the artifact

$$\lambda_{old} = \lambda_{new}(\mu^2)^{2-\omega} ,$$

where λ_{new} is dimensionless and μ , called 't Hooft mass parameter, is an arbitrary parameter that will play the role of mass scale.

We note how the introduction of this parameter is mandatory if we want to maintain the coupling constant dimensionless.

In the following we suppress the subscript in λ_{new} . Therefore, the action in 2ω dimension reads

$$S_w[\phi] = \int d^{2\omega}x \left[\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} (\mu^2)^{2-\omega} \phi^4 \right] . \quad (9.23)$$

The Feynman rules for this theory are the same as the ones for the theory in four-dimensions with three exceptions

- (i) the scalar product between vectors is summed over their 2ω components,
- (ii) the loop integrals are now in 2ω -dimension, that is

$$\int \frac{d^{2\omega} \ell}{(2\pi)^{2\omega}} ,$$

- (iii) the vertex strength $-\lambda/4!$ is replaced by $(\mu^2)^{2-\omega}(-\lambda/4!)$.

Let us evaluate the lowest order diagrams for this theory. We start with the ‘‘tadpole’’ diagram⁷

$$\text{---} \bigcirc \text{---} \equiv T = -\frac{1}{2}(\lambda)(\mu^2)^{2-\omega} \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{1}{l^2 + m^2} , \quad (9.24)$$

that by (9.17) corresponds to

$$T = -\frac{\lambda m^2}{2(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2} \right)^{2-\omega} \Gamma(1-\omega) , \quad (9.25)$$

where we kept m^2 in front because the diagram has dimension of mass squared.

⁷ In the following, except for particular cases, we will omit the external free propagators.

By expanding around $\omega = 2$, we get by (9.18)

$$T = -\frac{\lambda m^2}{32\pi^2} \left[1 + (2 - \omega) \log \frac{4\pi\mu^2}{m^2} + \dots \right] \left(-\frac{1}{2 - \omega} - \psi(2) + \dots \right), \quad (9.26)$$

where we used (9.18) and the relation $x^{2-\omega} = e^{(2-\omega)\log x}$. We can rewrite (9.26) in the following way

$$T = \frac{\lambda m^2}{32\pi^2} \left(\frac{1}{2 - \omega} + \psi(2) - \log \frac{m^2}{4\pi\mu^2} + \mathcal{O}(2 - \omega) \right). \quad (9.27)$$

We observe how the introduction of μ allows us to keep track of the dimension, moreover one of the two terms that survives the $\omega \rightarrow 2$ limit is due to the cancellation of the Γ pole with the zero in the expansion of $(4\pi\mu^2/m^2)^{2-\omega}$. We conclude that the divergence of T appears as a simple pole, and that the finite part of T is totally arbitrary as a change of μ^2 affects it.

The next diagram is the “fish”

$$\text{fish diagram} = F, \quad p_1 + p_2 + p_3 + p_4 = 0.$$

The Feynman rules give

$$F = \frac{\lambda^2 (\mu^2)^{4-2\omega}}{2} \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{1}{l^2 + m^2} \frac{1}{(l - p)^2 + m^2}. \quad (9.28)$$

Note that there are three possible channels

$$p = p_1 + p_2, \quad p = p_1 + p_3, \quad p = p_1 + p_4,$$

corresponding to the s -, t - and u -channel contributions.

When there is more than one propagator taking part in a loop integration, it is convenient to introduce the Feynman parametrisation. Even if the case of (9.28) is trivial, the Feynman parametrisation plays a key role in several contexts, so that it is worth considering the general formula

$$\frac{1}{A_1 \cdots A_m} = (m-1)! \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_m \frac{\delta(1 - u_1 - \dots - u_m)}{(A_1 u_1 + \dots + A_m u_m)^m}. \quad (9.29)$$

Let us illustrate the effect of the Dirac δ in such a formula. Let us rewrite (9.29) in the more general form

$$\int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_m \delta(1 - u_1 - \dots - u_m) f(u_1, \dots, u_m). \quad (9.30)$$

Integrating over u_m , the δ -function sets

$$u_m = 1 - u_1 - \dots - u_{m-1} ,$$

in f . Furthermore, note that if the peak of the delta function fall outside of the integration interval $[0, 1]$, then the integral over u_m is zero, otherwise it is one. This is expressed by

$$\begin{aligned} & \int_0^1 du_m \delta(1 - u_1 - \dots - u_m) f(u_1, \dots, u_m) \\ &= \int_0^1 du_m \delta(1 - u_1 - \dots - u_m) f(u_1, \dots, 1 - u_1 - \dots - u_{m-1}) \\ &= \theta(1 - u_1 - \dots - u_{m-1}) \theta(u_1 + \dots + u_{m-1}) f(u_1, \dots, 1 - u_1 - \dots - u_{m-1}) . \end{aligned}$$

The second step function fixes the condition $0 \leq u_1 + \dots + u_{m-1}$, which is automatically fulfilled in subsequent integrations, because all integrations starts from 0. The first step function is non-vanishing for $u_1 + \dots + u_{m-1} \leq 1$, setting the upper limit for the next integral (over u_{m-1}) to $1 - u_1 - \dots - u_{m-2}$, and so on. A simple example is

$$\begin{aligned} & \int_0^1 du_1 \int_0^1 du_2 \int_0^1 du_3 \delta(1 - u_1 - u_2 - u_3) f(u_1, u_2, u_3) \\ &= \int_0^1 du_1 \int_0^{1-u_1} du_2 f(u_1, u_2, 1 - u_1 - u_2) . \end{aligned}$$

A variation of (9.29) is⁸

$$\begin{aligned} \frac{1}{A_1 \cdots A_m} &= (m-1)! \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_{m-1} \\ &\quad \times \frac{u_1^{m-2} \cdots u_{m-2}}{[A_m u_1 \cdots u_{m-1} + A_{m-1} u_1 \cdots u_{m-2} (1 - u_{m-1}) + \dots + A_1 (1 - u_1)]^m} , \end{aligned} \tag{9.31}$$

that may be easier to integrate than the standard form (9.29), as all the integration limits are the same.

It is easy to prove (9.31) directly by induction. The case $m = 2$, which is the one we need for (9.28), is trivial

$$\frac{1}{AB} = \int_0^1 du \frac{1}{[Bu + A(1-u)]^2} . \tag{9.32}$$

⁸ Such a formula has been derived in http://kodu.ut.ee/~kkannike/english/science/physics/notes/feynman_param.pdf.

Therefore, the proof reduces to show that

$$\begin{aligned} & m! \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_m \\ & \quad \times \frac{u_1^{m-1} \cdots u_{m-1}}{[A_{m+1}u_1 \cdots u_m + A_m u_1 \cdots u_{m-1}(1-u_m) + \cdots + A_1(1-u_1)]^{m+1}} \\ & = \frac{1}{A_{m+1}} \times (9.31), \end{aligned}$$

assuming that (9.31) holds. Note that the variable u_m appears only in the first two terms in the denominator. Therefore, regrouping u_m and integrating over it, we get

$$\begin{aligned} & = m! \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_m \\ & \quad \times \frac{u_1^{m-1} \cdots u_{m-1}}{[(A_{m+1} - A_m)u_1 \cdots u_m + A_m u_1 \cdots u_{m-1} + \cdots + A_1(1-u_1)]^{m+1}} \\ & = \frac{m!}{m} \frac{1}{A_{m+1} - A_m} \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_{m-1} \frac{u_1^{m-1} \cdots u_{m-1}}{u_1 \cdots u_{m-1}} \left\{ \right. \\ & \quad \left. + [(A_{m+1} - A_m)u_1 \cdots u_{m-1} + A_m u_1 \cdots u_{m-1} + \cdots + A_1(1-u_1)]^{-m} \right. \\ & \quad \left. - [A_m u_1 \cdots u_{m-1} + \cdots + A_1(1-u_1)]^{-m} \right\} \\ & = (m-1)! \frac{1}{A_m - A_{m+1}} \int_0^1 du_1 \int_0^1 du_2 \cdots \int_0^1 du_{m-1} u_1^{m-2} \cdots u_{m-2} \left\{ \right. \\ & \quad \left. + [A_{m+1}u_1 \cdots u_{m-1} + A_{m-1}u_1 \cdots (1-u_{m-1}) + \cdots + A_1(1-u_1)]^{-m} \right. \\ & \quad \left. - [A_m u_1 \cdots u_{m-1} + A_{m-1}u_1 \cdots (1-u_{m-1}) + \cdots + A_1(1-u_1)]^{-m} \right\} \\ & = \frac{1}{A_m - A_{m+1}} \left(\frac{1}{A_{m+1}A_{m-1} \cdots A_1} - \frac{1}{A_m A_{m-1} \cdots A_1} \right) \\ & = \frac{1}{A_{m-1} \cdots A_1} \frac{1}{A_m - A_{m+1}} \left(\frac{1}{A_{m+1}} - \frac{1}{A_m} \right) \\ & = \frac{1}{A_1 \cdots A_{m+1}}, \end{aligned}$$

where the last integral has been evaluated with the help of (9.31).

Another useful formula for the parametrisation of Feynman integrals is the following

one

$$\frac{1}{D_1^{a_1} \cdots D_k^{a_k}} = \frac{\Gamma(a_1 + \cdots + a_k)}{\Gamma(a_1) \cdots \Gamma(a_k)} \int_0^1 \cdots \int_0^1 dx_1 \cdots dx_k \frac{\delta(1 - x_1 - \cdots - x_k) x_1^{a_1-1} \cdots x_k^{a_k-1}}{(D_1 x_1 + \cdots + D_k x_k)^{a_1 + \cdots + a_k}} .$$

Let us go back to (9.28). As we said, this case is trivial, and it is sufficient to consider (9.32)

$$\frac{1}{(l^2 + m^2)[(l-p)^2 + m^2]} = \int_0^1 dx \frac{1}{[l^2 + m^2 - 2l \cdot p(1-x) + p^2(1-x)]^2} .$$

The denominator can be rewritten in the form

$$l'^2 + m^2 + p^2 x(1-x) ,$$

where

$$l' = l - p(1-x) .$$

We have $d^{2\omega} l' = d^{2\omega} l$, so that

$$F = \frac{\lambda^2(\mu^2)^{4-2\omega}}{2} \int_0^1 dx \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{1}{[l^2 + m^2 + p^2 x(1-x)]^2} , \quad (9.33)$$

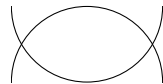
and by (A.11), that is

$$\int d^N l \frac{1}{(l^2 + a^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(a^2)^{A - N/2}} ,$$

we have

$$F = \frac{\lambda^2(\mu^2)^{4-2\omega}}{2} \int_0^1 dx \frac{\Gamma(2-\omega)}{(4\pi)^\omega} \frac{1}{[m^2 + p^2 x(1-x)]^{2-\omega}} . \quad (9.34)$$

The Laurent expansion of the integrand near $\omega = 2$, yields to $\mathcal{O}(2-\omega)$



$$\begin{aligned} &= \frac{\lambda^2(\mu^2)^{2-\omega}}{32\pi^2} \int_0^1 dx \left[\frac{1}{2-\omega} + \psi(1) - \log \left(\frac{m^2 + p^2 x(1-x)}{4\pi\mu^2} \right) \right] \\ &= \frac{\lambda^2(\mu^2)^{2-\omega}}{32\pi^2} \left[\frac{1}{2-\omega} + \psi(1) - \int_0^1 dx \log \left(\frac{m^2 + p^2 x(1-x)}{4\pi\mu^2} \right) \right] . \end{aligned}$$

Again, observe that this time the finite part depends not only on μ^2 , which is arbitrary, but also on the external momenta. Let us emphasise that this arbitrariness in the finite part is generic to the method because of the separation of a divergent expression into a divergence plus a finite part.

There remains to integrate over the Feynman parameter x . Since $x(1-x)$ is always positive over the range of integration, the argument of the logarithm is always positive, making the integral easy to evaluate. We use the following formula

$$\int_0^1 dx \log \left[1 + \frac{4}{a} x(1-x) \right] = -2 + \sqrt{1+a} \log \left(\frac{\sqrt{1+a} + 1}{\sqrt{1+a} - 1} \right), \quad a > 0. \quad (9.35)$$

The result is then

$$\begin{aligned} \text{Diagram} &= (\mu^2)^{2-\omega} \frac{\lambda^2}{32\pi^2} \left[\frac{1}{2-\omega} + \psi(1) + 2 + \log \frac{4\pi\mu^2}{m^2} \right. \\ &\quad \left. - \sqrt{1 + \frac{4m^2}{p^2}} \log \left\{ \frac{\sqrt{1 + \frac{4m^2}{p^2}} + 1}{\sqrt{1 + \frac{4m^2}{p^2}} - 1} \right\} + \mathcal{O}(2-\omega) \right]. \end{aligned}$$

In the evaluation of the four-point function, there will be three such contributions with $p = p_1 + p_2$, $p = p_1 + p_3$ and $p = p_1 + p_4$, corresponding to the s -, t - and u -channel contributions. Note that here all momenta are incoming.

This diagram is computed in the Euclidean domain; continuation to Minkowski space will entail changing the sign of p^2 and carefully interpreting the result. As it stands, however, the finite part has no interesting analytical structure as long as $p^2 > 0$.

Using the same techniques, we compute the “double scoop” diagram

$$\text{Diagram} = DS = \frac{\lambda^2 (\mu^2)^{4-2\omega}}{4} \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \frac{1}{l^2 + m^2} \int \frac{d^{2\omega} q}{(2\pi)^{2\omega}} \frac{1}{(q^2 + m^2)^2}. \quad (9.36)$$

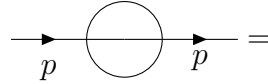
The two integrals are independent and can be solved separately using (A.11). By steps

similar to those carried out in the case of tadpole we obtain

$$\begin{aligned}
DS = & -\frac{\lambda^2 m^2}{1024\pi^4} \left\{ \frac{1}{(2-\omega)^2} + \frac{1}{2-\omega} \left(2 \log \frac{4\pi\mu^2}{m^2} + \psi(2) + \psi(1) \right) \right. \\
& + 2 \log^2 \frac{4\pi\mu^2}{m^2} + 2 \log \frac{4\pi\mu^2}{m^2} (\psi(2) + \psi(1)) + \frac{1}{2} [(\psi(2) + \psi(1))^2 \\
& \left. + \frac{2\pi^2}{3} - \psi'(2) - \psi'(1)] + \mathcal{O}(2-\omega) \right\}. \tag{9.37}
\end{aligned}$$

Note the appearance of a double pole and the arbitrariness of the residue of the simple pole and of the finite part.

Finally, we calculate the “setting sun” diagram



$$\Sigma(p) = \frac{\lambda^2 (\mu^2)^{4-2\omega}}{6} \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega} q}{(2\pi)^{2\omega}} \frac{1}{(l^2 + m^2)(q^2 + m^2)[(q+p-l)^2 + m^2]}. \tag{9.38}$$

It is interesting to note that $\Sigma(p, m^2)$ satisfies the scale relation

$$\Sigma(p, m^2) = (m^2)^{2\omega-3} \Sigma\left(\frac{p}{m}, 1\right). \tag{9.39}$$

Noticing that

$$\frac{\partial}{\partial m^2} = \frac{\partial \frac{p^\mu}{m}}{\partial m^2} \frac{\partial}{\partial \frac{p^\mu}{m}} = -\frac{p^\mu}{2m^3} \frac{\partial}{\partial \frac{p^\mu}{m}} = -\frac{k^\mu}{2m^2} \frac{\partial}{\partial k^\mu}, \quad k^\mu = \frac{p^\mu}{m}, \tag{9.40}$$

we have

$$\frac{\partial \Sigma(p, m^2)}{\partial m^2} = (2\omega - 3)(m^2)^{2\omega-4} \Sigma\left(\frac{p}{m}, 1\right) + (m^2)^{2\omega-3} \frac{\partial}{\partial m^2} \Sigma\left(\frac{p}{m}, 1\right). \tag{9.41}$$

Also note that by (9.39) we have

$$\Sigma\left(\frac{p}{m}, 1\right) = (m^2)^{3-2\omega} \Sigma(p, m^2), \tag{9.42}$$

so that by (9.41)

$$\Sigma(p, m^2) = \frac{m^2}{2\omega - 3} \frac{\partial \Sigma(p, m^2)}{\partial m^2} + \frac{(m^2)^{2\omega-3}}{2(2\omega - 3)} k^\mu \frac{\partial}{\partial k^\mu} \Sigma(k, 1). \tag{9.43}$$

The calculations to find the explicit expression of $\Sigma(p)$ are very lengthy. However,

the above scaling property of $\Sigma(p)$ is very useful to make the calculations shorter. In particular, in the Appendix are shown two different ways to derive

$$\Sigma(p) = -\frac{1}{2\omega-3} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{3m^2 + p^\mu (p+q-l)_\mu}{(l^2+m^2)(q^2+m^2)[(p+q-l)^2+m^2]^2}, \quad (9.44)$$

which is Eq.(4.4.22) in Ramond's book. It turns out that the derivation of (9.44) using the above scaling property is easier than the standard one. To find $\Sigma(p)$ requires further manipulations that we do not report. The final result is⁹

$$\Sigma(p) = -\frac{\lambda^2}{6(16\pi)^2} \left\{ \frac{3m^2}{2\epsilon^2} + \frac{3m^2}{\epsilon} \left[\frac{1}{2} + \psi(1) + \log\left(\frac{4\pi\mu^2}{m^2}\right) \right] + \frac{p^2}{4\epsilon} + \text{finite terms} \right\}. \quad (9.45)$$

Observe that we now have arbitrariness at the level of the simple pole, due to the mass scale μ , as well as at the level of the finite part. Also note the appearance of a new kind of pole whose residue is dependent on p^2 . As we will see later, this term leads to the renormalisation of ϕ . Therefore, in the ϕ_4^4 theory, the first term that leads to the wave function renormalisation appears at 2-loop, differently from what happens in the theory ϕ_6^3 , where the renormalisation of the wave function already appears at 1-loop.

9.3 Counterterms¹⁰

So far we have just regularised the divergences of the one and two-loops diagrams, i.e. we have written them in a more treatable way. What we are going to do now is to cancel them with the procedure of renormalisation. Set

$$\hat{\lambda} := \frac{\lambda}{16\pi^2}, \quad \hat{m}^2 := \frac{m^2}{4\pi\mu^2},$$

and consider the tadpole

$$\text{---}\bigcirc\text{---} = \frac{\hat{\lambda}m^2}{2} \left(\frac{1}{\epsilon} + \psi(2) - \log \hat{m}^2 + \mathcal{O}(\epsilon) \right), \quad (9.46)$$

$\epsilon = 2 - \omega \rightarrow 0$. In order to cancel the infinity given by $\hat{\lambda}m^2/(2\epsilon)$ we add the Feynman rule

$$\text{---}\times\text{---} = -\frac{\hat{\lambda}m^2}{4} \left(\frac{1}{\epsilon} + F_1(\epsilon, m^2) \right), \quad (9.47)$$

⁹ Such calculations are reported in Ramond's book.

¹⁰ Matteo Turco

where $F_1(\epsilon, m)$ is an arbitrary function, finite and analytic as $\epsilon \rightarrow 0$. By adding the counterterm, we are adding a contribution which is infinite in the $\epsilon \rightarrow 0$ limit, thus nobody forbids us to add an arbitrary term F_1 . Adding the counterterm (9.47) corresponds to modify the mass term in the Lagrangian

$$\frac{m^2}{2}\phi^2 \rightarrow \frac{m^2}{2}\left[1 + \frac{\hat{\lambda}}{2}\left(\frac{1}{\epsilon} + F_1(\epsilon, m^2)\right)\right]\phi^2,$$

so that now

$$\begin{aligned} \tilde{G}^{(2)}(p) &= \text{---} + \text{---}\bigcirc\text{---} + \text{---}\times\text{---} + \mathcal{O}(\lambda^2) \\ &= \frac{1}{p^2 + m^2} + \frac{1}{(p^2 + m^2)^2} \frac{\hat{\lambda}m^2}{2} (\psi(2) - \log \hat{m}^2 - F_1) + \mathcal{O}(\lambda^2), \end{aligned} \tag{9.48}$$

is finite even in the $\epsilon \rightarrow 0$ limit. In this respect, it is worth stressing the following mechanism. Namely, we formulated the theory in 2ω dimension. This forced us to introduce a mass scale, that is the 't Hooft parameter μ , that, apparently, one would need only for $\epsilon \neq 0$. Nevertheless, quantities such as μ^ϵ , that would disappear in the $\epsilon \rightarrow 0$ limit, survive because one first expands μ^ϵ and then takes the $\epsilon \rightarrow 0$ limit. So, for example

$$\frac{\mu^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \log \mu + \mathcal{O}(\epsilon).$$

It follows that, after subtracting the singularities by adding counterterms to the Lagrangian density, the Green's functions still have a μ -dependence, even in the $\epsilon \rightarrow 0$ limit.¹¹

As we said, the $\tilde{\Gamma}^{(N)}$'s are the building block of quantum field theory. In particular, renormalisation of $\tilde{\Gamma}^{(2)}$ and $\tilde{\Gamma}^{(4)}$ are sufficient to get finite Green's functions. We then consider these two functions. Concerning $\tilde{\Gamma}^{(2)}(p)$, we can use the relation $\tilde{G}^{(2)}(p)\tilde{\Gamma}^{(2)}(p) = 1$. To this end, we first rewrite $\tilde{G}^{(2)}(p)$ in the form

$$\tilde{G}^{(2)}(p) = (p^2 + m^2)^{-1} \left\{ 1 + (p^2 + m^2)^{-1} \left[\frac{\hat{\lambda}m^2}{2} (\psi(2) - \log \hat{m}^2 - F_1) \right] \right\} + \mathcal{O}(\lambda^2),$$

so that

$$\tilde{\Gamma}^{(2)}(p) = (p^2 + m^2) \left\{ 1 + (p^2 + m^2)^{-1} \left[\frac{\hat{\lambda}m^2}{2} (\psi(2) - \log \hat{m}^2 - F_1) \right] \right\}^{-1} + \mathcal{O}(\lambda^2).$$

¹¹ It is interesting to note that since besides μ the only parameter with the dimension of a mass is m , it follows that the only way in which μ may appear in the logarithmic terms is through the ratio m/μ . On the other hand, as we will see, the logarithmic terms are, in turn, connected with the so-called overlapping divergences. This shows an essential property of the renormalisation procedure, as it relates renormalisability to the singularity structures and the mass scale parameter μ .

Finally, using $(1+x)^{-1} \approx 1-x$, we get

$$\begin{aligned}\tilde{\Gamma}^{(2)}(p) &= (p^2 + m^2) \left\{ 1 - (p^2 + m^2)^{-1} \left[\frac{\hat{\lambda} m^2}{2} (\psi(2) - \log \hat{m}^2 - F_1) \right] \right\} + \mathcal{O}(\lambda^2) \\ &= p^2 + m^2 \left[1 - \frac{\hat{\lambda}}{2} (\psi(2) - \log \hat{m}^2 - F_1) \right] + \mathcal{O}(\lambda^2).\end{aligned}$$

Let us now consider the 4-point proper vertex function up to 1-loop

$$\begin{aligned}\tilde{\Gamma}^{(4)} &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \mathcal{O}(\lambda^3), \\ &= -\lambda \mu^{2\epsilon} \left[1 - \frac{3}{2} \hat{\lambda} \left(\frac{1}{\epsilon} + \psi(1) + 2 - \log \hat{m}^2 - \frac{1}{3} A(s, t, u) \right) + \mathcal{O}(\epsilon) \right] + \mathcal{O}(\lambda^3),\end{aligned}$$

with

$$\begin{aligned}A(s, t, u) &= \sum_{z=s, t, u} \sqrt{1 + \frac{4m^2}{z}} \log \left(\frac{\sqrt{1 + \frac{4m^2}{z}} + 1}{\sqrt{1 + \frac{4m^2}{z}} - 1} \right), \\ s &= (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2.\end{aligned}$$

If we want consider also 2-loop contributions, we have

$$\tilde{\Gamma}^{(4, 2\text{loops})} = \text{diagram 1} + \text{diagram 2} + \text{diagram 3}.$$

We modify the ϕ^4 term in the Lagrangian density to get rid of the divergent term $3\lambda \hat{\lambda} \mu^{2\epsilon} / (2\epsilon)$

$$\frac{\lambda \mu^{2\epsilon}}{4!} \phi^4 \longrightarrow \frac{\lambda \mu^{2\epsilon}}{4!} \left[1 + \frac{3\hat{\lambda}}{2} \left(\frac{1}{\epsilon} + G_1(\epsilon, m^2) \right) \right] \phi^4,$$

where G_1 is an arbitrary dimensionless function of ϵ , analytic as $\epsilon \rightarrow 0$. This new counterterm produces the new Feynman rule

$$\text{diagram 1} = -\frac{3}{2 \cdot 4!} \mu^{2\epsilon} \lambda \hat{\lambda} \left(\frac{1}{\epsilon} + G_1 \right), \quad (9.49)$$

and, with this new contribution, we have

$$\begin{aligned} \tilde{\Gamma}^{(4)} &= -\mu^{2\epsilon}\lambda\left[1 - \frac{3}{2}\hat{\lambda}\left(-G_1 + \psi(1) + 2 - \log \hat{m}^2 - \frac{1}{3}A(s, t, u)\right)\right] + \mathcal{O}(\lambda^3) \\ &= \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \mathcal{O}(\lambda^3), \end{aligned} \tag{9.50}$$

which is finite. This concludes the renormalisation up to one loop of $\tilde{\Gamma}^{(4)}$.

The counterterms that we have introduced induce new corrections to $\mathcal{O}(\lambda^2)$ to the self-energy

Here we see that we need to renormalise a theory at a given number of loops before moving on to a larger number of loops. In order to renormalise the two-point function at two loops we need to add counterterms that take into account the corrections coming from the counterterms at one-loop.

If we focus on the order λ^2 we have

$$= -\frac{\hat{\lambda}^2}{24\epsilon}p^2 + \frac{m^2\hat{\lambda}^2}{2}\left[\frac{1}{\epsilon^2} + \frac{1}{2\epsilon}(F_1 + 3G_1 + 1) + \dots\right], \tag{9.51}$$

where the dots stand for finite terms that we do not bother to write down. There are two kinds of infinities we have to deal with

- (1) $\frac{m^2\hat{\lambda}^2}{2}\left[\frac{1}{\epsilon^2} + \frac{1}{2\epsilon}(F_1 + 3G_1 + 1)\right]$,
- (2) $-\frac{\hat{\lambda}^2}{24\epsilon}p^2$, that contains p^2 coming from the “setting sun” diagram.

To cancel (1) we modify once again the mass term in the Lagrangian density by adding

the counterterm

$$\frac{m^2 \hat{\lambda}^2}{4} \left[\frac{1}{\epsilon^2} + \frac{1}{2\epsilon} (F_1 + 3G_1 + 1) + F_2 \right] \phi^2 ,$$

where F_2 is another arbitrary function of ϵ and m^2 , which is finite as $\epsilon \rightarrow 0$. The infinity (2) instead requires a modification to the kinetic term because of the presence of p^2

$$\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \longrightarrow \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \left[1 - \hat{\lambda}^2 \left(\frac{1}{24\epsilon} + H_2(\epsilon, m^2) \right) \right] ,$$

H_2 being arbitrary, finite and analytic as $\epsilon \rightarrow 0$.

All of this goes on and on at every order in λ . The very important thing to notice is that we have not added to the Lagrangian density terms that were absent in the original Lagrangian density.

An apparently critical situation in computing Feynman diagrams is when two divergent loops share a propagator. In that case one has the so-called *overlapping* divergence. The setting sun diagram is an example. It turns out that higher corrections may contain singularities with residues $\log p^2$. This would imply the addition of highly non-local terms in the Lagrangian densities. Nevertheless, it turns out that such kind of divergences cancel each other. A first signal of such a cancellation is the singularity proportional to

$$\frac{1}{\epsilon} \log \hat{m}^2 ,$$

in the double-scoop diagram (9.37). In fact, as (9.51) shows, in the total contributions at order λ^2 there is no trace of $\log \hat{m}^2$ in the singular terms.

The analysis of the overlapping divergences is the main point in the proof of the power-counting renormalisability theorem, due to Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ theorem).

9.4 About Feynman rules

Feynman rules follow from Wick's theorem. They can be obtained by the path integral formulation too¹²

$$Z[J] = \exp \left(- \int d^D x V \left(\frac{\delta}{\delta J(x)} \right) \right) \int \mathcal{D}\phi \exp \left(- S_0 + \int d^D x J(x) \phi(x) \right) ,$$

¹²In the following we omit the normalisation constants.


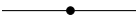
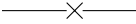
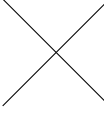
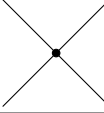
Term in \mathcal{L}_{ren}	Feynman symbols	Value
$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$		$\frac{1}{p^2}$
$\frac{m}{2}\phi^2$		$-\frac{m^2}{2}$
$\frac{m^2\hat{\lambda}}{4}\left(\frac{1}{\epsilon} + F_1\right)\phi^2$		$-\frac{m^2\hat{\lambda}}{4}\left(\frac{1}{\epsilon} + F_1\right)$
$\frac{\lambda\mu^{2\epsilon}}{4!}\phi^4$		$-\frac{\lambda\mu^{2\epsilon}}{4!}$
$\frac{3\lambda\hat{\lambda}\mu^{2\epsilon}}{2 \cdot 4!}\left(\frac{1}{\epsilon} + G_1\right)\phi^4$		$-\frac{3\lambda\hat{\lambda}\mu^{2\epsilon}}{2 \cdot 4!}\left(\frac{1}{\epsilon} + G_1\right)$

Table 9.1: Feynman rules corresponding to the term in \mathcal{L}_{ren} .

where S_0 is the free action. As we have seen, a shift of the field ϕ allows us to rewrite the free path integral in terms of the Feynman propagator:

$$Z[J] = \exp\left(-\int d^Dx V\left(\frac{\delta}{\delta J(x)}\right)\right) \exp\left(\frac{1}{2}\langle J(x)\Delta_F(x-y)J(y)\rangle\right).$$

Previously we expanded $\exp\left(-\int d^Dx V\left(\frac{\delta}{\delta J(x)}\right)\right)$ in power series of λ and obtained the Feynman rules

$$\begin{aligned} \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 &\longrightarrow \text{---} = \frac{1}{p^2 + m^2}, \\ \frac{\lambda}{4!}\phi^4 &\longrightarrow \text{X} = -\frac{\lambda}{4!}. \end{aligned}$$

The correspondence between the term $\frac{\lambda}{4!}$ and the vertex $-\frac{\lambda}{4!}$ is general: each term $c_n\phi^n$ in the Lagrangian density corresponds to the vertex $-c_n$. In particular, this is true for the quadratic term as well if we consider as free theory the massless free particle with Lagrangian density $\partial_\mu\phi\partial_\mu\phi/2$. In the case of the massive free particle we have

$$Z[J] = \exp\left(-\int d^Dx \frac{m^2}{2} \frac{\delta^2}{\delta J^2(x)}\right) \exp\left(\frac{1}{2}\langle J(x)\Delta_{F0}(x-y)J(y)\rangle\right),$$

where $\Delta_{F0}(x-y)$ is the massless Feynman propagator. This leads to the first two Feynman rules depicted in table 9.1; the only connected Green's functions in free theory

are thus given by

$$\begin{aligned}
 & \text{---} + \text{---} \text{---} \bullet \text{---} + \text{---} \text{---} \bullet \text{---} + \dots \\
 &= \frac{1}{p^2} + \frac{1}{p^2}(-m^2)\frac{1}{p^2} + \frac{1}{p^2}(-m^2)\frac{1}{p^2}(-m^2)\frac{1}{p^2} + \dots = \frac{1}{p^2} \sum_{k=0}^{\infty} \left(-\frac{m^2}{p^2} \right)^k = \frac{1}{p^2 + m^2} ,
 \end{aligned} \tag{9.52}$$

where the factors of $1/2$ in the vertices are removed by the symmetry factors of 2 due to the 2 possible ways of attaching two lines to a vertex.

Adding the counterterms, we get a new Lagrangian density giving the Feynman rules depicted in table 9.1

We now have to expand both in m^2 and λ , but the expansion in m^2 can easily be performed to all orders as in (9.52). In this way we get the same Feynman rules as in the previous section.

To conclude this section we make a comment about the Feynman rules used by Ramond. There exist in literature a different choice for assigning Feynman rules to vertices. In our convention we assigned the vertex $-c_n$ to the term $c_n\phi^n$ in the Lagrangian and computed the symmetry factor due to different possible ways of attaching lines to a vertex. The alternative is to assign the vertex $-n!c_n$, instead of $-c_n$. As explained in section 4.4 of [11], with this convention one should consider a slightly different analysis to find the weight associated to each diagram.¹³

9.5 Renormalisation group equation

Adding the counterterms defines a new Lagrangian density that, in the $\epsilon \rightarrow 0$ limit, leads to finite results. This is the renormalised Lagrangian density

$$\mathcal{L}_{\text{ren}}(\phi; m, \lambda, \mu, \epsilon) = \mathcal{L}(\phi; m, \lambda_{\text{old}}, \epsilon) + \mathcal{L}_{\text{ct}}(\phi; m, \lambda, \mu, \epsilon) ,$$

where

$$\lambda = \mu^{-2\epsilon} \lambda_{\text{old}} ,$$

$$\mathcal{L}(\phi; m, \lambda_{\text{old}}, \epsilon) = \frac{1}{2} \partial_{\mu} \phi \partial_{\mu} \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda_{\text{old}}}{4!} \phi^4 , \tag{9.53}$$

and

$$\mathcal{L}_{\text{ct}}(\phi; m, \lambda, \mu, \epsilon) = \frac{1}{2} A \partial_{\mu} \phi \partial_{\mu} \phi + \frac{1}{2} m^2 B \phi^2 + \frac{\lambda}{4!} \mu^{2\epsilon} C \phi^4 . \tag{9.54}$$

¹³ Ramond followed our convention for the vertex, while uses the alternative for the counterterms. Thus, all the counterterms in the book by Ramond have $-n!c_n$ as Feynman rule.

As we said, the term μ^ϵ , that appears both in the divergent diagrams and in the counterterms, is expanded according to¹⁴ $\mu^\epsilon = \exp(\epsilon \log \mu) = 1 + \epsilon \log \mu + \mathcal{O}(\epsilon^2)$. It follows that some positive power of ϵ in this expansion disappears because it is canceled by some pole in the divergent diagrams, so that we are left with powers of $\log \mu$ which are not multiplied by any power of ϵ . A consequence of such a mechanism is that the functional dependence on μ , that in the original Lagrangian density arises only through $\lambda_{\text{old}} = \mu^{2\epsilon} \lambda$, it is not maintained in the case of \mathcal{L}_{ct} . Another obvious reason for this is that the coefficients A , B and C of the counterterms in \mathcal{L}_{ct} depend on both λ and μ . This explains why in the case \mathcal{L}_{ct} , and therefore of \mathcal{L}_{ren} , we included in their arguments λ and μ separately. Also note that, as stressed above, this also shows that, while the μ dependence of \mathcal{L} disappears in the limit $\epsilon \rightarrow 0$, it survives both in the case of \mathcal{L}_{ren} and \mathcal{L}_{ct} .

We explicitly computed the coefficients A , B and C up to the first orders in λ in section 9.3. Setting

$$Z_\phi := 1 + A, \quad (9.55)$$

$$m_0^2 := \frac{1+B}{1+A} m^2 = (1+B) Z_\phi^{-1} m^2, \quad (9.56)$$

$$\lambda_0 := \frac{1+C}{(1+A)^2} \lambda \mu^{2\epsilon} = (1+C) Z_\phi^{-2} \lambda \mu^{2\epsilon}, \quad (9.57)$$

and

$$\phi_0 := Z_\phi^{1/2} \phi, \quad (9.58)$$

we get

$$\mathcal{L}_{\text{ren}}(\phi, m, \lambda, \mu, \epsilon) = \frac{1}{2} \partial_\mu \phi_0 \partial_\mu \phi_0 + \frac{m_0^2}{2} \phi_0^2 + \frac{\lambda_0}{4!} \phi_0^4. \quad (9.59)$$

Notice that, whereas $\lambda = \lambda_{\text{new}} = \lambda_{\text{old}} \mu^{-2\epsilon}$ is dimensionless, λ_0 , like λ_{old} , has the dimension of $\mu^{2\epsilon}$.

Eqs.(9.55), (9.56) and (9.57) express Z_ϕ , $(m/m_0)^2$ and $\lambda_0/\mu^{2\epsilon}$, in terms of¹⁵ λ , m/μ and

¹⁴ It is clear that, for dimensional reasons, the term $\log \mu$ always arises in the form $\log(\mu/m)$.

¹⁵ Since Z_ϕ , $(m/m_0)^2$ and $\lambda_0/\mu^{2\epsilon}$, like λ and ϵ , are dimensionless, it follows that their dependence on m is only through the ratio m/μ .

ϵ . Let us consider their Laurent series

$$\begin{aligned}\lambda_0 &= \mu^{2\epsilon} \left(a_0 \left(\lambda, \frac{m}{\mu}, \epsilon \right) + \sum_{k=1}^{\infty} \frac{a_k \left(\lambda, \frac{m}{\mu} \right)}{\epsilon^k} \right), \\ m_0^2 &= m^2 \left(b_0 \left(\lambda, \frac{m}{\mu}, \epsilon \right) + \sum_{k=1}^{\infty} \frac{b_k \left(\lambda, \frac{m}{\mu} \right)}{\epsilon^k} \right), \\ Z_\phi &= c_0 \left(\lambda, \frac{m}{\mu}, \epsilon \right) + \sum_{k=1}^{\infty} \frac{c_k \left(\lambda, \frac{m}{\mu} \right)}{\epsilon^k}.\end{aligned}$$

Note that a_0 , b_0 and c_0 are analytic as $\epsilon \rightarrow 0$, whereas a_k , b_k and c_k , $k \in \mathbb{N}_+$, are independent of ϵ . Furthermore, by (9.55), (9.56) and (9.57), it follows that such coefficients are determined by the coefficients A , B and C that define the counterterm Lagrangian density (9.54). We have

$$\begin{aligned}a_0 \left(\lambda, \frac{m}{\mu}, \epsilon \right) &= \lambda \left(1 + \frac{3}{2} \hat{\lambda} G_1 \right) + \mathcal{O}(\lambda^3), \\ b_0 \left(\lambda, \frac{m}{\mu}, \epsilon \right) &= 1 + \frac{1}{2} (\hat{\lambda} F_1 + \hat{\lambda}^2 F_2) + \hat{\lambda}^2 H_2 + \mathcal{O}(\lambda^3), \\ c_0 \left(\lambda, \frac{m}{\mu}, \epsilon \right) &= 1 - \hat{\lambda}^2 H_2 + \mathcal{O}(\lambda^3), \\ a_1 \left(\lambda, \frac{m}{\mu} \right) &= \frac{3}{2} \frac{\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3), \\ b_1 \left(\lambda, \frac{m}{\mu} \right) &= \frac{1}{2} \left[\hat{\lambda} + \frac{\hat{\lambda}^2}{4} (F_1 + 3G_1 + 1) \right] + \frac{\hat{\lambda}^2}{24} + \mathcal{O}(\lambda^3), \\ c_1 \left(\lambda, \frac{m}{\mu} \right) &= -\frac{\hat{\lambda}^2}{24} + \mathcal{O}(\lambda^3), \\ b_2 \left(\lambda, \frac{m}{\mu} \right) &= \frac{1}{2} \hat{\lambda}^2 + \mathcal{O}(\lambda^3), \\ &\vdots\end{aligned}$$

A crucial observation is that all the above coefficients are expected to depend on m/μ only through the arbitrary finite part F_1 , G_1 , \dots . This may be seen as a consequence of the fact that the counterterms are used to eliminate the divergences that occur in the UV limit, where mass terms should be irrelevant. If we take the arbitrary finite terms to be zero, then all the coefficients a_k , b_k and c_k are independent of m/μ .

Eq.(9.59) shows that the renormalised Lagrangian density has the same functional structure of $\mathcal{L}(\phi; m, \lambda_{\text{old}})$, that is

$$\mathcal{L}_{\text{ren}}(\phi; m, \lambda, \mu, \epsilon) = \mathcal{L}(\phi; m_0, \lambda_0, \epsilon).$$

Notice, in particular, that it is λ_{old} and not $\lambda = \lambda_{\text{new}}$ which is replaced by λ_0 . Let

$$S_{\text{ren}}[\phi; m, \lambda, \mu, \epsilon] = \int d^{2\omega}x \mathcal{L}_{\text{ren}}(\phi; m, \lambda, \mu, \epsilon) ,$$

be the renormalised action. We then have

$$S_{\text{ren}}[\phi; m, \lambda, \mu, \epsilon] = S[\phi_0; m_0, \lambda_0, \epsilon] , \quad (9.60)$$

where

$$S[\phi_0; m_0, \lambda_0, \epsilon] = \int d^{2\omega}x \mathcal{L}(\phi_0; m_0, \lambda_0, \epsilon) .$$

As follows by the renormalisation procedure, in order to get the renormalised, that is finite, Green's functions, it is necessary to consider the generating functional (in this case as well we omit the normalisation constant)

$$Z_{\text{ren}}[J; m, \lambda, \mu, \epsilon] = \int \mathcal{D}\phi \exp\left(-S_{\text{ren}}[\phi; m, \lambda, \mu, \epsilon] + \int d^{2\omega}x J(x)\phi(x)\right) .$$

In this regard it should be emphasised that the measure is $\mathcal{D}\phi$, not $\mathcal{D}\phi_0$, and the interaction with the source is via the field ϕ , not ϕ_0 . In particular, note that

$$(i) \quad \mathcal{D}\phi = N\mathcal{D}\phi_0 , \quad (9.61)$$

with N a constant that can be reabsorbed by redefining the normalisation constant of the generating functional,

(ii) there is a substantial difference in the interaction term with the source, because now it is the field ϕ , rather than ϕ_0 , that couples with the source J .

The renormalised Green's functions are

$$G_{\text{ren}}^{(N)}(\{x_k\}; m, \lambda, \mu, \epsilon) = \frac{1}{Z_{\text{ren}}[0, m, \lambda, \mu, \epsilon]} \frac{\delta^N Z_{\text{ren}}[J, m, \lambda, \mu, \epsilon]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0} .$$

Note that, as specified by the arguments of $G_{\text{ren}}^{(N)}$, we are considering the dependence of $G_{\text{ren}}^{(N)}$ on the parameters m , λ , μ and ϵ . Such a dependence can be made explicit in S_{ren} , using the expressions for Z_ϕ , m_0 and λ_0 in terms of these parameters.

Let us set

$$J_0 := Z_\phi^{-1/2} J ,$$

and note that (9.60) and (9.61) imply

$$\begin{aligned}
Z_{\text{ren}}[J; m, \lambda, \mu, \epsilon] &= \int \mathcal{D}\phi \exp\left(-S[\phi_0; m_0, \lambda_0, \epsilon] + \int d^{2\omega}x J(x)\phi(x)\right) \\
&= \int \mathcal{D}\phi_0 \exp\left(-S[\phi_0; m_0, \lambda_0, \epsilon] + \int d^{2\omega}x J(x)\phi(x)\right) \\
&= \int \mathcal{D}\phi_0 \exp\left(-S[\phi_0; m_0, \lambda_0, \epsilon] + \int d^{2\omega}x J_0(x)\phi_0(x)\right) \\
&= Z[J_0; m_0, \lambda_0, \epsilon] ,
\end{aligned}$$

that is the renormalised generating functional is just the original one, but with J , m and λ replaced by J_0 , m_0 and λ_0 , respectively. We then have

$$\begin{aligned}
G_{\text{ren}}^{(N)}(\{x_k\}; m, \lambda, \mu, \epsilon) &= \frac{1}{Z[0]} \frac{\delta^N Z[J_0]}{\delta J(x_1) \dots \delta J(x_N)} \Big|_{J=0} \\
&= \frac{Z_\phi^{-N/2}}{Z[0]} \frac{\delta^N Z[J_0]}{\delta J_0(x_1) \dots \delta J_0(x_N)} \Big|_{J_0=0} \\
&= Z_\phi^{-N/2} G^{(N)}(\{x_k\}; m_0, \lambda_0, \epsilon) ,
\end{aligned}$$

that is

$$G^{(N)}(\{x_k\}; m_0, \lambda_0, \epsilon) = Z_\phi^{N/2} G_{\text{ren}}^{(N)}(\{x_k\}; m, \lambda, \mu, \epsilon) . \quad (9.62)$$

Note that $G^{(N)}(\{x_k\}; m_0, \lambda_0, \epsilon)$ is the Green's function, which is divergent in the $\epsilon \rightarrow 0$ limit, obtained by the initial Lagrangian density \mathcal{L} , but now with m and λ_{old} replaced by m_0 and λ_0 respectively. Eq.(9.62) is a key relation between the renormalised Green's functions and the Green's functions which have *the same functional structure* of the original divergent ones, but now with *different arguments*. This implies that even in the limit $\epsilon \rightarrow 0$ and with m_0 and λ_0 considered as independent variables, $G^{(N)}(\{x_k\}; m_0, \lambda_0, \epsilon)$ has the same divergence structure of the one of $G^{(N)}(\{x_k\}; m, \lambda_{\text{old}}, \epsilon)$, with m and λ_{old} treated as independent variables.

The relation (9.62) is stable under Fourier transform and therefore is still true in momentum space. It is obvious that (9.62) is also satisfied by the connected Green's functions.

Observing that the relation $\tilde{\Gamma}^{(N)} \sim (\tilde{G}^{(2)})^{-N} \tilde{G}^{(N)}$ is stable under renormalisation and using (9.62), both in the case of generic N and $N = 2$, we see that

$$\tilde{\Gamma}^{(N)}(\{p_k\}; m_0, \lambda_0, \epsilon) = Z_\phi^{-N/2} \tilde{\Gamma}_{\text{ren}}^{(N)}(\{p_k\}; m, \lambda, \mu, \epsilon) . \quad (9.63)$$

Note that, once the arbitrary finite parts $F_1, G_1, H_1, F_2, \dots$, have been fixed, Eqs.(9.56) and (9.57) provide, for any fixed value of ϵ , two relations between five parameters. This

means that we remain with three free parameters, that is

$$\begin{aligned}\lambda_0 &= \lambda_0(m, \lambda, \mu) , \\ m_0 &= m_0(m, \lambda, \mu) .\end{aligned}$$

We can fix m_0 and λ_0 and look for which values of λ , m and μ we get the same value of λ_0 and m_0 , that is find λ' , m' and μ' , such that for any arbitrary but fixed λ_0 and m_0 one has $\lambda_0(m, \lambda, \mu) = \lambda_0(m', \lambda', \mu')$ and $m_0(m, \lambda, \mu) = m_0(m', \lambda', \mu')$, that provides a relation between λ , m and μ . It is natural to express such a relation by considering λ' and m' as dependent on the scale μ' , that is $\lambda_0(m, \lambda, \mu) = \lambda_0(m'(\mu'), \lambda'(\mu'), \mu')$ and $m_0(m, \lambda, \mu) = m_0(m'(\mu'), \lambda'(\mu'), \mu')$. Relabeling λ' , m' and μ' by λ , m and μ respectively, we have

$$\lambda_0 = \lambda_0(m(\mu), \lambda(\mu), \mu) := f_{\lambda_0}(\mu) , \quad (9.64)$$

$$m_0 = m_0(m(\mu), \lambda(\mu), \mu) := f_{m_0}(\mu) . \quad (9.65)$$

It follows that the functions $m(\mu)$ and $\lambda(\mu)$, defined in such that a variation of μ leaves both λ_0 and m_0 invariant, satisfy the equations

$$\mu \frac{d\lambda_0}{d\mu} = \mu \frac{d}{d\mu} f_{\lambda_0}(\mu) = \mu \left(\frac{\partial}{\partial \mu} + \frac{d\lambda}{d\mu} \frac{\partial}{\partial \lambda} + \frac{dm}{d\mu} \frac{\partial}{\partial m} \right) \lambda_0(m(\mu), \lambda(\mu), \mu) = 0 , \quad (9.66)$$

$$\mu \frac{dm_0}{d\mu} = \mu \frac{d}{d\mu} f_{m_0}(\mu) = \mu \left(\frac{\partial}{\partial \mu} + \frac{d\lambda}{d\mu} \frac{\partial}{\partial \lambda} + \frac{dm}{d\mu} \frac{\partial}{\partial m} \right) m_0(m(\mu), \lambda(\mu), \mu) = 0 . \quad (9.67)$$

This is the main idea underlying the renormalisation group. Such a strategy can be implemented also in the case of the 1PI amputated functions, by considering the left-hand side of (9.63) as independent, even implicitly, of μ . Actually, acting on (9.63) with

$$Z_\phi^{N/2} \mu \frac{d}{d\mu} ,$$

by keeping λ_0 and m_0 constant, we get the following differential equation involving only the renormalised 1PI amputated N -point function

$$\left(\mu \frac{\partial}{\partial \mu} + \mu \frac{d\lambda}{d\mu} \frac{\partial}{\partial \lambda} + \mu \frac{dm}{d\mu} \frac{\partial}{\partial m} - \frac{N}{2} \mu \frac{d \log Z_\phi}{d\mu} \right) \tilde{\Gamma}_{\text{ren}}^{(N)}(\{p_k\}, m, \lambda, \mu, \epsilon) = 0 . \quad (9.68)$$

Let us introduce the functions

$$\begin{aligned}\beta\left(\lambda, \frac{m}{\mu}, \epsilon\right) &:= \mu \frac{d\lambda}{d\mu}, \\ \gamma_m\left(\lambda, \frac{m}{\mu}, \epsilon\right) &:= \frac{1}{2} \mu \frac{d \log m^2}{d\mu}, \\ \gamma_d\left(\lambda, \frac{m}{\mu}, \epsilon\right) &:= \frac{1}{2} \mu \frac{d \log Z_\phi}{d\mu},\end{aligned}$$

and note that these are analytic for $\epsilon \rightarrow 0$. We then get the Callan-Symanzik equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta\left(\lambda, \frac{m}{\mu}, \epsilon\right) \frac{\partial}{\partial \lambda} + \gamma_m\left(\lambda, \frac{m}{\mu}, \epsilon\right) \frac{\partial}{\partial m} - N \gamma_d\left(\lambda, \frac{m}{\mu}, \epsilon\right)\right) \tilde{\Gamma}_{\text{ren}}^{(N)}(\{p_k\}, m, \lambda, \mu, \epsilon) = 0. \quad (9.69)$$

We now show that such an equation and Euler's theorem lead to another interesting equation for $\tilde{\Gamma}_{\text{ren}}^{(N)}$. According to Euler's theorem, we have

$$\begin{aligned}f(\alpha x_1, \dots, \alpha x_n) &= \alpha^G f(x_1, \dots, x_n), \quad \alpha > 0, \\ &\iff \\ \sum_{k=1}^n x_k \frac{\partial f}{\partial x_k} &= G f,\end{aligned}$$

where $G \in \mathbb{R}$ is called the *degree of homogeneity* of f . To exploit this theorem we compute the dimension of $\Gamma^{(N)}$. Let us first consider the case of $\tilde{G}^{(N)}$. Since¹⁶

$$\begin{aligned}(2\pi)^{2\omega} \delta^{(2\omega)}(p_1 + \dots + p_N) \tilde{G}^{(N)}(p_1, \dots, p_N) \\ = \int \left(\prod_{k=1}^N d^{2\omega} x_k e^{-ip_k \cdot x_k} \right) \langle \Omega | T \phi(x_1) \dots \phi(x_N) | \Omega \rangle,\end{aligned} \quad (9.70)$$

we get

$$M^{-2\omega} [\tilde{G}^{(N)}] = M^{-2\omega N} [\phi]^N.$$

Dimensional analysis of the kinetic term in the action gives $[\phi] = M^{\omega-1}$, which implies

$$[\tilde{G}^{(N)}] = M^{\omega(2-N)-N}.$$

Therefore, recalling that $\omega = 2 - \epsilon$, we have

$$[\tilde{\Gamma}^{(N)}] = [\tilde{G}^{(N)} (\tilde{G}^{(2)})^{-N}] = M^{\omega(2-N)+N} = M^{4-N+\epsilon(N-2)}.$$

This means that $\tilde{\Gamma}^{(N)}$ is homogeneous in its parameters with dimension M , with degree

¹⁶ Note that the Fourier transform of the Green functions is done with the asymmetric normalisation, that is it uses $\int d^D x \dots$ instead of $\int d^D x / (2\pi)^{D/2} \dots$

of homogeneity $4 - N + \epsilon(N - 2)$

$$\tilde{\Gamma}_{\text{ren}}^{(N)}(\{\alpha p_k\}, \alpha m, \lambda, \alpha \mu, \epsilon) = \alpha^{4-N+\epsilon(N-2)} \tilde{\Gamma}_{\text{ren}}^{(N)}(\{p_k\}, m, \lambda, \mu, \epsilon) .$$

By Euler theorem we then have

$$\left(\mu \frac{\partial}{\partial \mu} + s \frac{\partial}{\partial s} + m \frac{\partial}{\partial m} - [4 - N + \epsilon(N - 2)] \right) \tilde{\Gamma}_{\text{ren}}^{(N)}(\{sp_k\}, m, \lambda, \mu, \epsilon) = 0 . \quad (9.71)$$

To write this expression we have introduced s , a common scale for the momenta, and used $s \frac{\partial}{\partial s} = \sum p_k \frac{\partial}{\partial p_k}$, which holds when applied to a function whose dependence on s and the p_k 's is only through the set $\{sp_k\}$. Thanks to (9.71) we can remove the term $\mu \frac{\partial}{\partial \mu}$ from equation (9.69). Actually, this can be done because Eq.(9.69) is invariant if the momenta $\{p_k\}$ are replaced by any other set $\{p'_k\}$, so that, in particular, even $\tilde{\Gamma}_{\text{ren}}^{(N)}(\{sp_k\}, m, \lambda, \mu, \epsilon)$ is solution of (9.69). In this way, comparing (9.69) with (9.71), we get a differential equation for $\tilde{\Gamma}_{\text{ren}}^{(N)}(\{sp_k\}, m, \lambda, \mu, \epsilon)$ without the differential operator $\mu \frac{\partial}{\partial \mu}$. Furthermore, since all quantities are finite even at $\epsilon = 0$, we take the limit $\epsilon \rightarrow 0$. The resulting equation is the so-called Gell'Mann-Low equation or *renormalisation group equation*

$$\left[-s \frac{\partial}{\partial s} + \beta \left(\lambda, \frac{m}{\mu} \right) \frac{\partial}{\partial \lambda} + \left(\gamma_m \left(\lambda, \frac{m}{\mu} \right) - 1 \right) m \frac{\partial}{\partial m} - N \gamma_d \left(\lambda, \frac{m}{\mu} \right) + 4 - N \right] \tilde{\Gamma}_{\text{ren}}^{(N)}(\{sp_k\}, m, \lambda, \mu) = 0 . \quad (9.72)$$

Note that the coefficients of such an equation depend on the choice of the finite parts in the counterterms.

9.6 Renormalisation prescriptions

A renormalisation prescription corresponds to imposing some conditions to fix the arbitrary functions $F_1, G_1, H_1, F_2, \dots$. Several prescriptions have been used in literature to study different aspects of a theory and here we show some of them before focusing on the 't Hooft-Weinberg prescription.

a) We require the following conditions (from now on we omit the label ren)

$$\begin{aligned} \tilde{\Gamma}_{\text{ren}}^{(2)}(p)|_{p^2 \sim 0} &\approx p^2 + m_a^2 , \\ \tilde{\Gamma}_{\text{ren}}^{(4)}(p_1, \dots, p_4)|_{p_k=0} &= -\mu^{2\epsilon} \lambda_a , \end{aligned} \quad (9.73)$$

where m_a^2 and λ_a are fixed quantities. With some calculations one can find that

the conditions (9.73) imply

$$\begin{aligned} F_1^a &= \psi(2) - \log \frac{m_a^2}{4\pi\mu^2} , \\ G_1^a &= \psi(1) - \log \frac{m_a^2}{4\pi\mu^2} , \\ H_2^a &= 0 . \end{aligned}$$

Fixing the values of $\tilde{\Gamma}_{\text{ren}}^{(2)}$ and $\tilde{\Gamma}_{\text{ren}}^{(4)}$ at null external momenta may be dangerous in case of massless particles, because infrared divergences may arise.

- b) More in general, to avoid the infrared problem, we can do the same as in a) at an arbitrary value for the external momenta

$$\begin{aligned} \tilde{\Gamma}_{\text{ren}}^{(2)}(p)|_{p^2 \sim M^2} &= p^2 + m_b^2 , \\ \tilde{\Gamma}_{\text{ren}}^{(4)}(p_1, \dots, p_4) &= -\mu^{2\epsilon} \lambda_b \quad \text{at} \quad p_j \cdot p_k = M^2 \left(\delta_{jk} - \frac{1}{4} \right) . \end{aligned}$$

Note that the p_k 's satisfy

$$(p_j + p_k)^2 = M^2 ,$$

so that $s = t = u = M^2$. One may check that the above prescriptions imply

$$\begin{aligned} F_1^b &= \psi(2) - \log \hat{m}_b^2 , \\ G_1^b &= \psi(1) - \log \hat{m}_b^2 - \int_0^1 dx \log \left[1 + \frac{M^2}{m_b^2} x(1-x) \right] , \\ H_2^b &= 0 . \end{aligned}$$

- c) A very convenient choice is the one introduced by 't Hooft and Weinberg, also called minimal subtraction scheme, or MS scheme. The prescription consists in setting all the arbitrary functions to zero, order by order in λ

$$F_1^c = G_1^c = H_1^c = F_2^c = \dots = 0 .$$

In this way, as previously observed, the renormalisation group coefficients, a_k , b_k and c_k , are mass independent and the β , γ_d and γ_m functions are easy to compute.

- d) A widely used prescription is the so-called modified minimal subtraction, or MS-bar ($\overline{\text{MS}}$) scheme. The only difference with respect to the MS scheme concerns the following rescaling of the 't Hooft parameter

$$\mu^2 \longrightarrow \mu^2 \frac{e^{\gamma_E}}{4\pi} ,$$

that cancels a constant that always appears together with the singularities.

In the following we focus on the MS scheme. Note that, in such a prescription, we have

$$a_0 = \lambda + \mathcal{O}(\lambda^3) .$$

First of all we will explicitly find the expression for the β function. Consider the Laurent expansion in ϵ for λ_0 in this prescription

$$\lambda_0 = \mu^{2\epsilon} \left(\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k} \right) .$$

As we said, we consider λ_0 as independent variable, so that deriving with respect to μ we get

$$0 = 2\epsilon \left(\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k} \right) + \mu \frac{d\lambda}{d\mu} \left(1 + \sum_{k=1}^{\infty} \frac{a'_k(\lambda)}{\epsilon^k} \right) , \quad (9.74)$$

where

$$a'_k(\lambda) \equiv \frac{\partial a_k(\lambda)}{\partial \lambda} ,$$

$k \in \mathbb{N}$. Since $\beta = \mu d\lambda/d\mu$ is analytic for $\epsilon \rightarrow 0$, we can set $\beta = A + B\epsilon$ neglecting higher orders terms in ϵ . Therefore, in equation (9.74) and get

$$\begin{aligned} 0 &= 2\epsilon \left(\lambda + \sum_{k=1}^{\infty} \frac{a_k(\lambda)}{\epsilon^k} \right) + (A + B\epsilon) \left(1 + \sum_{k=1}^{\infty} \frac{a'_k(\lambda)}{\epsilon^k} \right) \\ &= (B + 2\lambda)\epsilon + 2a_1 + A + Ba'_1 + \sum_{k=1}^{\infty} \frac{2a_{k+1} + Aa'_k + Ba'_{k+1}}{\epsilon^k} . \end{aligned}$$

Order by order in ϵ the right-hand side must be zero, so that

$$\begin{aligned} \epsilon : \quad & B + 2\lambda = 0 , \\ \epsilon^0 : \quad & 2a_1 + A + Ba'_1 = 0 , \\ \epsilon^{-k} : \quad & 2a_{k+1} + Aa'_k + Ba'_{k+1} = 0 . \end{aligned}$$

The first two equations imply

$$B = -2\lambda , \quad A = -2 \left(1 - \lambda \frac{d}{d\lambda} \right) a_1 ,$$

so that, for $\epsilon \rightarrow 0$,

$$\beta(\lambda) = -2 \left(1 - \lambda \frac{d}{d\lambda} \right) a_1 , \quad (9.75)$$

and by

$$a_1 = \frac{3\lambda^2}{32\pi^2} + \mathcal{O}(\lambda^3) ,$$

we get

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3). \quad (9.76)$$

9.7 Scaling property of the coupling constant and Landau pole

Integrating (9.76)

$$\mu \frac{d\lambda}{d\mu} = \beta(\lambda) = \frac{3\lambda^2}{16\pi^2},$$

we find the dependence of λ on μ up to $\mathcal{O}(\lambda^2)$ of λ on μ

$$\frac{d\lambda}{\lambda^2} = \frac{3}{16\pi^2} \frac{d\mu}{\mu},$$

from which

$$\begin{aligned} -\frac{1}{\lambda(\mu)} &= \frac{3}{16\pi^2} \log \frac{\mu}{\mu_s} - \frac{1}{\lambda_s}, & \lambda_s &:= \lambda(\mu_s) \\ &\iff \\ \lambda(\mu) &= \frac{\lambda_s}{1 - \frac{3\lambda_s}{16\pi^2} \log \frac{\mu}{\mu_s}}. \end{aligned}$$

In this particular case, starting from μ_s , λ increases with μ , from the value λ_s at $\mu = \mu_s$ up to $+\infty$ when

$$\mu = \mu_s \exp\left(\frac{16\pi^2}{3\lambda_s}\right).$$

Of course, approaching this point, λ grows larger and larger and perturbation theory is not trustable anymore. This is the Landau pole.

Let us forget for a moment about ϕ^4 and examine the behaviour of the coupling constant in a few cases.

- (i) $\beta(\lambda) > 0, \forall \lambda$: λ always increases with μ . If β diverges for some λ , then λ itself is infinite. This is, as in the case of ϕ_4^4 , the Landau pole.
- (ii) $\beta(\lambda) < 0, \forall \lambda$: λ always decreases with μ , as in the case of QCD and in general of non-Abelian theories. If $\lambda \rightarrow 0$ for $\mu \rightarrow +\infty$, then we have that at high energy the theory behaves like a free theory, this is the *asymptotic freedom*. If $\lambda \rightarrow +\infty$ for $\mu \rightarrow 0$, then we have the *confinement*, but this is a non-perturbative phenomenon. As an example we can take $\beta(\lambda) = -A\lambda^2, A > 0$. Integrating it we obtain

$$\lambda(\mu) = \frac{\lambda_s}{1 + A\lambda_s \log \frac{\mu}{\mu_s}}.$$

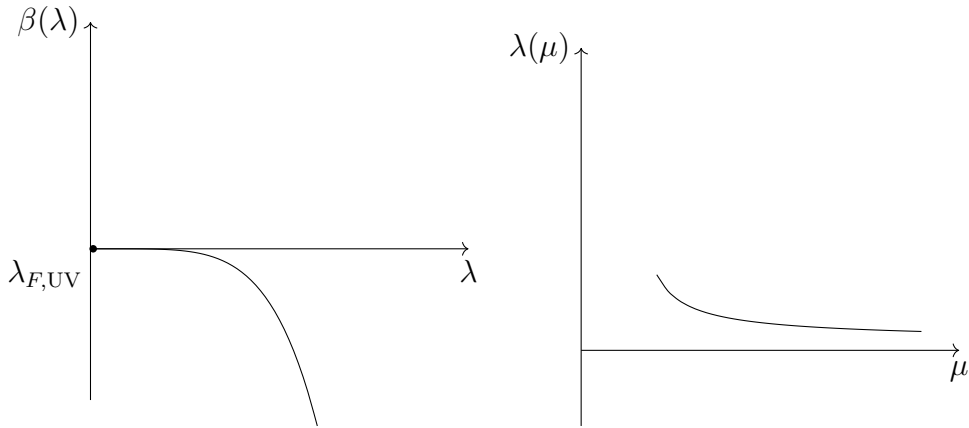


Figure 9.1: Behaviour of $\beta(\lambda)$ and $\lambda(\mu)$ corresponding to the case (ii).

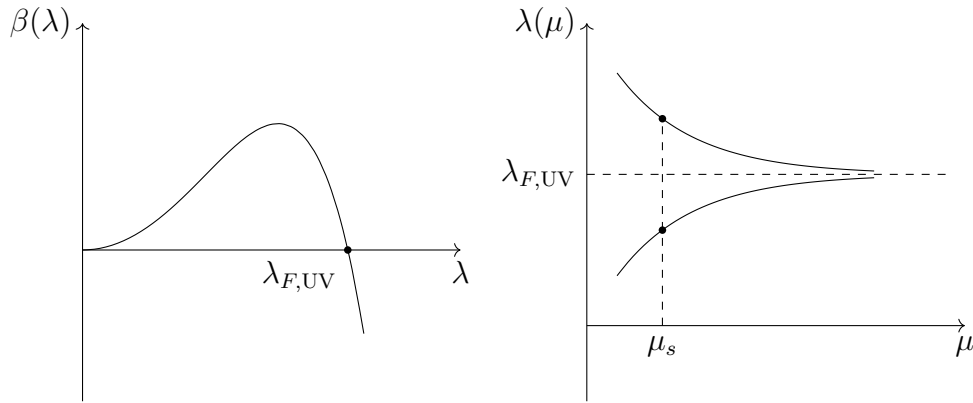


Figure 9.2: Behaviour of $\beta(\lambda)$ and $\lambda(\mu)$ corresponding to the case (iii).

Observe that, differently from the case of ϕ_4^4 , we have $\lambda \rightarrow 0$ for $\mu \rightarrow +\infty$. Instead we have a pole at small scale, $\mu = \mu_s e^{-\frac{1}{A\lambda_s}}$, that is λ is larger at big distances. The behaviour of β and λ in this case is depicted in figure 9.1.

(iii) $\beta(\lambda) > 0$ for small λ and then $\beta(\lambda) < 0$ when $\lambda > \lambda_F$, for some λ_F . We then have

$$\beta(\lambda_F) = 0 ,$$

and β is decreasing in a neighbourhood of λ_F , that is

$$\beta'(\lambda_F) < 0 .$$

The behaviour of β in this case is showed in figure 9.2.

Let us expand β near λ_F

$$\beta(\lambda) \approx (\lambda - \lambda_F)\beta'(\lambda_F) ,$$

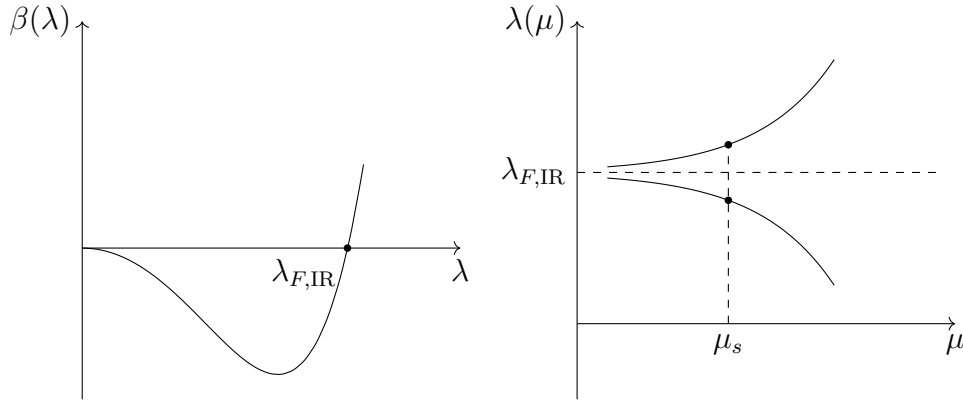


Figure 9.3: Behaviour of $\beta(\lambda)$ and $\lambda(\mu)$ corresponding to the case (iv).

that is

$$\int_{\lambda_s}^{\lambda} \frac{d\lambda'}{\lambda - \lambda_F} = \beta'(\lambda_F) \int_{\mu_s}^{\mu} \frac{d\mu'}{\mu'},$$

equivalent to

$$\frac{\lambda - \lambda_F}{\lambda_s - \lambda_F} = \left(\frac{\mu}{\mu_s} \right)^{\beta'(\lambda_F)}.$$

The sign of $\beta'(\lambda_F)$ is crucial. In the present $\beta'(\lambda_F) < 0$, so that, both in the case $\lambda_s < \lambda_F$ and $\lambda_s > \lambda_F$ we have (see also figure 9.2)

$$\lim_{\mu \rightarrow \infty} \lambda(\mu) = \lambda_F.$$

For this reason we say λ_F is a UV fixed point. If $\lambda_F \ll 1$ and $\lambda_s < \lambda_F$ we always are in the perturbative regime.

- (iv) $\beta(\lambda) < 0$ for small λ and then positive for $\lambda > \lambda_F$: the situation is similar to the case iii, but now the sign of $\beta'(\lambda_F)$ is positive. All what happened for $\mu \rightarrow +\infty$ now happens for $\mu \rightarrow 0$ and we say that λ_F is a IR fixed point. All this is represented in figure 9.3.

9.8 Prescription dependence of the renormalisation group coefficients

We can derive a general expression for $\beta(\lambda)$ without choosing any prescription. Start with

$$\lambda_0 = \mu^{2\epsilon} \left(a_0 + \sum_{k=1}^{\infty} \frac{a_k}{\epsilon^k} \right),$$

and take the total derivative with respect to μ

$$2\epsilon \left(a_0 + \sum_{k=1}^{\infty} \frac{a_k}{\epsilon^k} \right) + \mu \frac{d\lambda}{d\mu} \left(a'_0 + \sum_{k=1}^{\infty} \frac{a'_k}{\epsilon^k} \right) + \mu \frac{d\left(\frac{m}{\mu}\right)}{d\mu} \left(\dot{a}_0 + \sum_{k=1}^{\infty} \frac{\dot{a}_k}{\epsilon^k} \right) = 0, \quad (9.77)$$

where

$$\dot{a}_k \equiv \frac{\partial a_k}{\partial(m/\mu)},$$

$k \in \mathbb{N}$. We have

$$\mu \frac{d}{d\mu} \left(\frac{m}{\mu} \right) = \frac{m}{\mu} (\gamma_m - 1).$$

Note that a complete analysis, would require to consider also the relation one gets by setting to 0 the total derivative of m_0^2 with respect to μ . This will give another relation between λ , m and μ . Comparing such a relation with (9.77) will lead to $\lambda(\mu)$ and $m(\mu)$.

We do not consider such an issue here, and continue to investigate the relation (9.77) by setting $\beta = A + B\epsilon$

$$\begin{aligned} 2a_0 + Ba'_0 &= 0, \\ 2a_1 + Aa'_0 + Ba'_1 + \frac{m}{\mu}(\gamma_m - 1)\dot{a}_0 &= 0, \end{aligned}$$

so that, up to order λ^2

$$\begin{aligned} B &= -2\lambda \left(1 + \frac{3\lambda}{32\pi^2} G_1 \right) \left(1 + \frac{3\lambda}{16\pi^2} G_1 \right)^{-1} \approx -2\lambda \left(1 - \frac{3\lambda}{32\pi^2} G_1 \right), \\ Aa'_0 &= -\frac{3\lambda^2}{16\pi^2} + 2\lambda \left(1 - \frac{3\lambda}{32\pi^2} G_1 \right) \frac{3\lambda}{16\pi^2} - \frac{m}{\mu} (\gamma_m - 1) \frac{3\lambda^2}{32\pi^2} \dot{G}_1. \end{aligned}$$

In the limit $\epsilon \rightarrow 0$ we have

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \frac{m}{\mu} \frac{3\lambda^2}{32\pi^2} \dot{G}_1,$$

explicitly showing the prescription dependence of β unless the mass can be neglected. In particular, if $m \ll \mu$

$$\beta(\lambda) \approx \frac{3\lambda^2}{16\pi^2}.$$

It is interesting to see what happens if we change prescription. The parameters that appear in two different prescriptions should be related by a finite renormalisation

$$\lambda' = f \left(\lambda, \frac{m}{\mu} \right) = \lambda + \mathcal{O}(\lambda^2).$$

Therefore, since

$$\beta' = \mu \frac{d\lambda'}{d\mu} = \mu \frac{d\lambda}{d\mu} \frac{\partial f}{\partial \lambda} + \mu \frac{d\left(\frac{m}{\mu}\right)}{d\mu} f = \beta \frac{\partial f}{\partial \lambda} + \frac{m}{\mu} (\gamma_m - 1) f ,$$

it follows that if we neglect the mass, then

$$\beta'(\lambda') = \beta(\lambda) \frac{\partial f}{\partial \lambda} ,$$

so that

$$\beta'(\lambda') = 0 \iff \beta(\lambda) = 0 .$$

Since in the UV, that is for large momentum, it is expected that the mass has no role, it follows that a UV fixed point in the prescription with λ' corresponds to a UV fixed point in the prescription with λ . In other words, the presence of a UV fixed point is prescription independent. It turns out that even the sign of the derivative of β is prescription independent, which means that even the kind of the fixed point is prescription independent.

9.9 $\tilde{\Gamma}_{\text{ren}}^{(n)}$ scaling and anomalous dimension

As we said, in the MS scheme, the functions β , γ_m and γ_d are independent of the mass. So that, Eq.(9.72) reduces to

$$\left[-s \frac{\partial}{\partial s} + \beta(\lambda) \frac{\partial}{\partial \lambda} + (\gamma_m(\lambda) - 1) m \frac{\partial}{\partial m} + d_n - n \gamma_d(\lambda) \right] \tilde{\Gamma}_{\text{ren}}^{(n)}(\{sp_k\}; m, \lambda, \mu) = 0 , \quad (9.78)$$

$d_n = 4 - n$. In the following we show the explicit steps from this equation, which is Eq.(4.6.28) of the Ramond book, to equation (4.6.31), that expresses a key scaling property of $\tilde{\Gamma}_{\text{ren}}^{(n)}$.

The proof of (4.6.31) uses an adaptation of the method of *characteristic curves* that reduces a linear, or quasilinear,¹⁷ partial differential equation (PDE) to a system of first order ordinary differential equations (ODE). An excellent reference for this method is the text by Courant and Hilbert, *Methods of Mathematical Physics II*, pp. 28-32.

Consider the linear PDE

$$\sum_{k=1}^n a_k(\mathbf{x}) u_{x_k} + b(\mathbf{x}) u = 0 , \quad (9.79)$$

where

$$u_x := \frac{\partial u}{\partial x} .$$

¹⁷ A quasilinear PDE corresponds to (9.79) with a_k and b also depending on u itself, that is Eq.(9.79) with the substitutions $a_k(\mathbf{x}) \rightarrow a_k(\mathbf{x}, u)$, $b(\mathbf{x}) \rightarrow b(\mathbf{x}, u)$.

Denote the initial condition for u in the form

$$h(\mathbf{x}, u)|_{\mathbf{x} \in M} = 0, \quad (9.80)$$

where M is a codimension one subspace of \mathbb{R}^n . Let us consider the coordinate transformation

$$(x_1, \dots, x_n) \longrightarrow (s, t_1, \dots, t_{n-1}),$$

defined by some set of functions f_k , $k = 1, \dots, n$, that is

$$x_k = f_k(s, t_1, \dots, t_{n-1}). \quad (9.81)$$

The method of characteristic curves is based on the observation that imposing

$$\frac{dx_k}{ds} = a_k(\mathbf{x}), \quad (9.82)$$

$k = 1, \dots, n$, it follows that the total derivative of¹⁸ u

$$\frac{du}{ds} = \frac{\partial u}{\partial s} + \sum_{k=1}^n \frac{dx_k}{ds} \frac{\partial u}{\partial x_k} = \sum_{k=1}^n \frac{dx_k}{ds} \frac{\partial u}{\partial x_k}, \quad (9.83)$$

coincides with the left-hand side of (9.79). We then have that (9.79) is equivalent to the system of ODE's (9.82) together with

$$\frac{du}{ds} + b(\mathbf{x})u = 0. \quad (9.84)$$

The other key point is to impose that the values of \mathbf{x} defining M correspond to $s = 0$, that is

$$x_k|_M = f_k(0, t_1, \dots, t_{n-1}), \quad (9.85)$$

$k = 1, \dots, n$. Such conditions on the $f_k(0, t_1, \dots, t_{n-1})$'s fix, together with (9.82), the coordinate transformation, that is the $f_k(s, t_1, \dots, t_{n-1})$. Equation (9.80) becomes

$$h(\{f_k(0, t_1, \dots, t_{n-1})\}, v(0)) = 0, \quad (9.86)$$

where

$$v(s) := u(f_1(s, t_1, \dots, t_{n-1}), \dots, f_n(s, t_1, \dots, t_{n-1})). \quad (9.87)$$

For each fixed set of values of¹⁹ t_1, \dots, t_{n-1} , the solution

$$g_k(s) := f_k(s, t_1, \dots, t_{n-1}), \quad (9.88)$$

¹⁸ Note that, by construction, all the partial derivatives of u with respect to s, t_1, \dots, t_{n-1} vanish.

¹⁹ s, t_1, \dots, t_{n-1} are called *characteristic coordinates*.

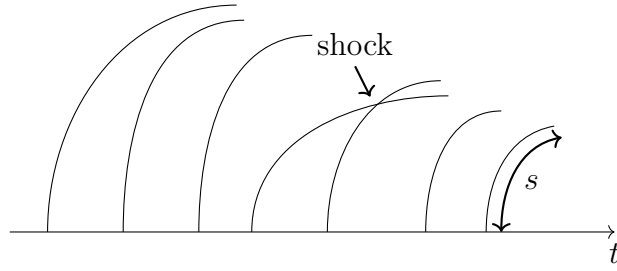


Figure 9.4: Depiction of the characteristics for a quasilinear equation.

$k = 1, \dots, n$ of the system (9.82) corresponds to a curve, called *characteristic curve*, parameterised by s that, as follows by (9.85), originates in M . Different values of t_1, \dots, t_{n-1} correspond to different characteristic curves (see figure 9.4).

A simple example is the PDE

$$\left(a(x, t) \frac{\partial}{\partial x} + b(x, t) \frac{\partial}{\partial t} + c(x, t) \right) u(x, t) = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \quad (9.89)$$

with initial condition

$$u(x, 0) = f(x). \quad (9.90)$$

Denote by $s \geq 0$ and $\tau \in \mathbb{R}$ the characteristic coordinates, imposing that the coordinate transformation be invertible. Next, consider the total derivative with respect to s

$$\frac{du}{ds} = \left(\frac{\partial}{\partial s} + \frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dt}{ds} \frac{\partial}{\partial t} \right) u(x, t) = \left(\frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dt}{ds} \frac{\partial}{\partial t} \right) u(x, t). \quad (9.91)$$

Setting

$$\frac{dx}{ds} = a(x, t), \quad \frac{dt}{ds} = b(x, t), \quad (9.92)$$

implies that the ODE

$$\frac{du}{ds} + c(x, t)u = 0, \quad (9.93)$$

corresponds, together with (9.92), to (9.89). Both (9.89) and (9.93) are constrained by the initial condition (9.90). Eq.(9.93) must be interpreted as an ODE for the function

$$f_\tau(s) = u(x(s, \tau), t(s, \tau)),$$

considered at fixed τ . Different values of τ define different functions $f_\tau(s)$. In other words, τ plays the role of modulo of the functional structure of u in (9.93). Therefore, for each fixed value τ_0 of τ , the equation (9.93) corresponds to equation (9.89) *restricted* to the curve

$$\gamma_{\tau_0}(s) := \{[x(s, \tau_0), t(s, \tau_0)] | 0 \leq s < \infty\},$$

so that

$$\{\gamma_\tau(s) | \tau \geq 0\} ,$$

is the set of characteristic curves associated to (9.89) and (9.90).

As an explicit example we consider the equation

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + 2\right)u = 0 , \quad x \in \mathbb{R} , \quad t \geq 0 , \quad (9.94)$$

$$u(x, 0) = \sin x . \quad (9.95)$$

In this case, we have

$$\frac{dx}{ds} = 1 , \quad \frac{dt}{ds} = 1 , \quad (9.96)$$

that is

$$x(s, \tau) - x(s_0, \tau) = s - s_0 , \quad t(s, \tau) - t(s_0, \tau) = s - s_0 . \quad (9.97)$$

Each τ identifies a different characteristic curve. The general form of the solution of (9.97) is

$$\begin{aligned} x(s, \tau) &= s + f(\tau) , \\ t(s, \tau) &= s + g(\tau) . \end{aligned} \quad (9.98)$$

On the other hand, requiring that the coordinate transformation be invertible means that there no values of s and τ for which the Jacobian vanishes. This gives $f'(\tau) \neq g'(\tau)$ for all τ . The natural solution is $f(\tau) = \tau$ and $g(\tau) = 0$. Furthermore, choosing $s_0 = 0$, we have

$$x(s, \tau) = s + \tau , \quad t(s, \tau) = s . \quad (9.99)$$

It follows that the characteristic curves are straight lines

$$x = t + \tau ,$$

one for each value of τ .

Note that the initial condition for u , that is $u(x, 0) = \sin x$, is given at $t = 0$. On the other hand, by (9.99) it follows that $t = 0$ corresponds to $s = 0$. Therefore, we have $x(0, \tau) = \tau$ and $u(x(0, \tau), t(0, \tau)) = u(\tau, 0) = \sin \tau$. It follows that (9.95) is equivalent to the ODE

$$\frac{du}{ds} + 2u = 0 , \quad s \geq 0 , \quad (9.100)$$

together with the condition

$$u(x(0, \tau), t(0, 0)) = u(\tau, 0) = \sin \tau . \quad (9.101)$$

The solution of (9.100) is $u = ce^{-2s}$, with the s -independent function c fixed by (9.101), that is $c = \sin \tau$, so that

$$f_\tau(s) = \bar{u}(s, \tau) = e^{-2s} \sin \tau , \quad (9.102)$$

where

$$\bar{u}(s, \tau) := u(x(s, \tau), t(s, \tau)) . \quad (9.103)$$

Finally, by (9.99) and (9.102) it follows that the solution of (9.95) is

$$u(x, t) = e^{-2t} \sin(x - t) . \quad (9.104)$$

To check the role of s and τ , it is useful to see how, besides (9.94) and (9.95), $u(x, t) = e^{-2t} \sin(x - t)$ also solves Eq.(9.100)

$$\left(\frac{d}{ds} + 2 \right) u = \left(\frac{dx}{ds} \frac{\partial}{\partial x} + \frac{dt}{ds} \frac{\partial}{\partial t} + 2 \right) e^{-2t} \sin(x - t) = 0 ,$$

that is

$$\left(\frac{dx}{ds} \cos(x - t) + \frac{dt}{ds} + 2 \sin(x - t) \right) e^{-2t} = 0 . \quad (9.105)$$

We now apply a variation of the above method to equation (9.78). We first consider the general case without assuming any prescription. As such the functions β , γ_m and γ_d depend both on $\lambda(\mu)$ and $m(\mu)/\mu$.

First of all note that with respect to the method of characteristics curves, in the equation (9.78) there is the partial derivative with respect to s , variable that we want to use as one of the characteristic coordinates. On the other hand, noticing that for any differentiable function f one has

$$s \frac{\partial}{\partial s} f(\{sp_k\}) = \sum_k p_k \frac{\partial}{\partial p_k} f(\{sp_k\}) , \quad (9.106)$$

so that

$$s \frac{\partial}{\partial s} f(\{sp_k\})|_{s=1} = \sum_k p_k \frac{\partial}{\partial p_k} f(\{p_k\}) ,$$

it follows that (9.78) evaluated at $s = 1$ is equivalent to

$$\left(- \sum_k p_k \frac{\partial}{\partial p_k} + \beta \left(\lambda, \frac{m}{\mu} \right) \frac{\partial}{\partial \lambda} + \delta_m \left(\lambda, \frac{m}{\mu} \right) m \frac{\partial}{\partial m} + c_d \left(\lambda, \frac{m}{\mu} \right) \right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu) = 0 , \quad (9.107)$$

where²⁰

$$\delta_m \left(\lambda, \frac{m}{\mu} \right) := \gamma_m \left(\lambda, \frac{m}{\mu} \right) - 1, \quad c_d \left(\lambda, \frac{m}{\mu} \right) := d_n - n\gamma_d \left(\lambda, \frac{m}{\mu} \right). \quad (9.108)$$

We now consider the following variation of the method of characteristic curves. First, we introduce two new variables parameterised by s

$$\bar{m} = \bar{m}(s, \lambda, m), \quad \bar{\lambda} = \bar{\lambda}(s, \lambda, m), \quad (9.109)$$

and consider m e λ as the values of the initial conditions for \bar{m} e $\bar{\lambda}$

$$\bar{m}(1, \lambda, m) = m, \quad \bar{\lambda}(1, \lambda, m) = \lambda. \quad (9.110)$$

Consider the equation²¹

$$\left(s \frac{\partial}{\partial s} + \beta \left(\bar{\lambda}, \frac{\bar{m}}{\mu} \right) \frac{\partial}{\partial \bar{\lambda}} + \delta_m \left(\bar{\lambda}, \frac{\bar{m}}{\mu} \right) \bar{m} \frac{\partial}{\partial \bar{m}} + c_d \left(\bar{\lambda}, \frac{\bar{m}}{\mu} \right) \right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) = 0, \quad (9.111)$$

and note that for $s = 1$ it reduces to (9.107). We also note that such an equation has the same form of the original renormalisation group equation Eq.(9.78), but written in an arbitrary prescription

$$\left(s \frac{\partial}{\partial s} + \beta \left(\lambda, \frac{m}{\mu} \right) \frac{\partial}{\partial \lambda} + \delta_m \left(\lambda, \frac{m}{\mu} \right) m \frac{\partial}{\partial m} + c_d \left(\lambda, \frac{m}{\mu} \right) \right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; m, \lambda, \mu) = 0. \quad (9.112)$$

Let us consider the total derivative

$$s \frac{d}{ds} = s \frac{\partial}{\partial s} \Big|_{\bar{\lambda}, \bar{m}} + s \frac{d\bar{\lambda}}{ds} \frac{\partial}{\partial \bar{\lambda}} \Big|_{s, \bar{m}} + s \frac{d\bar{m}}{ds} \frac{\partial}{\partial \bar{m}} \Big|_{s, \bar{\lambda}}, \quad (9.113)$$

and impose the constraints

$$s \frac{d\bar{\lambda}}{ds} = \beta \left(\bar{\lambda}, \frac{\bar{m}}{\mu} \right), \quad s \frac{d \log \bar{m}}{ds} = \delta_m \left(\bar{\lambda}, \frac{\bar{m}}{\mu} \right). \quad (9.114)$$

Eqs.(9.111), (9.113) and (9.114) imply

$$\left(s \frac{d}{ds} + c_d(\bar{\lambda}(s), \bar{m}(s)/\mu) \right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) = 0. \quad (9.115)$$

The great simplification due to the 't Hooft and Weinberg prescription, which implies

²⁰ Recall that, in any renormalisation prescription, both λ and m always depend on μ .

²¹ In the following we will frequently omit to explicitly write the dependence of both \bar{m} and $\bar{\lambda}$ on s , λ and m , and sometimes will denote only the dependence on s .

the mass independence of β , γ_m e γ_d , concerns the solutions of the equations (9.114) that now reduce to

$$s \frac{d\bar{\lambda}}{ds} = \beta(\bar{\lambda}) , \quad s \frac{d \log \bar{m}}{ds} = \delta_m(\bar{\lambda}) . \quad (9.116)$$

Since $\beta(\lambda)$ is m -independent, it follows that even $\beta(\bar{\lambda})$ is both \bar{m} - and m -independent. Therefore, by (9.116), it follows that even $\bar{\lambda}$ is m -independent. This means that in the 't Hooft and Weinberg prescription we have the map²²

$$m \longrightarrow \bar{m}(s, \lambda, m) = m f_m(s, \lambda) , \quad \lambda \longrightarrow \bar{\lambda}(s, \lambda) , \quad (9.117)$$

for some function $f_m(s, \lambda)$, satisfying the condition $f_m(1, \lambda) = 1$. Note that in such a transformation s plays the role of modulus for the functions \bar{m} and $\bar{\lambda}$, that is different values of s define different variables. In this way one has a parametrisation of coordinate transformations from $(m, \lambda) \in \mathbb{R}^2$ to $(\bar{m}(s, \lambda), \bar{\lambda}(s, \lambda)) \in \mathbb{R}^2$.

Eq.(9.116) admits the separation of variables

$$\frac{ds}{s} = \frac{d\bar{\lambda}}{\beta(\bar{\lambda})} , \quad \frac{d\bar{m}}{\bar{m}} = \frac{ds}{s} \delta_m(\bar{\lambda}(s, \lambda)) , \quad (9.118)$$

whose solutions are

$$s = \exp \left(\int_{\lambda}^{\bar{\lambda}(s, \lambda)} \frac{d\lambda'}{\beta(\lambda')} \right) , \quad (9.119)$$

and

$$\bar{m}(s) = \frac{m}{s} \exp \left(\int_1^s ds' \frac{\gamma_m(\bar{\lambda}(s'))}{s'} \right) = \frac{m}{s} \exp \left(\int_{\lambda}^{\bar{\lambda}(s, \lambda)} d\lambda' \frac{\gamma_m(\lambda')}{\beta(\lambda')} \right) . \quad (9.120)$$

²² It is worth recalling that under a change of variables $x_k \rightarrow y_k$, $k \in [1, n]$,

$$dy_j = \sum_{k=1}^n \frac{dy_j}{dx_k} dx_k ,$$

$j \in [1, n]$. In particular, the components of each one of the two sets $\{x_k\}$ and $\{y_k\}$ must be considered as independent variables, that is

$$\frac{dx_j}{dx_k} = \delta_{jk} , \quad \frac{dy_j}{dy_k} = \delta_{jk} .$$

It follows that, from the point of view of the differential calculus, the transformation (9.117) cannot be interpreted as a change of variables from s, λ, m to $s, \bar{\lambda}, \bar{m}$. This means that one should first consider the map from s, λ, m to $t, \bar{\lambda}, \bar{m}$, and then, after that the derivatives have been computed, one can set $t = s$. The difference arises when one considers each triplet of variables as independent ones. In particular, a change of variables implies that the derivatives between each distinct pair of \bar{m} , $\bar{\lambda}$ and t are vanishing. On the other hand, $d\bar{\lambda}/ds \neq 0$ and $d\bar{m}/ds \neq 0$. A simple example is provided by (9.99), where $x = s + \tau$ and $t = s$, so that $0 = dx/dt \neq dx/ds = 1$.

Since

$$0 = \frac{ds}{d\lambda} = \frac{\partial s}{\partial \lambda} + \frac{ds}{d\bar{\lambda}} \frac{\partial \bar{\lambda}}{\partial \lambda} ,$$

we have that the total derivative of (9.119) with respect to λ yields

$$\frac{d\bar{\lambda}}{d\lambda} = \frac{\beta(\bar{\lambda})}{\beta(\lambda)} . \quad (9.121)$$

Similarly, deriving (9.120) with respect to m and λ we have

$$\frac{d\bar{m}}{dm} = \frac{\bar{m}}{m} , \quad (9.122)$$

and

$$\frac{d\bar{m}}{d\lambda} = \frac{d\bar{\lambda}}{d\lambda} \frac{\gamma_m(\bar{\lambda})}{\beta(\bar{\lambda})} - \frac{\gamma_m(\lambda)}{\beta(\lambda)} = \frac{1}{\beta(\lambda)} (\gamma_m(\bar{\lambda}) - \gamma_m(\lambda)) , \quad (9.123)$$

respectively.

Since in the 't Hooft and Weinberg prescription even γ_d is independent of m , we have

$$\int_1^s \frac{ds'}{s'} \gamma_d(\bar{\lambda}(s')) = \int_\lambda^{\bar{\lambda}(s,\lambda)} d\lambda' \frac{\gamma_d(\lambda')}{\beta(\lambda')} . \quad (9.124)$$

Now note that (9.115) reduces to

$$\left(s \frac{d}{ds} + c_d(\bar{\lambda}(s)) \right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) = 0 , \quad (9.125)$$

that is

$$s \frac{d}{ds} f(s, s_0) = 0 , \quad (9.126)$$

where

$$f(s, s_0) := \exp \left(\int_{s_0}^s \frac{ds'}{s'} c_d(\bar{\lambda}(s')) \right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) . \quad (9.127)$$

Eq.(9.126) means that $f(s, s_0)$ is s -independent. By (9.110) it follows that

$$f(1, 1) = \tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu) ,$$

so that the s -independence implies $f(s, 1) = f(1, 1)$. Therefore, recalling that

$$c_d(\lambda) = 4 - n(1 + \gamma_d(\lambda)) ,$$

we have

$$\tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu) = s^{4-n} \exp\left(-n \int_1^s \frac{ds'}{s'} \gamma_d(\bar{\lambda}(s'))\right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) . \quad (9.128)$$

An explicit and instructive check of this relation is to verify that the right-hand side satisfies the same PDE of the left-hand side, which is Eq.(9.107) in the MS scheme. The additional constraint, concerning the initial condition, is trivially satisfied because for $s = 1$ Eq.(9.128) is an identity.

In order to check that the right-hand side of (9.128) satisfies equation (9.107) in the 't Hooft-Weinberg prescription, we first consider the action of the three differential operators in Eq.(9.107) on both sides of (9.128). Let us start with the action of $-\sum_k p_k \partial_{p_k}$

$$-\sum_k p_k \frac{\partial}{\partial p_k} \tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu) = -g(s, \lambda) \sum_k p_k \frac{\partial}{\partial p_k} \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) , \quad (9.129)$$

where

$$g(s, \lambda) := s^{4-n} \exp\left(-n \int_1^s \frac{ds'}{s'} \gamma_d(\bar{\lambda}(s'))\right) = s^{4-n} \exp\left(-n \int_\lambda^{\bar{\lambda}(s, \lambda)} d\lambda' \frac{\gamma_d(\lambda')}{\beta(\lambda')}\right) .$$

Next, we apply $\beta(\lambda)\partial_\lambda$ to both sides of (9.128). By (9.121), (9.122) and (9.123) we have

$$\begin{aligned} & \beta(\lambda) \frac{\partial}{\partial \lambda} \tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu) \\ &= g(s, \lambda) \beta(\lambda) \left[n \left(\frac{\gamma_d(\lambda)}{\beta(\lambda)} - \frac{\gamma_d(\bar{\lambda})}{\beta(\bar{\lambda})} \frac{d\bar{\lambda}}{d\lambda} \right) + \frac{d\bar{\lambda}}{d\lambda} \frac{\partial}{\partial \bar{\lambda}} + \frac{d\bar{m}}{d\lambda} \frac{\partial}{\partial \bar{m}} \right] \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) \\ &= g(s, \lambda) \left[n(\gamma_d(\lambda) - \gamma_d(\bar{\lambda})) + \beta(\bar{\lambda}) \frac{\partial}{\partial \bar{\lambda}} + (\gamma_m(\bar{\lambda}) - \gamma_m(\lambda)) \bar{m} \frac{\partial}{\partial \bar{m}} \right] \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) . \end{aligned} \quad (9.130)$$

Finally, we have

$$\begin{aligned}
& (\gamma_m(\lambda) - 1)m \frac{\partial}{\partial m} \tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu) \\
&= g(s, \lambda)(\gamma_m(\lambda) - 1)m \left(\frac{d\bar{\lambda}}{dm} \frac{\partial}{\partial \bar{\lambda}} + \frac{d\bar{m}}{dm} \frac{\partial}{\partial \bar{m}} \right) \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) \\
&= g(s, \lambda)(\gamma_m(\lambda) - 1)\bar{m} \frac{\partial}{\partial \bar{m}} \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) , \tag{9.131}
\end{aligned}$$

where we used

$$\frac{d\bar{\lambda}}{dm} = 0 ,$$

that, as proven before, is one of the consequences of the 't Hooft-Weinberg prescription.

Now note that in the sum of (9.130) and (9.131), the term

$$g(s, \lambda)\gamma_m(\lambda)\bar{m} \frac{\partial}{\partial \bar{m}} \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) ,$$

cancels because it appears with opposite signs.

To get the differential equation satisfied by $\tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu)$ we must add to (9.129), (9.130) and (9.131), the relation (9.128) multiplied by $(4 - n - n\gamma_d(\lambda))$. We then proved that if

$$\left[-\sum_k p_k \frac{\partial}{\partial p_k} + \beta(\lambda) \frac{\partial}{\partial \lambda} + (\gamma_m(\lambda) - 1)m \frac{\partial}{\partial m} + 4 - n(\gamma_d(\lambda) + 1) \right] \tilde{\Gamma}_{\text{ren}}^{(n)}(\{p_k\}; m, \lambda, \mu) = 0 , \tag{9.132}$$

then, the relation (9.128) is satisfied if and only if we also have

$$\left[-\sum_k p_k \frac{\partial}{\partial p_k} + \beta(\bar{\lambda}) \frac{\partial}{\partial \bar{\lambda}} + (\gamma_m(\bar{\lambda}) - 1)\bar{m} \frac{\partial}{\partial \bar{m}} + 4 - n(\gamma_d(\bar{\lambda}) + 1) \right] \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; \bar{m}(s), \bar{\lambda}(s), \mu) = 0 . \tag{9.133}$$

To prove that such an equation is satisfied, note that if a function $f(\{x_k\})$ satisfies the ODE

$$\sum_k x_k \frac{\partial}{\partial x_k} f(\{x_k\}) = g(\{x_k\}) ,$$

then, since for any invertible matrix M we have

$$\sum_k (Mx)_k \frac{\partial}{\partial (Mx)_k} = \sum_k x_k \frac{\partial}{\partial x_k} ,$$

it follows that we also have

$$\sum_k x_k \frac{\partial}{\partial x_k} f(\{(Mx)_k\}) = g(\{(Mx)_k\}) .$$

Therefore, (9.132) is equivalent to

$$\left[-\sum_k p_k \frac{\partial}{\partial p_k} + \beta(\lambda) \frac{\partial}{\partial \lambda} + (\gamma_m(\lambda) - 1)m \frac{\partial}{\partial m} + 4 - n(\gamma_d(\lambda) + 1) \right] \tilde{\Gamma}_{\text{ren}}^{(n)}(\{(Mp)_k\}; m, \lambda, \mu) = 0 .$$

In particular, (9.132) is equivalent to

$$\left[-\sum_k p_k \frac{\partial}{\partial p_k} + \beta(\lambda) \frac{\partial}{\partial \lambda} + (\gamma_m(\lambda) - 1)m \frac{\partial}{\partial m} + 4 - n(\gamma_d(\lambda) + 1) \right] \tilde{\Gamma}_{\text{ren}}^{(n)}(\{s^{-1}p_k\}; m, \lambda, \mu) = 0 ,$$

which is just Eq.(9.133), the difference only being a matter of notation's choice, \bar{m} and $\bar{\lambda}$, instead of m and λ , respectively. In other words, if the PDE

$$\hat{O}(m, \lambda)F(m, \lambda) = 0 ,$$

is satisfied, then we also have

$$\hat{O}(\bar{m}, \bar{\lambda})F(\bar{m}, \bar{\lambda}) = 0 .$$

The s -dependence of $\bar{\lambda}$ and \bar{m} is irrelevant because there are no s -derivatives in $\hat{O}(\bar{m}, \bar{\lambda})$.

9.10 Computation of γ_m and γ_d

As an exercise we can compute γ_m and γ_d in the 't Hooft-Weinberg prescription. For γ_m we start from the Laurent expansion of m_0^2 ($b_0 = 1$ in the 't Hooft-Weinberg prescription),

$$m_0^2 = m^2 \left(1 + \sum_{k=1}^{\infty} \frac{b_k}{\epsilon^k} \right) ,$$

and differentiate with respect to μ

$$0 = 2m \frac{dm}{d\mu} \left(1 + \sum_{k=1}^{\infty} \frac{b_k}{\epsilon^k} \right) + m^2 \frac{d\lambda}{d\mu} \sum_{k=1}^{\infty} \frac{b'_k}{\epsilon^k} .$$

Remembering that

$$\gamma_m = \frac{\mu}{m} \frac{dm}{d\mu} ,$$

we have

$$0 = 2\gamma_m \left(1 + \sum_{k=1}^{\infty} \frac{b_k}{\epsilon^k} \right) + \beta(\lambda) \sum_{k=1}^{\infty} \frac{b'_k}{\epsilon^k}.$$

Recalling that $\beta(\lambda) = A + B\epsilon$ and Eq. (9.75), we get, order by order,

$$\begin{aligned} \epsilon^0 : \quad & 2\gamma_m - 2\lambda \frac{db_1}{d\lambda} = 0, \\ \epsilon^{-k} : \quad & 2\gamma_m b_k - 2 \frac{db_k}{d\lambda} \left(1 - \lambda \frac{d}{d\lambda} \right) a_1 - 2\lambda \frac{db_{k+1}}{d\lambda} = 0. \end{aligned}$$

Thus, using

$$b_1 = \frac{\lambda}{32\pi^2} - \frac{5}{24} \left(\frac{\lambda}{16\pi^2} \right)^2 + \mathcal{O}(\lambda^3),$$

we find

$$\gamma_m = \lambda \frac{db_1}{d\lambda} = \frac{\lambda}{32\pi^2} - \frac{5}{12} \left(\frac{\lambda}{16\pi^2} \right)^2 + \mathcal{O}(\lambda^3).$$

Similarly for γ_d we start from

$$Z_\phi = 1 + \sum_{k=1}^{\infty} \frac{c_k}{\epsilon^k}. \quad (9.134)$$

The definition of γ_d is

$$\gamma_d = \frac{1}{2} \mu \frac{d \log Z_\phi}{d\mu} = \frac{1}{2Z_\phi} \mu \frac{dZ_\phi}{d\mu},$$

so that

$$Z_\phi \gamma_d = \frac{1}{2} \mu \frac{dZ_\phi}{d\mu} = \frac{1}{2} \mu \frac{d\lambda}{d\mu} \sum_{k=1}^{\infty} \frac{c'_k}{\epsilon^k} = \frac{1}{2} \beta(\lambda) \sum_{k=1}^{\infty} \frac{c'_k}{\epsilon^k}.$$

Using again equation (9.134) and $\beta(\lambda) = A + B\epsilon$ we have, order by order in ϵ

$$\begin{aligned} \epsilon^0 : \quad & \gamma_d = -\lambda \frac{dc_1}{d\lambda}, \\ \epsilon^{-k} : \quad & \lambda \frac{dc_{k+1}}{d\lambda} = \lambda c_k \frac{dc_1}{d\lambda} - \frac{dc_k}{d\lambda} \left(1 - \lambda \frac{d}{d\lambda} \right) a_1. \end{aligned}$$

Since

$$c_1 = -\frac{1}{24} \left(\frac{\lambda}{16\pi^2} \right)^2 + \mathcal{O}(\lambda^3),$$

we find

$$\gamma_d = \frac{1}{12} \left(\frac{\lambda}{16\pi^2} \right)^2 + \mathcal{O}(\lambda^3).$$

Finally, we check the behaviour of $\tilde{\Gamma}^{(N)}$ if ϕ_4^4 has a UV fixed point at $\lambda = \lambda_F$. Since for $\mu \rightarrow +\infty$ (equivalent to $s \rightarrow +\infty$) $\lambda \rightarrow \lambda_F$, we assume that $\gamma_m \rightarrow \gamma_m(\lambda_F)$ and

$\gamma_d \rightarrow \gamma_d(\lambda_F)$. Thus, the second equation in (9.118) becomes

$$s \frac{d\bar{m}}{ds} = \bar{m}(\gamma_m(\lambda_F) - 1) ,$$

so that (9.120) reduces to

$$\bar{m}(s) = m s^{\gamma_m(\lambda_F) - 1} .$$

Similarly

$$\int_1^s \frac{ds'}{s'} \gamma_d(\lambda_F) = \gamma_d(\lambda_F) \log s ,$$

and for large s Eq.(9.128) reads

$$\tilde{\Gamma}_{\text{ren}}^{(N)}(\{sp_k\}, m, \lambda, \mu) \rightarrow s^{4-N-N\gamma_d(\lambda_F)} \tilde{\Gamma}_{\text{ren}}^{(N)}(\{p_k\}, m s^{\gamma_m(\lambda_F) - 1}, \lambda_F, \mu) .$$

If $1 - \gamma_m(\lambda_F) > 0$, the mass term in the right-hand side can be neglected.

Chapter 10

Fermionic Path Integral¹

We saw that scalar quantum fields correspond to commuting classical fields in the path integral formulation. In the fermion case, from the spin-statistics theorem, we know that the field obey the anticommutation relations

$$\{\psi(x), \psi(y)\}|_{x^0=y^0} = 0 .$$

We will see that, in the path integral formulation, fermionic quantum fields correspond to Grassmannian classical fields. Let us start by introducing the *Grassmann algebra*. The main references for this chapter are [7, 11, 36, 8, 37].

10.1 Grassmann calculus

The n -dimensional Grassmann algebra G_n is the one whose generators θ_k , with $k = 1, \dots, n$ fulfill the following anticommutation relation

$$\{\theta_j, \theta_k\} = 0 ,$$

which, in particular, implies

$$\theta_k^2 = 0 .$$

It follows that the expansion of a function of the θ_k 's only contains a finite number of terms. For instance, in the case of a two variables function, we have

$$\begin{aligned} f(\theta_1, \theta_2) &= a_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_1\theta_2 \\ &= a_0 + a_1\theta_1 + a_2\theta_2 - a_3\theta_2\theta_1 . \end{aligned}$$

¹ Elia de Sabbata and Pietro Oreglia

where a_0 , a_1 , a_2 and a_3 are ordinary c -numbers. The next step is to define the *left derivative*: this is the standard derivative posed on the *left* of the function and acting on the *right*

$$\begin{aligned}\frac{\partial f}{\partial \theta_1} &= \frac{\partial^L f}{\partial \theta_1} = a_1 + a_3 \theta_2 , \\ \frac{\partial f}{\partial \theta_2} &= \frac{\partial^L f}{\partial \theta_2} = a_2 - a_3 \theta_1 .\end{aligned}\tag{10.1}$$

Another notation for the *left* derivative is to use the symbol of derivative with the *right* arrow.² We then have the following equivalent symbols for the *left* derivative

$$\frac{\partial f}{\partial \theta_k} = \frac{\overrightarrow{\partial}}{\partial \theta_k} = \frac{\partial^L f}{\partial \theta_k} .$$

The *right derivative* is the standard derivative posed on the *right* of the function and acting on the *left*

$$\begin{aligned}\frac{\partial^R f}{\partial \theta_1} &= a_1 - a_3 \theta_2 , \\ \frac{\partial^R f}{\partial \theta_2} &= a_2 + a_3 \theta_1 .\end{aligned}$$

Even in this case we have equivalent notations

$$\frac{f \overleftarrow{\partial}}{\partial \theta_k} = \frac{\partial^R f}{\partial \theta_k} .$$

Note that

$$\left(\theta_k \frac{\partial}{\partial \theta_k} + \frac{\partial}{\partial \theta_k} \theta_k \right) f = f ,$$

that can be rewritten as

$$\left\{ \theta_j, \frac{\partial}{\partial \theta_k} \right\} = \delta_{jk} , \quad \left\{ \frac{\partial}{\partial \theta_j}, \frac{\partial}{\partial \theta_k} \right\} = 0 .\tag{10.2}$$

A property, which is crucial in several contexts, such as Poincaré invariance in bosonic theory, concerns the translation invariance arising in integrations on the full real axis. Since Grassmann numbers are not taking values in \mathbb{R} , we should find the way to get such a property in Grassmanian integration. As a first step we define the infinitesimal $d\theta_j$ such that

$$\{\theta_j, d\theta_k\} = 0 , \quad \{d\theta_j, d\theta_k\} = 0 .\tag{10.3}$$

² We introduced left and right derivatives always in treating fermions, that is in (4.4).

We then requires translation invariance

$$\int f(\theta)d\theta = \int f(\theta + \eta)d\theta , \quad (10.4)$$

with η any element of the Grassmann algebra. Consider the case

$$f = a_1 + a_2\theta .$$

By (10.4) we get

$$(a_1 + a_2\eta) \int d\theta = 0 ,$$

that is

$$\int d\theta = 0 .$$

However, such a property, and therefore translation invariance, is already implicit in the Grassmann algebra. To see this, note that if we interpret multiple integrals iteratively, that is, for example

$$\int f(\theta_1, \theta_2) d\theta_1 d\theta_2 = \int \left(\int f(\theta_1, \theta_2) d\theta_1 \right) d\theta_2 ,$$

then, by

$$\begin{aligned} \left(\int d\theta_1 \right)^2 &= \int d\theta_1 \int d\theta_2 = \int d\theta_1 d\theta_2 \\ &= - \int d\theta_2 d\theta_1 = - \left(\int d\theta_1 \right)^2 , \end{aligned}$$

we have

$$\int d\theta_1 = \int d\theta_2 = 0 .$$

We are left with the freedom of choosing

$$\int \theta d\theta = 1 .$$

We then have

$$\begin{aligned} \int f d\theta_1 &= \int (a_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_1\theta_2) d\theta_1 \\ &= a_0 \int d\theta_1 + a_1 \int \theta_1 d\theta_1 + a_2\theta_2 \int d\theta_1 + a_3\theta_2 \int d\theta_1 \theta_1 \\ &= a_1 + a_3\theta_2 . \end{aligned}$$

Comparing this with (10.1), we see that differentiation and integration give the same

result.

The above analysis extends to the case of G_n with arbitrary n . In particular, integration formulas in the n -dimensional case simply reads

$$\int \theta_j d\theta_k = \delta_{jk} ,$$

$k = 1, \dots, n$.

Note that each element of G_n can be expressed in terms of the 2^n monomials

$$1, \theta_1, \dots, \theta_n, \theta_1\theta_2, \dots, \theta_{n-1}\theta_n, \dots, \theta_1\theta_2 \dots \theta_n .$$

This implies that any element $f(\theta) \in G_n$ corresponds to an element of the vector space spanned by such a basis, that is

$$f(\theta) = f_0^{(0)} + \sum_{j_1=1}^n f_{j_1}^{(1)} \theta_{j_1} + \frac{1}{2!} \sum_{j_1, j_2=1}^n f_{j_1 j_2}^{(2)} \theta_{j_1} \theta_{j_2} + \dots + \frac{1}{n!} \sum_{j_1, \dots, j_n=1}^n f_{j_1 \dots j_n}^{(n)} \theta_{j_1} \dots \theta_{j_n} . \quad (10.5)$$

Note that

- (i) the coefficients $f_{j_1 \dots j_k}^{(k)}$, $k \geq 2$, can be chosen completely antisymmetric in its indexes.
- (ii) In particular, by (10.5) it follows that acting with the left derivatives on $f(\theta)$, we have

$$f_{j_1 \dots j_k}^{(k)} = \frac{\partial}{\partial \theta_{j_k}} \dots \frac{\partial}{\partial \theta_{j_1}} f(\theta) |_{\theta_{j_1} = \dots = \theta_{j_k} = 0} ,$$

with $j_1 > \dots > j_k = 1, k \geq 1$.

- (iii) The last term in the right-hand side of (10.5) is equivalent to

$$f_{1 \dots n}^{(n)} \theta_1 \dots \theta_n .$$

- (iv) Eq.(10.5) shows that the elements of the G_n -algebra are polynomials.

- (v) $f(\theta)$ can be decomposed in the form

$$f(\theta) = f_-(\theta) + f_+(\theta) ,$$

where $f_-(\theta)$ ($f_+(\theta)$) is the sum of odd (even) monomials in the θ_k 's of $f(\theta)$. If a function is a sum of odd (even) monomials only, it is said to have definite degree odd (even).

- (vi) Elements of even degree are in the centre of the Grassmann algebra, in fact, they

commute with all of its generators (it is sufficient to check the result for monomials)

$$\theta_1 \dots \theta_{2n} \theta_k = (-1)^{2n} \theta_k \theta_1 \dots \theta_{2n} .$$

Moreover, the derivative with respect to a generator of an element of odd degree is an element of even degree and vice versa (provided the derivative is nonzero).

(vii) The Leibniz rule for Grassmann calculus is

$$\frac{\partial}{\partial \theta_k} (fg) = \frac{\partial f}{\partial \theta_k} g + (f_+ - f_-) \frac{\partial g}{\partial \theta_k} .$$

This identity can be easily proved when f and g are monomials, the result follows then by linearity.

(viii) One has

$$\frac{\partial}{\partial \theta_k} f_+ = -f_+ \overleftarrow{\frac{\partial}{\partial \theta_k}} , \quad \frac{\partial}{\partial \theta_k} f_- = f_- \overleftarrow{\frac{\partial}{\partial \theta_k}} .$$

Also his identity can be easily checked for monomials, and the result follows then by linearity.

(ix) Since the maximum power in θ_k in the expansion of $f(\theta)$ is one, from $\int d\theta = 0$ it follows

$$\int \frac{\partial f(\theta)}{\partial \theta_k} d\theta_k = 0 .$$

(x) From (vii) and (ix) it follows

$$\int \mp f_{\pm}(\theta) \frac{\partial g(\theta)}{\partial \theta_k} d\theta_k = \int \frac{\partial f_{\pm}(\theta)}{\partial \theta_k} g(\theta) d\theta_k .$$

(xi) From (viii) and (x) follows the formula for integration by parts

$$\int f(\theta) \frac{\partial g(\theta)}{\partial \theta_k} d\theta_k = \int f(\theta) \overleftarrow{\frac{\partial}{\partial \theta_k}} g(\theta) d\theta_k .$$

Another property of the Grassmann calculus, is that it admits the Fourier transform

$$\tilde{f}(\eta) = \int e^{\eta \cdot \theta} f(\theta) d\theta_n \dots d\theta_1 ,$$

where $\eta \cdot \theta := \sum_k \eta_k \theta_k$, with $\{\theta_k\} \cup \{\eta_k\}$, the $2n$ generators of G_{2n} . In turn, this defines the Dirac-Grassmann δ

$$\delta(\eta) = \int e^{\eta \cdot \theta} d\theta_n \dots d\theta_1 ,$$

so that, considering the expansion of the exponential $e^{\eta \cdot \theta}$, we get

$$\delta(\eta) = \eta_n \dots \eta_1 ,$$

and it can be proved that

$$f(\theta) = \int \delta(\eta - \theta) d\eta_1 \dots d\eta_n f(\eta) .$$

Let us consider a complex linear combination of Grassmann variables

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}} , \quad \theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}} .$$

Since our Grassmann calculus should mimic several properties of the Dirac field, including the action of Hermitian conjugation, we define the complex conjugation of a product of Grassmann variables, to reverse the order

$$(\theta\eta)^* := \eta^* \theta^* = -\theta^* \eta^* .$$

This implies that if $\theta^* = \theta$ and $\eta^* = \eta$, then $(\theta\eta)^* = -\theta\eta$. Note that

$$\int \theta \theta^* d\theta^* d\theta = 1 ,$$

showing that, like in the case of the complex numbers $z = x + iy$ and $z^* = x - iy$, for which $\partial_{z^*} z = 0$, even θ and θ^* can be considered as anticommuting independent quantities. In other words,

$$\frac{\partial \theta}{\partial \theta^*} = 0 .$$

Let us consider the Gaussian integral

$$\int d\theta^* d\theta e^{-\theta^* a \theta} = \int d\theta^* d\theta (1 - \theta^* a \theta) = a .$$

Grassmann variables have properties which are analogous to the wedge product for bosonic variables. For example, in considering the transformation properties of the volume form under a linear transformation $x'_j = \sum_k \Lambda_{jk} x_k$, we have

$$dx'_1 \wedge \dots \wedge dx'_n = (\det \Lambda) dx_1 \wedge \dots \wedge dx_n .$$

In the case of complex Grassmann variables, we are interested if U is a unitary matrix, then we have the invariance

$$\int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n f(\theta') = \int d\theta_1^* d\theta_1 \dots d\theta_n^* d\theta_n f(\theta) , \quad (10.6)$$

where

$$\theta'_j = \sum_{k=1}^n U_{jk} \theta_k .$$

To prove this, we first note that both the left-hand side and the right-hand side of (10.6) are proportional to $(\prod_k \theta_k)(\prod_k \theta_k^*)$. Next, we consider the identity

$$\theta'_1 \dots \theta'_n = \frac{1}{n!} \epsilon^{j_1 \dots j_n} \theta_{j_1} \dots \theta_{j_n} .$$

Then note that the determinant of a matrix Λ_{jk} can be expressed in two equivalent ways

$$\det \Lambda = \epsilon_{j_1 \dots j_n} \Lambda_{1j_1} \dots \Lambda_{nj_n} = \frac{1}{n!} \epsilon_{j_1 \dots j_n} \epsilon_{k_1 \dots k_n} \Lambda_{j_1 k_1} \dots \Lambda_{j_n k_n} .$$

We then have

$$\begin{aligned} \theta'_1 \dots \theta'_n &= \frac{1}{n!} \epsilon^{j_1 \dots j_n} U_{j_1 k_1} \theta_{k_1} \dots U_{j_n k_n} \theta_{k_n} \\ &= \frac{1}{n!} \epsilon^{j_1 \dots j_n} U_{j_1 k_1} \theta_{k_1} \dots U_{j_n k_n} \theta_{k_n} \epsilon^{k_1 \dots k_n} \theta_1 \dots \theta_n \\ &= (\det U) \theta_1 \dots \theta_n . \end{aligned}$$

The proof of (10.6) then follows by observing that

$$\left(\prod_k \theta'_k \right) \left(\prod_k \theta'_k{}^* \right) = (\det U) (\det U)^* \left(\prod_k \theta_k \right) \left(\prod_k \theta_k^* \right) ,$$

and then using $(\det U)(\det U)^* = (\det U)(\det U^t)^* = (\det U)(\det U^{-1}) = 1$. We can now use such an invariance to diagonalise a Hermitian matrix A in the Gaussian integral. Let $\{a_k\}$ the set of eigenvalues of A . We have

$$\left(\prod_j \int d\theta_j^* d\theta_j \right) e^{-\theta_i^* A_{ik} \theta_k} = \left(\prod_j \int d\theta_j^* d\theta_j \right) e^{-\theta_k^* a_k \theta_k} = \prod_k a_k = \det A , \quad (10.7)$$

where in the second equality we used

$$\int d\theta^* d\theta e^{-\theta^* a \theta} = \int d\theta^* d\theta (1 - \theta^* a \theta) = \int d\theta^* d\theta (1 + \theta \theta^* a) = a .$$

A similar analysis, together with

$$\int d\theta^* d\theta \theta \theta^* e^{-\theta^* a \theta} = \int d\theta^* d\theta \theta \theta^* (1 + \theta \theta^* a) = 1 ,$$

shows that

$$\left(\prod_l \int d\theta_l^* d\theta_l \right) \theta_j \theta_k^* e^{-\theta_m^* A_{mn} \theta_n} = \det A (A^{-1})_{jk} . \quad (10.8)$$

As a last thing before applying to the path integral we need to introduce *infinite dimensional Grassmann algebra* using as generator the fields

$$\psi(x) = \sum_k \psi_k \phi_k(x) ,$$

where the ψ_k 's are Grassmann variables, while the $\phi_k(x)$'s are a basis of Dirac spinors, for example the eigenfunctions of the Dirac operator. Note that we used a discrete index, which is not the case if we work in \mathbb{R}^4 . A possible solution is to work in some compactification of \mathbb{R}^4 . We have the relations

$$\{\psi(x), \psi(y)\} = 0 ,$$

$$\frac{\delta\psi(x)}{\delta\psi(y)} = \delta^{(4)}(x - y) ,$$

and

$$\int d\psi(x) = 0 , \quad \int \psi(x) d\psi(x) = 1 .$$

10.2 Path integral for fermions

We now have all the machinery needed to evaluate functional integrals, and hence correlation functions, involving fermions. For example, the Dirac two point function is given by³

$$\langle 0 | T\psi(x_1)\bar{\psi}(x_2) | 0 \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left[i \int_{-T}^T d^4x \bar{\psi}(i\rlap{\not{D}} - m)\psi \right] \psi(x_1)\bar{\psi}(x_2)}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left[i \int_{-T}^T d^4x \bar{\psi}(i\rlap{\not{D}} - m)\psi \right]} .$$

In the following, we use the functional version of the Grassmann calculus exploited in the previous section. In particular, note that the functional version of (10.7) implies

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left\{ - \int_{-\infty}^{\infty} d^4x \bar{\psi}[-i(i\rlap{\not{D}} - m)]\psi \right\} = \det \left[-i(i\rlap{\not{D}} - m) \right] .$$

It then follows by (10.8) that

$$\langle 0 | T\psi(x_1)\bar{\psi}(x_2) | 0 \rangle = \{[-i(i\rlap{\not{D}} - m)]^{-1}\}_{12} .$$

³ We write $\mathcal{D}\bar{\psi}$ instead of $\mathcal{D}\psi^*$ for convenience; the two are unitarily equivalent.

Now, in analogy to what we have done in the scalar case, we define the generating functional for the free Dirac fields. Introducing the Grassmann source fields

$$\eta_\alpha(x) , \quad \bar{\eta}_\alpha(x) := \eta_\beta^*(x) \gamma_{\beta\alpha}^0 ,$$

we have⁴

$$Z_0[\eta, \bar{\eta}] = \frac{1}{N} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[i \int d^4x (\bar{\psi}(x)(i\cancel{\partial} - m)\psi(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)) \right] .$$

Our aim is now to express this functional in a form like the one of the bosonic case, that is

$$Z_0[J] = e^{-\frac{i}{2}\langle J \Delta_F J \rangle} .$$

Let us set

$$S_F^{-1} := i\cancel{\partial} - m , \tag{10.9}$$

so that

$$Z_0[\eta, \bar{\eta}] = \frac{1}{N} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[i \int d^4x (\bar{\psi} S_F^{-1} \psi + \bar{\eta} \psi + \bar{\psi} \eta) \right] .$$

Let us find the values ψ_m and $\bar{\psi}_m$ of ψ and $\bar{\psi}$, respectively, that minimise

$$Q(\psi, \bar{\psi}) := \bar{\psi} S_F^{-1} \psi + \bar{\eta} \psi + \bar{\psi} \eta .$$

The equation $\delta Q = 0$, that is

$$\frac{\delta Q}{\delta \psi} + \frac{\delta Q}{\delta \bar{\psi}} = 0 \implies \frac{\delta Q}{\delta \psi} = \frac{\delta Q}{\delta \bar{\psi}} = 0 ,$$

has solutions

$$\psi_m = -S_F \eta , \quad \bar{\psi}_m = -\bar{\eta} S_F .$$

We then have

$$Q = Q_m + (\bar{\psi} - \bar{\psi}_m) S_F^{-1} (\psi - \psi_m) ,$$

where

$$Q_m = Q(\psi_m, \bar{\psi}_m) = -\bar{\eta} S_F \eta .$$

⁴ Recall that the complex conjugation acts in the product of Grassmann variables by reversing the order, this was done to mimic the property of the Hermitian conjugation. In this way, the source terms in the generating functional have similar properties of the other bilinears, $m\bar{\psi}\psi$ and $i\bar{\psi}\cancel{\partial}\psi$.

It follows that the generating functional can be expressed in the form

$$\begin{aligned} Z_0[\eta, \bar{\eta}] &= \frac{1}{N} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left\{ i \int d^4x \left[Q_m + (\bar{\psi} - \bar{\psi}_m) S_F^{-1} (\psi - \psi_m) \right] \right\} \\ &= \frac{1}{N} \exp \left(- i \int d^4x d^4y \bar{\eta}(x) S_F(x-y) \eta(y) \right) \det (- i S_F^{-1}) , \end{aligned}$$

where we factorised the exponential (note that the two terms in the exponential are of even degree and thus they commute), and used (10.7) extended to the functional case. Note that

$$N = Z_0[0, 0] = \det (- i S_F^{-1}) ,$$

so that

$$Z_0[\eta, \bar{\eta}] = \exp \left(- i \int d^4x d^4y \bar{\eta}(x) S_F(x-y) \eta(y) \right) .$$

By (10.9) we see that⁵

$$S_F(x) = (i\rlap{/}\partial + m) \Delta_F(x) ,$$

whose check reads

$$\begin{aligned} S_F^{-1\alpha}{}_{\beta} S_F(x)^{\beta}{}_{\gamma} &= (i\rlap{/}\partial - m)^{\alpha}{}_{\beta} S_F(x)^{\beta}{}_{\gamma} = (i\rlap{/}\partial - m)^{\alpha}{}_{\beta} (i\rlap{/}\partial + m)^{\beta}{}_{\gamma} \Delta_F(x) \\ &= -((\gamma^{\mu})^{\alpha}{}_{\beta} (\gamma^{\nu})^{\beta}{}_{\gamma} \partial_{\mu} \partial_{\nu} + m \delta^{\alpha}{}_{\gamma}) \Delta_F(x) \\ &= -\delta^{\alpha}{}_{\gamma} (\square + m) \Delta_F(x) \\ &= \delta^{\alpha}{}_{\gamma} \delta^{(4)}(x) . \end{aligned}$$

Extending the derivation of (10.2) to the functional case, we get

$$\left\{ \frac{\delta}{\delta\eta(x)}, \frac{\delta}{\delta\eta(y)} \right\} = 0 ,$$

and

$$\frac{\delta}{\delta\eta(z)} (\eta(x) \eta(y)) = \delta^{(4)}(z-x) \eta(y) - \delta^{(4)}(z-y) \eta(x) .$$

We now show that these rules lead to a factor of -1 for each closed fermion loop. We have

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \frac{1}{i^2} \frac{\delta}{\delta\bar{\eta}(x)} (-1) \frac{\delta}{\delta\eta(y)} Z_0[\eta, \bar{\eta}] \Big|_{\eta=\bar{\eta}=0} ,$$

where the (-1) comes from the left derivative acting on the source term $\bar{\psi}\eta$. The third

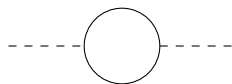
⁵ We recall that if \hat{O} is an operator, $\hat{O}(x)$ stands for $\delta(x)\hat{O}$, i.e. the integral kernel whose convolution with a function $\psi(x)$ gives $(\hat{O}\psi)(x)$. In fact, one has

$$\int dy \hat{O}(x-y) \psi(y) = \int dy \delta(x-y) \hat{O}\psi(y) = \int dy \delta(x-y) (\hat{O}\psi(y)) = (\hat{O}\psi)(x) .$$

term in the expansion of $Z_0[\eta, \bar{\eta}]$ reads

$$-\frac{1}{2} \int d^4x d^4y d^4x' d^4y' \bar{\eta}(x) S_F(x-y) \eta(y) \bar{\eta}(x') S_F(x'-y') \eta(y') .$$

We can represent the correction to the boson propagator due to a fermion loop like



which corresponds to a term like

$$\frac{\delta^2}{\delta \bar{\eta}_\alpha(z) \delta \eta_\beta(z)} \frac{\delta^2}{\delta \bar{\eta}_\gamma(z') \delta \eta_\delta(z')} Z[\eta, \bar{\eta}] .$$

Applying what we have found before about anticommutation relation of derivative of infinite dimensional Grassmann variables we obtain that this term becomes

$$+S_{F\alpha\delta}(z-z') S_{F\gamma\beta}(z'-z) .$$

The overall sign would be $-$ if the fields were scalar, but this is not the case so another -1 (and so an overall $+$ sign) appears.

Chapter 11

Gauge theories¹

Our aim now is to extend the formalism we have developed so that we can work also with gauge fields. As we will see, we should also consider non-physical degrees of freedom. However, despite the difficulties in getting rid of these spurious states, the path integral formalism displays in gauge theories its whole power. The main references for this chapter are [7, 11, 36, 8, 37, 36].

11.1 Lie algebras and gauge groups

Even if we already introduced the main properties of Lie algebras and Lie groups, it is useful to summarise the main features we need to investigate theories which are invariant under non-Abelian gauge transformations.

Common examples of Lie algebras are $so(n)$, $su(n)$ and $sp(2n, \mathbb{R})$. Let us recall that the elements of $so(n)$ are skew-symmetric square real matrices, with Lie bracket the commutator. The elements of $su(n)$ correspond to traceless $n \times n$ anti-Hermitian matrices, with Lie bracket the commutator. The elements of $sp(2n, \mathbb{R})$ are square real matrices A satisfying $JA + A^T J = 0$ with J is the standard square skew-symmetric matrix

$$J = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix},$$

with \mathbb{I}_n the unit matrix of order n .

The above definition of the $so(n)$, $su(n)$ and $sp(2n, \mathbb{R})$ algebras is the one that refers to the standard normalisation of the generators of the Lie algebra used in the mathematical literature, that is the one given in (2.6). Usually, in physics literature, there is a difference of a factor i with respect to (2.6). Namely, the physicist choice is to replace

¹ Marco Zecchinato

each T_k in (2.6) by $-iT_k$, so that

$$[T^a, T^b] = if^{abc}T^c . \quad (11.1)$$

Note that, setting $[A, B] = AB - BA$ is an additional request that also requires the notion of product.

Let us recall that a Lie algebra is said simple if has no non-trivial ideals,² and semi-simple if it is the direct sum of simple algebras.

Any element of the Lie Group continuously connected to the identity can be expressed in the form³

$$U = \exp(-i\Lambda^a T^a) .$$

The number of generators of the algebras is the dimension of the corresponding Lie groups. In particular,

$$\begin{array}{lll} SU(n) & \dim = n^2 - 1 & n \geq 2 , \\ SO(n) & \dim = \frac{n(n-1)}{2} & n \geq 2 , \\ Sp(2n, \mathbb{R}) & \dim = n(2n+1) & n \geq 1 , \end{array}$$

where $SU(n)$ is the group of square complex $n \times n$ unitary matrices with unit determinant,⁴ $SO(n)$ denotes the group of square real orthogonal matrices with determinant 1. Finally, $Sp(2n, \mathbb{R})$ is the group of square real symplectic matrices, that is such that $M^T J M = J$.

$SO(n)$ is a compact and connected (but not simply connected) group, $SU(n)$ is compact and simply connected. The symplectic group $Sp(2n, \mathbb{R})$ is connected and not compact. Other interesting examples of Lie algebras are the *exceptional Lie algebras* $G_2(14)$, $F_4(52)$, $E_6(78)$, $E_7(138)$, $E_8(248)$, where the numbers in brackets are the corresponding dimensions. A theorem by Cartan shows that the algebras $so(n)$, $su(n)$, $sp(n)$ and the exceptional Lie algebras, are the only finite-dimensional simple Lie algebras. Such algebras are important in several fields of physics, for example the direct sum $su(3) \oplus$

² An *ideal* is a subalgebra \mathfrak{h} of \mathfrak{g} such that $[h, g] \in \mathfrak{h}$, $\forall h \in \mathfrak{h}$, $\forall g \in \mathfrak{g}$.

³ Note that with the normalisation used in the mathematical literature, the exponential map corresponds to $\exp(\Lambda^a T^a)$.

⁴ Let us stress that $SU(n)$ is not a complex Lie group. Similarly, $su(n)$ is not a complex Lie algebra. In this respect, it is worth recalling that a complex Lie group is a complex-analytic manifold that is also a group in such a way that the map

$$G \times G \rightarrow G , \quad (x, y) \mapsto xy^{-1} ,$$

is holomorphic. An example is the general linear group over the complex numbers, denoted $GL(n, \mathbb{C})$.

$su(2) \oplus u(1)$ is the Lie algebra of the Standard Model, whose gauge symmetry is given by the group $SU(3) \otimes SU(2) \otimes U(1)$.

For our applications it is useful to consider the relations between the fundamental and adjoint representations. The adjoint representation corresponds to matrices acting on the vector space of the Lie algebra, that is the one generated by the T^a 's. In all other representations the corresponding vector spaces are different.

The matrices in the adjoint representation are

$$(T_A^a)^{bc} = -if^{abc} .$$

Let X_i be an element of the R representation space. This means that under an infinitesimal transformation

$$X_i \mapsto X'_i = X_i - i\Lambda^a (T_R^a)_{ij} X_j . \quad (11.2)$$

Let us now consider the mentioned group action in the adjoint representation on the vector space of its Lie algebra. In this case the element of the representation vector space should be denoted by X^a . Set

$$X = X^a T_R^a ,$$

and

$$U_R = \exp(-i\Lambda_R) \in G ,$$

where $\Lambda = \Lambda^a T_R^a$.

The transformation

$$X \mapsto X' = U_R X U_R^{-1} , \quad (11.3)$$

is particularly relevant because it is an automorphism of the Lie algebra; in particular, it preserves the Lie brackets. Let us show that such a transformation corresponds to the transformation of X^a in the adjoint representation. We have

$$\begin{aligned} X^a T_R^a &\mapsto (1 - i\Lambda^a T_R^a) X^b T_R^b (1 + i\Lambda^c T_R^c) \\ &\approx X^a T_R^a - i\Lambda^a X^b [T_R^a, T_R^b] \\ &= X^a T_R^a + f^{abc} \Lambda^a X^b T_R^c \\ &= (X^a - f^{cab} \Lambda^c X^b) T_R^a , \end{aligned} \quad (11.4)$$

that is

$$X^a \mapsto X^{a'} = X^a + f^{abc} \Lambda^b X^c , \quad (11.5)$$

that, according to (11.2), means that X^a is a vector in the adjoint representation space.

Let us now exponentiate the transformation (11.5)

$$X^{a'} = S^{ab} X^b = \exp(-\Lambda^c f^c)^{ab} X^b, \quad (11.6)$$

where

$$S^{ab} = \exp(-i\Lambda^c T_A^c)^{ab} = \exp(-\Lambda^c f^c)^{ab},$$

which is an element of $SO(n)$, $n = \dim \mathfrak{g}$.

Note that comparing (11.6) with (11.3), we get the mentioned adjoint action of the Lie group on its algebra. In fact, we have

$$\begin{aligned} U X U^{-1} &= (U_{ik} T^a U_{kj}^{-1}) X^a = T^b X^{b'} \\ &= T_{ij}^b \exp(-\Lambda^c f^{cba}) X^a, \end{aligned}$$

with U and T^a belonging to the same (arbitrary) representation that is

$$\exp(-\Lambda^c f^{cba}) T_{ij}^b = U_{ik} T_{kl}^a U_{lj}^{-1}, \quad (11.7)$$

that can be proved by using the Baker-Campbell-Hausdorff type formula

$$\exp(A) \exp(B) \exp(-A) = \sum_{k=0}^{\infty} \frac{1}{k!} [A, [A, [\dots [A, B] \dots]]].$$

We note again that by (11.6) it follows that the transformation property of X^a is independent of the chosen representation. The only condition is that the representation of U_R , in the transformation rule $X' = U_R X U_R^{-1}$, must be the same representation of T_R^a that defines X .

An equivalent proof of (11.7) follows by observing that the transformation of a vector in the adjoint representation is equivalent to a rotation of the T^a 's. Such an invariance can be formulated by introducing the scalar product in the vector space defined by the Lie algebra

$$\langle X|Y \rangle := \text{Tr}(XY) = X^a Y^b \text{Tr}(T^a T^b) = t X^a Y^a, \quad (11.8)$$

where

$$t := \text{Tr}(T^a T^b).$$

Such a scalar product is clearly invariant under rotations of the basis T^a or, equivalently, of the components X^a, Y^a , of the vectors X and Y . Then, the proof of (11.7) simply follows by

$$\langle X'|Y' \rangle = \text{Tr}(U X Y U^{-1}) = \langle X|Y \rangle,$$

and the observation that rotations correspond to $SO(n)$ transformations.

11.2 Main features of Yang-Mills theories

Yang-Mills theories are theories which are invariant under non-Abelian gauge transformations. Let us consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^T \partial^\mu \Phi - \frac{1}{2}m^2 \Phi^T \Phi ,$$

where Φ is a scalar-field with n -components. One can easily see that the theory is invariant under the global transformations

$$\begin{cases} \Phi \mapsto \Phi' = U\Phi , \\ \partial_\mu \Phi \mapsto (\partial_\mu \Phi)' = U(\partial_\mu \Phi) , \end{cases}$$

with $U \in O(n)$. By Noether's theorem, a conserved current corresponds to each one of the generators

$$J_\mu^a = i\partial_\mu \Phi^T T^a \Phi .$$

To obtain what is properly known as Yang-Mills theory we have to include transformations which are *local*, i.e.

$$U \mapsto U(x) := \exp(-i\Lambda(x)) ,$$

where

$$\Lambda(x) := \Lambda^a(x)T^a .$$

To keep the theory invariant we must introduce *covariant derivative* is required , i.e. we have to substitute

$$\partial_\mu \mapsto D_\mu := \partial_\mu - igA_\mu ,$$

where $A_\mu = A_\mu^a T^a$ is the gauge field transforming in the adjoint representation with an inhomogeneous term⁵

$$A_\mu \mapsto A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} ,$$

whose infinitesimal version reads

$$A_\mu^a \mapsto A_\mu^{a'} = A_\mu^a + f^{abc}\Lambda^b(x)A_\mu^c(x) - \frac{1}{g}\partial_\mu \Lambda^a(x) . \quad (11.9)$$

⁵ As shown in section (11.1), the transformation in the adjoint representation, corresponds to (11.3), with U in an arbitrary representation. The only condition is that U must be in the same representation of the T^a 's chosen to define $X = X^a T^a$. In the case at hand, U and the T^a 's should be in the same representation used to define the transformation of the field Φ . This follows by consistency with the definition of the covariant derivative. More generally, one may have different representations of the covariant derivative, each one fixed by the representation defining the gauge transformation of the field interacting with A_μ^a .

We have

$$D_\mu \Phi \mapsto (D_\mu \Phi)' = U(x)(D_\mu \Phi) .$$

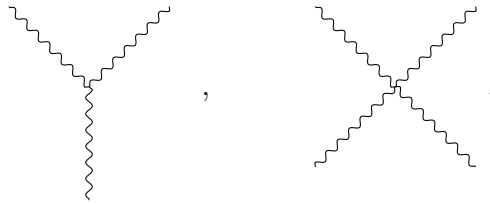
The above shows that to get gauge invariance one should add the gauge field. As a consequence, the Lagrangian density must be completed to introduce the kinetic part for A_μ^a . To this end, we introduce the strength-tensor

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \sum_{b,c} f^{abc} A_\mu^b A_\nu^c ,$$

and the kinetic term for A_μ^a

$$\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} ,$$

which is called Yang-Mills Lagrangian density. A key difference with respect to the Abelian case is that now the gauge-fields are self-interacting. In particular, there are the three- and four-point self-interactions



Setting $F_{\mu\nu} = F_{\mu\nu}^a T^a$, we see that \mathcal{L}_{YM} is proportional to $\text{Tr}(F^{\mu\nu} F_{\mu\nu})$, that is the kinetic term is proportional to the invariant length of the vector $F_{\mu\nu}$. For example, in the case of the Lie algebra $su(n)$, we can choose the standard normalisation

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab} ,$$

so that

$$\mathcal{L}_{YM} = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu}) .$$

The associated equations of motion are

$$(D^\mu F_{\mu\nu})^a = 0 .$$

The field tensor can be also expressed in terms of commutator of covariant derivatives

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu] .$$

A useful relation is the *Bianchi identity*

$$(D_\mu F_{\nu\rho})^a + (D_\rho F_{\mu\nu})^a + (D_\nu F_{\rho\mu})^a = 0 ,$$

that, by

$$[D_\mu, F_{\nu\rho}^a] = D_\mu F_{\nu\rho}^a ,$$

is equivalent to the Jacobi identity

$$[D_\mu, [D_\nu, D_\rho]] + [D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] = 0 .$$

The Bianchi identity can also be written in the form

$$D_\mu \tilde{F}^{\mu\nu} = 0 ,$$

where

$$\tilde{F}^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} ,$$

is the dual field tensor.

11.3 Gauge-fixing for Abelian gauge theories

As usual we start from the generating functional $Z[J]$, which in case of an Abelian gauge theory⁶ (we can think for instance at QED) takes the form

$$Z[J] = \int \mathcal{D}A_\mu e^{i \int d^4x (\mathcal{L}_0 + J^\mu A_\mu)} , \quad (11.10)$$

with

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

which is invariant under the *gauge transformations*

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu \Lambda(x) .$$

Since the equations of motion of the theory are

$$\partial_\mu F^{\mu\nu} = 0 \longleftrightarrow (g_{\mu\nu} \square - \partial_\mu \partial_\nu) A^\nu = 0 ,$$

⁶ A more fundamental expression of the path integral is that in terms of the field and its conjugate momentum, and with $\dot{\varphi}\pi - \mathcal{H}$ instead of \mathcal{L} . However, the canonical path integral reduces to the usual Lagrangian formulation after integration on momenta if the Hamiltonian density is quadratic in π with constant coefficient, and if there are no constraints. Actually, gauge theories (especially Non-Abelian ones) *have* constraints: nevertheless, Faddeev-Popov technique (see below) takes one out of troubles, paying the prices of a little bit more formalism, so we can keep on using the usual Lagrangian formulation.

it follows that integrating by parts and neglecting surface terms, we can rewrite the Lagrangian density in the form

$$\mathcal{L}_0 = \frac{1}{2} A^\mu (g_{\mu\nu} \square - \partial_\mu \partial_\nu) A^\nu . \quad (11.11)$$

The propagator of the theory is the inverse of the operator

$$\widehat{O}_{\mu\nu} := g_{\mu\nu} \square - \partial_\mu \partial_\nu .$$

We can see that the propagator is not well-defined. Let $D^{\mu\nu}(x-y)$ be the inverse of $\widehat{O}_{\mu\nu}$, that is such that

$$(g_{\mu\nu} \square - \partial_\mu \partial_\nu) D^{\nu\lambda}(x-y) = \delta_\mu^\lambda \delta^{(4)}(x-y) .$$

Deriving with respect to x^μ

$$\begin{aligned} \partial^\mu (g_{\mu\nu} \square - \partial_\mu \partial_\nu) D^{\nu\lambda}(x-y) &= \partial^\mu \delta_\mu^\lambda \delta^{(4)}(x-y) , \\ \Rightarrow (\partial_\nu - \partial_\nu) \square D^{\nu\lambda}(x-y) &= \partial^\lambda \delta^{(4)}(x-y) , \end{aligned}$$

showing that $\widehat{O}_{\mu\nu}$ is singular. For example, applying this operator to $\partial^\mu \Lambda$ we get zero

$$(g_{\mu\nu} \square - \partial_\mu \partial_\nu) \partial^\mu \Lambda = (\partial_\nu \square - \square \partial_\nu) \Lambda = 0 .$$

Our first attempt to find the propagator of the theory has failed. Let us notice that in the generating functional (11.10) we are integrating over all fields A_μ , including those which are related one another by a gauge transformation $A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda(x)$: this means an infinite contribution to Z and thus also to the Green's functions. If we want to avoid this infinite and redundant contributions we can choose to fix a particular gauge: for instance we can choose the *Lorenz gauge*⁷

$$\partial_\mu A^\mu = 0 ,$$

so that

$$Z[0] = \int_{A_\mu \in \mathcal{A}} \mathcal{D}A_\mu e^{i \int d^4x \mathcal{L}} ,$$

where the Lagrangian density is

$$\mathcal{L} = \frac{1}{2} A^\mu g_{\mu\nu} \square A^\nu , \quad (11.12)$$

⁷ Such a condition, frequently mistakenly called the Lorentz gauge, has been introduced by the Danish physicist and mathematician Ludvig Valentin Lorenz. Presumably, the mistake originated by the (Hendrikus Albertus) Lorenz invariance property of the Lorenz condition.

and the integration is performed only over those fields such that $\partial_\mu A^\mu = 0$. The propagator is now the inverse of the operator $g_{\mu\nu}\square$, which indeed is not singular. One can easily show that

$$D_{\mu\nu}(x-y) = -g_{\mu\nu}\Delta_F(x,y; m=0) .$$

The problem of redundant degrees of freedom is also seen in the canonical quantisation. Consider the canonical commutation relations

$$[A_\mu(t, \mathbf{x}), \pi_\nu(t, \mathbf{y})] = ig_{\mu\nu}\delta^{(3)}(\mathbf{x}-\mathbf{y}) , \quad (11.13)$$

$$[A_\mu(t, \mathbf{x}), A_\nu(t, \mathbf{y})] = 0 , \quad [\pi_\mu(t, \mathbf{x}), \pi_\nu(t, \mathbf{y})] = 0 ,$$

where

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} ,$$

is the canonical momentum. The problem arises because the Lagrangian density does not contain the $\partial_0 A_0$ term, so that

$$\pi^0 = 0 .$$

This means, in particular, that the A_0 commutes with π_0 , so that the equal time commutation relations cannot be satisfied. As a consequence, even covariance is lost because the g_{00} term in (11.13) is multiplied by zero. Again, one should modify the Lagrangian density and simultaneously fix a gauge condition. Note that at the level of the Lagrangian density, Eq.(11.12) is obtained, integrating by parts, adding to (11.11) the so-called *gauge fixing* term

$$\mathcal{L}_{GF} = -\frac{1}{2}(\partial_\mu A^\mu)^2 .$$

We then have

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{GF} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 .$$

The propagators associated to \mathcal{L}_0 and \mathcal{L} are the inverse of

$$g_{\mu\nu}\square - \partial_\mu\partial_\nu \mapsto -g_{\mu\nu}k^2 + k_\mu k_\nu , \quad (11.14)$$

and

$$g_{\mu\nu}\square \mapsto -g_{\mu\nu}k^2 , \quad (11.15)$$

respectively. One can see that (11.14) has no inverse: if it had inverse, it would be of the form

$$Ag_{\mu\nu} + Bk_\mu k_\nu ,$$

where $A = A(k^2)$ and $B = B(k^2)$. Therefore,

$$(-k^2 g_{\mu\nu} + k_\mu k_\nu)(A g^{\nu\lambda} + B k^\nu k^\lambda) = \delta_\mu^\lambda ,$$

$$\Rightarrow -A k^2 \delta_\mu^\lambda + A k_\mu k^\lambda = \delta_\mu^\lambda \rightsquigarrow \nexists \text{ solutions .}$$

On the other hand, in the Lorenz gauge

$$\tilde{D}_{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2} .$$

More generally, we can rewrite

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 = \frac{1}{2} A^\mu \left[g_{\mu\nu} \square + \left(\frac{1}{\alpha} - 1 \right) \partial_\mu \partial_\nu \right] A^\nu , \quad (11.16)$$

with $\alpha \in \mathbb{R}$ finite. In momentum space, the expression in the square brackets reads

$$-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\alpha} \right) k_\mu k_\nu ,$$

so that the photon propagator reads

$$\tilde{D}_{\mu\nu}(k) = -\frac{1}{k^2} \left[g_{\mu\nu} + (\alpha - 1) \frac{k_\mu k_\nu}{k^2} \right] . \quad (11.17)$$

Common choices of α are

$$\begin{cases} \alpha = 1 : \text{Feynman gauge ,} \\ \alpha = 0 : \text{Landau gauge .} \end{cases}$$

By gauge invariance we have

Claim. *Physics is unaffected by the value of α .*

Equivalent gauge conditions Let us consider a gauge transformation A_μ

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \Lambda ,$$

such that

$$\square \Lambda = -\partial_\mu A^\mu .$$

In this way we see that imposing $\square\Lambda = 0$ gives

$$\partial_\mu A^\mu \longrightarrow \partial_\mu A'^\mu = \partial_\mu A^\mu + \square\Lambda = \partial_\mu A^\mu - \partial_\mu A^\mu = 0 ,$$

which is the Lorenz gauge. This means that rather than constraining the field A_μ , it is possible to impose some conditions on $\Lambda(x)$, obtaining the same result. In particular, we have seen that

$$\square\Lambda = -\partial_\mu A^\mu \longleftrightarrow \partial_\mu A^\mu = 0 .$$

11.4 Gauge fixing and the Faddeev-Popov method

We aim to extend what we have discussed up to now also to Yang-Mills theories, trying to find some general rule for finding the gauge-field propagator. We have seen above that the generating functional Z is infinite, because we integrate over all possible A_μ fields, even over those related by a gauge transformation: therefore we have redundancy. Schematically we can write each gauge-field as

$$A_\mu \sim \bar{A}_\mu, \Lambda(x) ,$$

expressing the idea that a given A_μ can be reached starting from inequivalent gauge-fields \bar{A}_μ performing a gauge transformation of the parameter $\Lambda(x)$. The \bar{A}_μ are inequivalent because they cannot be related by gauge transformations. We can therefore split the integral

$$Z[J_\mu] = \int \mathcal{D}A_\mu e^{iS} \sim \int \mathcal{D}\bar{A}_\mu e^{iS} \int \mathcal{D}\Lambda . \quad (11.18)$$

The overcounting lies in the integration $\int \mathcal{D}\Lambda$: thanks to the Faddeev-Popov method one can show rigorously that the separation (11.18) holds and that one can forget about the infinite redundant contribution.

In order to introduce the Faddeev-Popov method, we need some results concerning the measure on the gauge groups.

In the case of locally compact groups,⁸ it is possible to define a measure, called Haar measure. We are interested in the case of compact groups G . In this case we have⁹

(i)

$$\text{Vol}(G) := \int_G dU = \text{finite} .$$

⁸ A topological space is said locally compact if for each point there exists a neighbourhood whose closure is a compact set.

⁹ For a proof, see M. Hamermesh, "Group Theory and its Application to Physical Problems", page 313.

(ii) Invariance of the measure

$$dU = d(U'U) = d(UU') ,$$

for all $U' \in G$.

In the case of gauge transformations, we should use the functional measure

$$\mathcal{D}U := \prod_{x \in \mathbb{R}^4} dU(x) ,$$

leading to infinite volume. Since the group elements are parameterised in the form $U(x) = \exp(-i\Lambda^a(x)T^a)$, the independent variables are $\Lambda^a(x)$. Then, rather than using DU , it is more appropriate to use $D\Lambda$.

Let us consider the effect of a gauge transformations on the measure of the fields. In the case of a generic field transforming

$$\phi(x) \mapsto \phi'(x) = U(x)\phi ,$$

with U in some representation of G , we have

$$\mathcal{D}\phi' = \prod_x |\det U(x)| \mathcal{D}\phi ,$$

so that, if, as is the generality of the cases of physical interest, $\det U = \pm 1$, then we will have

$$\mathcal{D}\phi' = \mathcal{D}\phi .$$

Let us consider the case of the gauge field measure $\mathcal{D}A_\mu^a$. Note that since the inhomogeneous term in

$$A_\mu \mapsto A_\mu^U := UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger , \quad (11.19)$$

is independent of A_μ^a , we should care only of the term $UA_\mu U^\dagger$. In this respect, note that by (11.6)

$$A_\mu^a(x) \rightarrow A_\mu^{a'}(x) = \exp(-\Lambda^c(x)f^c)^{ab} A_\mu^b(x) + f(x) , \quad (11.20)$$

with $f(x)$ the inhomogeneous term. It follows that

$$\frac{\delta A_\mu^{a'}(x)}{\delta A_\nu^b(y)} = \exp(-\Lambda^c(x)f^c)^{ab} \delta_\mu^\nu \delta^{(4)}(x-y) ,$$

and since $\exp(-\Lambda^c(x)f^c)^{ab} \in SO(n)$, we have that the Jacobian of the transformation

is the identity. Therefore,

$$\mathcal{D}A_\mu^{a'} = \mathcal{D}A_\mu^a .$$

In the following we denote a gauge fixing condition in the form

$$F^a[A_\mu] = 0 .$$

For example, in the Lorenz gauge

$$\partial^\mu A_\mu^a = 0 \longleftrightarrow F^a = \partial^\mu A_\mu^a .$$

Let us set

$$\Delta_F^{-1}[A_\mu] := \int \mathcal{D}U \delta[F[A_\mu^U]] , \quad (11.21)$$

where

$$\delta[F[A_\mu]] = \prod_{x^\mu, a} \delta(F^a[A_\mu(x)]) ,$$

is the *delta functional*. This is the product of delta functions at each point of space-time and for each value of a .

Let us now prove the gauge invariance of $\Delta_F^{-1}[A_\mu]$. Since $\mathcal{D}U = \mathcal{D}U''$, this implies that, in the case of compact groups, $\Delta_F^{-1}[A_\mu]$ is gauge invariant. Indeed, by (11.21)

$$\Delta_F^{-1}[A_\mu^{U'}] = \int \mathcal{D}U \delta[F[A_\mu^{U'U}]] ,$$

and setting $U'' = U'U$, we have

$$\Delta_F^{-1}[A_\mu^{U'}] = \int \mathcal{D}U'' \delta[F[A_\mu^{U''}]] = \Delta_F^{-1}[A_\mu] .$$

The next step of the Faddeev-Popov construction is the insertion of

$$1 = \Delta_F[A_\mu] \Delta_F^{-1}[A_\mu] = \Delta_F[A_\mu] \int \mathcal{D}U \delta[F[A_\mu^U]] ,$$

in the expression of $Z := Z[0]$

$$Z = \int \mathcal{D}A_\mu \Delta_F[A_\mu] \int \mathcal{D}U \delta[F[A_\mu^U]] e^{iS} .$$

We perform a gauge transformation taking A_μ^U to A_μ and use the fact that $\mathcal{D}A_\mu$ is the same as $\mathcal{D}A_\mu^U$. Furthermore, since the action is gauge invariant and, as we have just

shown, even $\Delta_F[A_\mu]$ is, we have

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \Delta_F[A_\mu] \int \mathcal{D}U \delta[F[A_\mu]] e^{iS} \\ &= \left(\int \mathcal{D}U \right) \int \mathcal{D}A_\mu \Delta_F[A_\mu] \delta[F[A_\mu]] e^{iS} . \end{aligned}$$

Since the integrand does not depend on U , it follows that the contribution of $\int \mathcal{D}U$ consists in an overall multiplicative constant and therefore can be ignored.

11.5 Faddeev-Popov determinant

Note that for fixed A_μ , there is at least one $\bar{A}_\mu := A_\mu^{\bar{U}}$ such that $F^a[\bar{A}_\mu] = 0$. We use the invariance on the functional Haar measure, to write

$$\Delta_F^{-1}[A_\mu] := \int \mathcal{D}U \delta[F[\bar{A}_\mu^U]] .$$

Now note that the argument of the δ -functional is vanishing when U is the identity, that is for $\Lambda = 0$, we then can consider the expansion of F^a at the first order in Λ

$$\begin{aligned} F^a[\bar{A}_\mu^U] &\approx F^a[\bar{A}_\mu^b + \mathcal{D}_\mu(\bar{A})\Lambda^b] \\ &= F^a[\bar{A}_\mu^b] + \frac{\partial F^a}{\partial \bar{A}_\mu^b} \mathcal{D}_\mu(\bar{A})\Lambda^b \\ &= \frac{\partial F^a}{\partial \bar{A}_\mu^c} (\partial_\mu \Lambda^c - f^{cdb} \bar{A}_\mu^d \Lambda^b) , \end{aligned} \tag{11.22}$$

that is

$$F^a[\bar{A}_\mu^U] = M^{ab}(\bar{A}_\mu) \Lambda^b ,$$

where

$$M^{ab}(\bar{A}_\mu) := \frac{\partial F^a[\bar{A}_\mu^U]}{\partial \Lambda^b} \Big|_{\Lambda=0} = \frac{\partial F^a}{\partial \bar{A}_\mu^c} (\partial_\mu \delta^{cb} - f^{cdb} \bar{A}_\mu^d) .$$

We then have

$$\begin{aligned} \Delta_F^{-1}[A_\mu] &= \int \mathcal{D}\Lambda \delta[F[\bar{A}_\mu^U]] \\ &= \int \mathcal{D}\Lambda \delta[M^{ab}(\bar{A}_\mu) \Lambda^b] \\ &= \det^{-1} M(\bar{A}_\mu) , \end{aligned} \tag{11.23}$$

where, as explained, we used $\mathcal{D}\Lambda$ rather than $\mathcal{D}U$. Next, observe that since $F[\bar{A}_\mu] = 0$, we can express (11.23) in the equivalent form

$$\int \mathcal{D}U \delta[F(A_\mu^U)] \det M(A_\mu^U) = 1 . \quad (11.24)$$

$\det M(A_\mu)$ is called Faddeev-Popov determinant. Let us stress that the relation $\Delta_F[A_\mu] = \det M(\bar{A}_\mu)$ shows that for each A_μ the associated determinant is the one for which $F^a[\bar{A}_\mu] = 0$. We obtained \bar{A}_μ by a gauge transformation $A_\mu \mapsto A_\mu^{\bar{U}}$. Therefore, \bar{U} depends on the initial A_μ .

Now consider iM instead of M , so that

$$\det(iM) = \int \mathcal{D}c \mathcal{D}\bar{c} \exp \left(-i \int d^4x \bar{c}^a M_{ab} c^b \right) .$$

It is worth stressing that the fields c and \bar{c} are anticommuting scalars. As a consequence of the spin-statistics theorem, they are unphysical, so that they cannot appear in the external legs of Feynman diagrams. For this reason these fields are called *ghosts*.

Next, we modify the gauge fixing term $\delta[F^a[A_\mu]]$: for example, instead of the Lorenz gauge condition we may consider

$$F^a = \partial^\mu A_\mu^a + C^a ,$$

with $C^a(x)$ an arbitrary function. We then have

$$Z = \int \mathcal{D}A_\mu \det(iM) \delta[F[A_\mu] - C] \exp(iS) , \quad (11.25)$$

where

$$\delta[F[A_\mu] - C] = \prod_{x^\mu, a} \delta(F^a[A_\mu(x)] - C^a(x)) ,$$

Note that the C^a 's are independent of A_μ , so that, since $\Delta_F[A_\mu]$ depends on the functional derivative of F^a with respect to U , it follows that $\Delta_F[A_\mu]$, and therefore $\Delta_F^{-1}[A_\mu]$, remains invariant if in its expression one replaces $\delta[F[A_\mu]]$ by $\delta[F[A_\mu] - C]$. This implies that Z is independent of the C^a 's. Therefore, we can safely insert in (11.25) the term

$$\exp \left(-\frac{i}{2\alpha} \int d^4x C^2(x) \right) ,$$

where $C^2 = \sum_a C^{a2}$. Integrating over the C^a 's, we get

$$\begin{aligned} Z &= \int \mathcal{D}A_\mu \mathcal{D}C \det(iM) \delta[F[A_\mu] - C] \exp\left(iS - \frac{i}{2\alpha} \int d^4x C^2(x)\right) \\ &= N \int \mathcal{D}A_\mu \mathcal{D}\bar{c} \mathcal{D}c \exp\left[i \int d^4x \left(\mathcal{L} - \frac{1}{2\alpha} F^2 - \bar{c}^a M_{ab} c^b\right)\right] \\ &= N \int \mathcal{D}A_\mu \mathcal{D}\bar{c} \mathcal{D}c \exp\left(i \int d^4x \mathcal{L}_{\text{eff}}\right), \end{aligned}$$

where the effective Lagrangian density is given by

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \mathcal{L} - \frac{1}{2\alpha} F^2 - \bar{c}^a M_{ab} c^b \\ &= \mathcal{L} + \mathcal{L}_{GF} + \mathcal{L}_{FPG}, \end{aligned} \tag{11.26}$$

with \mathcal{L}_{GF} the gauge-fixing term we met before, while \mathcal{L}_{FPG} is the so-called Faddeev-Popov *ghost* term. Therefore, since in the Lorenz gauge the Faddeev-Popov determinant is independent of A_μ , such a determinant can be absorbed in the normalisation constant of the functional generator.

A related issue concerns the physical degrees of freedom. As we know, the only physical components of A_μ are the two transverse ones. Up to now we have imposed only one condition, which is not enough to get rid of all spurious degrees of freedom. One way to proceed is to consider the *Coulomb* or *radiation gauge*

$$A_0 = 0, \quad \nabla \cdot \mathbf{A} = 0.$$

Certainly now we deal only with physical degrees of freedom, but in this way we have the unpleasant consequence that the manifest Lorenz covariance is lost. However, for QED this is not a problem because photons always couple to conserved currents. In the case of non-Abelian gauge theories the problem of eliminating unphysical degrees of freedom, preserving manifest Lorentz covariance, is solved by applying the Becchi-Rouet-Stora (BRS) quantisation.

As we said, the Lorenz gauge concerns only one condition, so there remains a spurious degree of freedom. Let us consider the matrix M in the Lorenz gauge, that is

$$M(x, y) = -\frac{1}{e} \square \delta^{(4)}(x - y),$$

which is independent of A_μ , so that it can be absorbed in the overall normalisation constant of the generating functional. Let us consider the following decomposition of the gauge field

$$A_\mu = A_\mu^T + A_\mu^L,$$

where A_μ^T and A_μ^L denote the transverse and longitudinal components of A_μ , respectively. We have

$$A_\mu^T(x) = P_{\mu\nu}A^\nu, \quad A_\mu^L = (g_{\mu\nu} - P_{\mu\nu})A^\nu,$$

with

$$P_{\mu\nu} = g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\square},$$

the projection operator. Now note that since we have the identity

$$\partial^\mu A_\mu^T = 0,$$

it follows that the functional distribution $\delta[\partial_\mu A^\mu]$ has effect on the longitudinal components of the gauge fields only. On the other hand, the longitudinal component of A_μ is a pure gauge, that is

$$A_\mu^L(x) = \partial_\mu\Lambda(x),$$

so that the effect of the $\delta[\partial_\mu A^\mu]$ constraint is just

$$\square\Lambda = 0.$$

This result is in contradiction with the assumption that $F[A_\mu^U]$ has, for any A_μ , a unique solution. The way out is to observe that such a residual invariance can be removed by a suitable choice of the boundary conditions. On the other hand, this in fact what we usually do in perturbation theory, since we usually assume that the fields vanish at infinity, which is a necessary condition to eliminate such fields once one integrates by parts. We then remove the residual gauge freedom assuming that A_μ vanishes at infinity.

We now may give the QED Feynman rules in the Lorenz gauge

▷ boson propagator

$$\begin{array}{c} \xrightarrow{k} \\ \mu \text{---} \text{---} \text{---} \nu \end{array} = -\frac{i}{k^2} \left[g_{\mu\nu} + (\alpha - 1) \frac{k_\mu k_\nu}{k^2} \right],$$

▷ vertex

$$\begin{array}{c} \mu \\ | \\ \alpha \text{---} \text{---} \beta \end{array} = -ie\gamma_{\alpha\beta}^\mu (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3),$$

▷ fermion propagator¹⁰

$$\begin{array}{c} \xrightarrow{p} \\ \xrightarrow{\quad} \\ \alpha \quad \beta \end{array} = \left(\frac{i}{\not{p} - m + i\epsilon} \right)_{\alpha\beta} ,$$

▷ each fermion loop carries an overall minus sign.

11.6 The Becchi-Rouet-Stora transformation

In the previous section, the discussion has been simplified by the fact that in QED we can ignore the ghost term. Let us now consider the more general situation of non-Abelian gauge theories.

We continue to denote by Z the generating functional with the external currents for the gauge and ghost fields, set to zero. We have

$$Z = N \int \mathcal{D}A_\mu \mathcal{D}\bar{c} \mathcal{D}c e^{i \int d^4x \mathcal{L}_{\text{eff}}} ,$$

where the effective Lagrangian density reads

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \mathcal{L}_{GF} + \mathcal{L}_{FPG} .$$

If we choose the Lorenz gauge, the gauge-fixing term is

$$\mathcal{L}_{GF} = -\frac{1}{2\alpha} (\partial_\mu A^{a\mu})^2 ,$$

and one can show that

$$\mathcal{L}_{FPG} = -\bar{c}^a \partial^\mu D_\mu c^a + \text{total derivative} ,$$

and the total derivative can be ignored because it only contributes with surface terms to the action, terms that we assume vanish. Our goal is to understand the behaviour of the effective Lagrangian density under a gauge transformation

$$\begin{aligned} A'_\mu &= U A_\mu U^\dagger - \frac{i}{g} (\partial_\mu U) U^\dagger , \\ \delta A^a_\mu &= \frac{1}{g} \partial_\mu \Lambda^a + f^{abc} A^b_\mu \Lambda^c = \frac{1}{g} (D_\mu \Lambda)^a . \end{aligned}$$

¹⁰ Here we are using a rather symbolic notation, whose definition is

$$\left(\frac{i}{\not{p} - m + i\epsilon} \right)_{\alpha\beta} := i \left[\left(\not{p} - m + i\epsilon \right)^{-1} \right]_{\alpha\beta} .$$

Since we have the freedom to choose any gauge transformation we want, we choose

$$\Lambda^a(x) = -c^a(x)\lambda ,$$

where λ is a Grassmann constant. Therefore,

$$\delta A^a_{\mu} = -\frac{1}{g}(D_{\mu}c^a)\lambda . \quad (11.27)$$

We can also fix¹¹

$$\delta c^a = -\frac{1}{2}f^{abc}c^bc^c\lambda , \quad (11.28)$$

$$\delta \bar{c}^a = -\frac{1}{\alpha g}(\partial_{\mu}A^{a\mu})\lambda . \quad (11.29)$$

In the following we show that

$$\delta \mathcal{L}_{\text{eff}} = 0 .$$

The equations (11.27), (11.28) and (11.29) are known as *Becchi-Rouet-Stora (BRS) transformations*. Now we have to prove that the effective Lagrangian density is indeed invariant under BRS transformations. We have

▷ the free gauge-gauge field term $\mathcal{L}_0 = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ is trivially invariant

$$\delta \mathcal{L}_0 = 0 ,$$

▷ for the gauge fixing term one has

$$\delta \mathcal{L}_{GF} = \frac{1}{\alpha}(\partial^{\mu}A^a_{\mu})\frac{1}{g}(\partial^{\nu}D_{\nu}c^a)\lambda ,$$

▷ for the ghost term

$$\begin{aligned} \delta \mathcal{L}_{FPG} &= -(\delta \bar{c}^a)\partial^{\mu}D_{\mu}c^a - \bar{c}^a\partial^{\mu}(\delta D_{\mu}c^a) \\ &= \frac{1}{\alpha g}(\partial^{\mu}A^a_{\mu})\lambda(\partial^{\nu}D_{\nu}c^a) = -\frac{1}{\alpha g}(\partial^{\mu}A^a_{\mu})(\partial^{\nu}D_{\nu}c^a)\lambda , \end{aligned}$$

where in passing from the first to the second line, we dropped the term $\bar{c}^a\partial^{\mu}(\delta D_{\mu}c^a)$ since it vanishes.¹²

Therefore, the effective Lagrangian density is invariant under BRS transformations. Using the BRS transformations one gets the *Taylor-Slavnov identities*, which are the

¹¹ Notice that the transformations of c^a and \bar{c}^a are different from each other, which is consistent with the fact these are *independent* fields.

¹² See Ryder's book [36, p. 272] for an explicit proof.

non-Abelian analogue of QED's Ward-Takahashi identities.

Chapter 12

Quantum electrodynamics¹

We now specialise our discussion about gauge theories to the case of quantum electrodynamics. Though at first sight QED seems much simpler than the others theories of fundamental interactions (and, indeed, it is), it has several interesting features. Moreover, it is a sort of laboratory to see how we can apply the general techniques we have already discussed. The main references for this chapter are [7, 11, 36, 8, 37].

12.1 Ward-Takahashi identities

The Ward-Takahashi identities are *exact* relations between vertex functions and propagators, thus true at all order of perturbation theory. They are a consequence of gauge invariance of QED, and are the key to renormalise the theory. As usual, our starting point is the generating functional

$$Z[J_\mu, \eta, \bar{\eta}] = N \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \mathcal{L}_{\text{eff}}},$$

where now the effective Lagrangian density (11.26) reads

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}(\not{\partial} + ie\not{A})\psi - m\bar{\psi}\psi - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + J_\mu A^\mu + \bar{\eta}\psi + \bar{\psi}\eta,$$

and it contains a free photon and fermion (electron) part, the covariant derivative instead of the usual derivative, the gauge fixing term for the electromagnetic field and the source term for fermions and electromagnetic field. What is missing is the Faddeev-Popov term: nevertheless, in the Lorenz gauge the ghosts do not couple with the electromagnetic field, therefore their contribution is an overall constant which can be reabsorbed into N .

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Before proceeding it is worth mentioning a property of the two-point gauge and fermion functions. In this respect, recall that since in the scalar theories with even potential densities the $2k + 1$ -point functions vanish, and since the only possible non-connected components of the 2-point function is given by the product of two 1-point functions, it follows that the two-point Green's functions are connected. As we will see below, by Furry theorem, the Green functions in which the only external fields are an odd number of photon fields vanish (as we will see this does not mean that the single diagram vanishes). It follows that the photon propagator is connected

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \langle 0|TA_\mu(x)A_\nu(y)|0\rangle_c .$$

The case of the fermion propagators is more similar to the case of the even scalar potential densities: the fact that the Lagrangian density is bilinear in the fermion fields implies that there are no 1-point fermion functions. Therefore, even in this case, we have

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle_c .$$

We recall that this Lagrangian density is gauge invariant *only if* we remove the source terms and the gauge fixing terms. However, this last term (as well as the ghost term) is necessary if we want to find the propagator. Yet we have to remember that physical predictions of the theory must be independent from the chosen gauge, hence Z as well must be gauge invariant. Let us see what this requirement implies. We apply an infinitesimal gauge transformation²

$$\begin{aligned} A_\mu &\longrightarrow A'_\mu = A_\mu + \partial_\mu\Lambda , \\ \psi &\longrightarrow \psi - ie\Lambda\psi , \quad \bar{\psi} \longrightarrow \bar{\psi} + ie\Lambda\bar{\psi} , \end{aligned}$$

where e is the fermion charge. Note that here $e = |e| > 0$, so that, following the standard notation, we have³

$$\text{positron charge } e > 0 , \quad \text{electron charge } -e < 0 ,$$

implying that terms such as e^{2n+1} , that may appear, e.g., in the Feynman diagrams, should correspond to $-e^{2n+1}$ if referred to the electron.⁴ As mentioned above, only the first three terms in \mathcal{L}_{eff} are invariant. The other terms instead transform so that the

² Here we are representing the $U(1)$ group elements by $U = e^{-ie\Lambda}$ instead of $U = e^{-i\Lambda}$.

³ A more correct notation would be to replace e by q . Furthermore, since e is usually associated to the charge of the electron, it should be considered negative. Then, one should use e to denote the electron charge and $-e > 0$ to denote the positron charge.

⁴ This notation choice may cause some tricky wrong signs.

integrand of Z acquires a factor

$$\exp\left\{i \int d^4x \left[-\frac{1}{\alpha}(\partial_\mu A^\mu)\square\Lambda + J^\mu\partial_\mu\Lambda - ie\Lambda(\bar{\eta}\psi - \bar{\psi}\eta)\right]\right\},$$

but since we are considering an infinitesimal transformation, this may be written as

$$1 + i \int d^4x \left[-\frac{1}{\alpha}\square(\partial_\mu A^\mu) - \partial_\mu J^\mu - ie(\bar{\eta}\psi - \bar{\psi}\eta)\right]\Lambda(x),$$

after integration by parts to remove derivation from Λ . We then have

$$\left[\frac{i}{\alpha}\square\partial_\mu\frac{\delta}{\delta J_\mu} - i\partial_\mu J^\mu - e\left(\bar{\eta}\frac{\delta}{\delta\bar{\eta}} - \eta\frac{\delta}{\delta\eta}\right)\right]Z[\eta, \bar{\eta}, J_\mu] = 0,$$

so that, since $Z = e^{iW}$, we get

$$\frac{1}{\alpha}\square\partial_\mu\frac{\delta W}{\delta J_\mu} + i\partial_\mu J^\mu + ie\left(\bar{\eta}\frac{\delta W}{\delta\bar{\eta}} - \eta\frac{\delta W}{\delta\eta}\right) = 0. \quad (12.1)$$

Such an identity has several important consequences. Let us start by deriving an identity for the longitudinal part of the 2-point gauge functions, that follows by differentiating (12.1) with respect to J_μ , and then setting the sources to zero

$$\frac{1}{\alpha}\square_x\frac{\partial}{\partial x_\mu}\frac{\delta^2 W[0]}{\delta J^\mu(x)\delta J^\nu(y)} = -\frac{\partial}{\partial x^\nu}\delta^{(4)}(x-y),$$

which is a Ward identity. In momentum space such an equation reads

$$\frac{i}{\alpha}k^2 k^\mu \tilde{G}_{\mu\nu}(k) = k_\nu, \quad (12.2)$$

where $\tilde{G}_{\mu\nu}$ is the Fourier transform of the 2-point photon function

$$G_{\mu\nu}(x-y) = \langle 0|TA_\mu(x)A_\nu(y)|0\rangle.$$

As usual, such a function is i times the exact propagators $D'_{\mu\nu}(x-y)$

$$G_{\mu\nu}(x-y) = iD'_{\mu\nu}(x-y).$$

The solution of (12.2) reads

$$\tilde{G}_{\mu\nu}(k) = -i\alpha\frac{k_\mu k_\nu}{k^4} + \tilde{G}_{\mu\nu}^T(k).$$

where $\tilde{G}_{\mu\nu}^T(k)$ is the transverse projection, that is the solution of the homogeneous

version of (12.2). This is given by

$$\tilde{G}_{\mu\nu}^T(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) f(k^2) ,$$

with f an arbitrary function. The Ward identity (12.2) is an exact one, so that it provides a constraint to any order of perturbation theory.

As we will see, the renormalisation procedure leads to a bare Lagrangian density which has the same functional form of the original one. It follows that the Ward identities extend to the renormalised theory. Now observe that the gauge dependent longitudinal part is exactly the same of the original one, given in (11.17). This implies that the renormalisation leave invariant such a term, that is the gauge fixing term in the effective Lagrangian density does not require any counterterms. This will be explicitly seen in considering the renormalisation of QED.

Eq.(12.2) implies other interesting identities. Let us consider the effective action

$$\Gamma[\tilde{\psi}, \tilde{\bar{\psi}}, A_\mu] = W[\eta, \bar{\eta}, J_\mu] - \int d^4x (\bar{\eta}\tilde{\psi} + \tilde{\bar{\psi}}\eta + J^\mu \tilde{A}_\mu) , \quad (12.3)$$

where

$$\tilde{O} := \langle 0|O|0\rangle_{J_\mu, \eta, \bar{\eta}} ,$$

with O any one of the three field operators A_μ , ψ and $\bar{\psi}$. Note that

$$\frac{\delta W}{\delta J_\mu(x)} = \tilde{A}^\mu(x) , \quad \frac{\delta W}{\delta \bar{\eta}(x)} = \tilde{\psi}(x) , \quad \frac{\delta W}{\delta \eta(x)} = -\tilde{\bar{\psi}}(x) .$$

One may also check that

$$\frac{\delta \Gamma}{\delta \tilde{A}_\mu(x)} = -J^\mu(x) , \quad \frac{\delta \Gamma}{\delta \tilde{\psi}(x)} = \bar{\eta}(x) , \quad \frac{\delta \Gamma}{\delta \tilde{\bar{\psi}}(x)} = -\eta(x) .$$

Eq.(12.1) now reads

$$-\frac{\square}{\alpha} \partial_\mu \tilde{A}^\mu + i \partial_\mu \frac{\delta \Gamma}{\delta \tilde{A}_\mu} + i e \tilde{\psi} \frac{\delta \Gamma}{\delta \tilde{\psi}} - i e \tilde{\bar{\psi}} \frac{\delta \Gamma}{\delta \tilde{\bar{\psi}}} = 0 . \quad (12.4)$$

Such an equation shows that Γ is not gauge invariant, rather, the gauge invariant functional is⁵

$$\Gamma_\alpha[\psi, \bar{\psi}, A_\mu] := \Gamma[\psi, \bar{\psi}, A_\mu] + \frac{1}{2\alpha} \int d^4x (\partial_\mu A^\mu)^2 .$$

Gauge invariance of $\Gamma_\alpha[\psi, \bar{\psi}, A_\mu]$ rather than of $\Gamma[\psi, \bar{\psi}, A_\mu]$, can be also argued that in the classical limit $\Gamma[\psi, \bar{\psi}, A_\mu]$ is the classical action, whose gauge invariance is broken

⁵ In the following the tilde over the fields is omitted.

just by the gauge fixing term.

Deriving (12.4) with respect to ψ and $\bar{\psi}$, and then putting $\psi = \bar{\psi} = A_\mu = 0$, we get

$$\begin{aligned} \partial_x^\mu \frac{\delta^3 \Gamma[0]}{\delta \bar{\psi}(x_1) \delta \psi(y_1) \delta A^\mu(x)} &= \\ &= ie \left(\delta^{(4)}(x - x_1) \frac{\delta^2 \Gamma[0]}{\delta \bar{\psi}(x_1) \delta \psi(y_1)} - \delta^{(4)}(x - y_1) \frac{\delta^2 \Gamma[0]}{\delta \bar{\psi}(x_1) \delta \psi(y_1)} \right). \end{aligned} \quad (12.5)$$

As done in ϕ_4^4 even here it is useful to consider the 1PI amputated functions. Let us consider the proper vertex function $\Gamma_\mu(p, q, p')$ by

$$\int d^4x d^4x_1 d^4y_1 e^{i(p'x_1 - py_1 - qx)} \frac{\delta^3 \Gamma[0]}{\delta \bar{\psi}(x_1) \delta \psi(y_1) \delta A^\mu(x)} = (2\pi)^4 \delta^{(4)}(p' - p - q) \Gamma_\mu(p, q, p').$$

As in the case of ϕ_4^4 , we have that the amputated 1PI two-point function is proportional to the inverse of the two-point function, that is

$$\frac{\delta^2 \Gamma[0]}{\delta \bar{\psi}(x) \delta \psi(y)} = i(iS'_F(x - y))^{-1}, \quad (12.6)$$

where $S'_F(x - y)$ is the exact propagator in configuration space. In momentum space we have

$$(2\pi)^4 \delta^{(4)}(p' - p) S'^{-1}_F(p) = - \int d^4x d^4y \frac{\delta^2 \Gamma[0]}{\delta \bar{\psi}(x) \delta \psi(y)} e^{i(p'x - py)}.$$

Multiplying (12.5) by $\exp[i(p'x_1 - py_1 - qx)]$, and then integrating over x, x_1 and y_1 , we get the *Ward-Takahashi identity*

$$q^\mu \Gamma_\mu(p, q, p + q) = e [S'^{-1}_F(p + q) - S'^{-1}_F(p)], \quad (12.7)$$

which can be pictorially represented as

$$q^\mu \Gamma_\mu(p, q, p + q) = e \left[\left(\text{fermion}(p+q) \text{---} \text{circle} \text{---} \text{fermion}(p+q) \right)^{-1} - \left(\text{fermion}(p) \text{---} \text{circle} \text{---} \text{fermion}(p) \right)^{-1} \right].$$

Taking the limit $q^\mu \rightarrow 0$ we get instead the *Ward identity*

$$e \frac{\partial S'^{-1}_F(p)}{\partial p^\mu} = \Gamma_\mu(p, 0, p). \quad (12.8)$$

This relation holds at all orders in perturbation theory. Let us consider the first orders. For instance, to first order, S'_F is just the free propagator S_F , hence

$$S_F^{-1}(p) = \gamma_\mu p^\mu - m \rightarrow \frac{\partial S_F^{-1}}{\partial p^\mu} = \gamma_\mu . \quad (12.9)$$

Expanding the exact propagator

$$\begin{aligned} iS'_F &= iS_F + iS_F \frac{\Sigma}{i} iS_F + iS_F \frac{\Sigma}{i} iS_F \frac{\Sigma}{i} iS_F + \dots \\ &= iS_F \left(1 - \frac{\Sigma}{i} iS_F \right)^{-1} , \end{aligned}$$

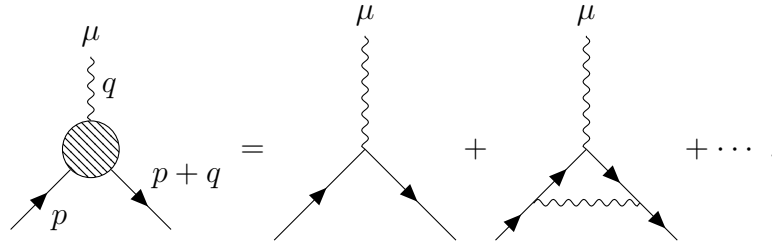
we see that

$$S'^{-1}_F(p) = S_F^{-1}(p) - \Sigma(p) ,$$

where $\Sigma(p)/i$ is the electron self-energy. By (12.9) it follows that

$$\begin{aligned} \frac{\partial S'^{-1}_F}{\partial p^\mu} &= \frac{\partial S_F^{-1}}{\partial p^\mu} - \frac{\partial \Sigma}{\partial p^\mu} \\ &= \gamma_\mu - \frac{\partial \Sigma}{\partial p^\mu} . \end{aligned} \quad (12.10)$$

Note that the expansion of $\Gamma_\mu(p, q, p+q)$ has the form



Let us denote by $\Lambda_\mu(p, q, p+q)$ the function $\Gamma_\mu(p, q, p+q)$ without the zero-loop contribution, that is

$$\Gamma_\mu(p, q, p+q) = e\gamma_\mu + e\Lambda_\mu(p, q, p+q) .$$

Note that Λ_μ represents the non-trivial 1PI contributions to Γ_μ . The Ward identity (12.8) and (12.10) imply

$$\Lambda_\mu(p, 0, p) = -\frac{\partial \Sigma(p)}{\partial p^\mu} . \quad (12.11)$$

Now notice that differentiating with respect to p^μ the identity $S_F(p)S_F^{-1}(p) = 1$ we get

$$\begin{aligned} \frac{\partial S_F(p)}{\partial p^\mu} &= -S_F(p) \frac{\partial S_F^{-1}(p)}{\partial p^\mu} S_F(p) \\ &= -S_F(p) \gamma_\mu S_F(p) , \end{aligned}$$

which means that the differentiation of the propagator with respect to p^μ corresponds to the insertion of a zero momentum photon line into the internal electron line.

12.2 Furry theorem

This is a useful theorem especially in loop calculations, since it ensures that under a certain condition we can get rid of some loops.

Theorem 12.2.1 (Furry). *Any scattering amplitude with no external fermions, and an odd number of external photons, is zero.*

Proof. Such an amplitude would come from a correlation function

$$\langle \Omega | T A^{\mu_1}(x_1) \dots A^{\mu_n}(x_n) | \Omega \rangle ,$$

inserted into the LSZ reduction formula. To see that this vanishes, we insert $\mathbb{I} = \mathcal{C}\mathcal{C}^{-1}$, with \mathcal{C} the charge conjugation operator, between each pair of fields, and on the far left and far right. Since the vacuum is unique, it must be invariant under charge conjugation

$$\mathcal{C}^{-1} | \Omega \rangle = | \Omega \rangle \quad \text{and} \quad \langle \Omega | \mathcal{C} = \langle \Omega | .$$

Since the T product of the gauge fields is a linear combination of such fields involving the Heaviside θ functions, it follows that T and \mathcal{C} commute. Using

$$\mathcal{C}^{-1} A^\mu(t, \mathbf{x}) \mathcal{C} = -A^\mu(t, \mathbf{x}) ,$$

we see that this correlation function is equal to $(-1)^n$ times itself, and so must vanish if n is odd. Note that this means that the amplitude vanishes even if the photon momenta are off shell, so it also does not appear as an off-shell Green's function in some other more complicated process.

It should be observed that this theorem concerns the Green's functions and does not in general imply that a singular diagram vanishes, since in considering the Green's functions one has to consider the sum of diagrams rather than a singular contributions. In particular, adding the contribution due to reversed orientation of each fermionic loop, gives, each one, an overall minus sign. Note that opposite fermionic loop orientations

correspond to opposite directions of the charge flow. In other words, the content of the Furry theorem is that for each Feynman diagram with an odd number of external photon legs, there is another Feynman diagram with the opposite value.

□

12.3 Superficial degree of divergence

We will see that by applying the same systematic procedures we developed for the ϕ_4^4 -theory, we can handle also divergent QED integrals.

Let us begin by recalling the Feynman rules for QED

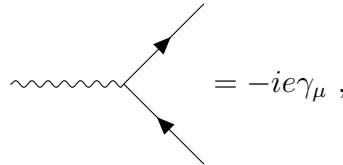
▷ fermion propagator

$$\begin{array}{c} \xrightarrow{p} \\ \longrightarrow \end{array} = \frac{i}{\not{p} - m} = i \frac{\not{p} + m}{p^2 - m^2} ,$$

▷ photon propagator

$$\begin{array}{c} \xrightarrow{k} \\ \sim \end{array} = -\frac{i}{k^2} \left[g^{\mu\nu} + (\alpha - 1) \frac{k^\mu k^\nu}{k^2} \right] ,$$

▷ vertex⁶



$$= -ie\gamma_\mu ,$$

▷ each fermion loop brings a factor of (-1) .

From now on we will use the Feynman gauge, i.e. we will set $\alpha = 1$.

The general formula for the superficial degree of divergence of a diagram is given by

$$D = dL - 2P_i - E_i ,$$

where

$$\left\{ \begin{array}{l} L := \text{number of loops,} \\ P_i := \text{number of internal photon lines,} \\ E_i := \text{number of internal electron lines,} \\ d := \text{dimension.} \end{array} \right.$$

⁶ Note that, as done in the case of the Feynman rules for ϕ_4^4 , we did not include the factor $(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3)$. On the other hand, momentum conservation is easily implemented at each vertex, so that it remains to integrate on the unfixed internal momenta. In this way, one avoids the insertion of several δ 's and trivial integrations.

In addition, let us introduce

$$\begin{cases} n := \text{number of vertices,} \\ P_e := \text{number of external photon lines,} \\ E_e := \text{number of external electron lines.} \end{cases}$$

We recall that the number of loops L , i.e. the number of internal independent momenta, is given by

$$L = I - (n - 1) ,$$

where $I :=$ total number of internal lines $= P_i + E_i$: we have to subtract n because of the δ -functions enforcing the momentum conservation at each vertex, whose effect is to decrease the number of integrations. The $+1$ is due to the fact that one of this δ -functions enforces the conservation of the external momenta. Each vertex has two electron legs: if they are internal they count twice, otherwise once, so that

$$2n = E_e + 2E_i .$$

In the case of photons, instead,

$$n = P_e + 2P_i .$$

It follows that we can rewrite the superficial degree of divergence in the form

$$\begin{aligned} D &= (d - 1)E_i + (d - 2)P_i - d(n - 1) \\ &= d + n\left(\frac{d}{2} - 2\right) - \left(\frac{d - 1}{2}\right)E_e - \left(\frac{d - 2}{2}\right)P_e , \end{aligned}$$

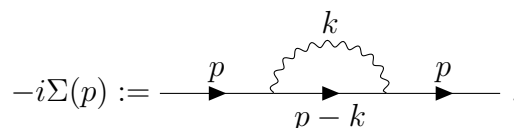
that in four dimension reads

$$D = 4 - \frac{3}{2}E_e - P_e ,$$

and there is no dependence on n , so that, in particular, D does not increase with n , as required for any renormalisable theory.

Typical one-loop QED diagrams are

- ▷ Electron self-energy



The diagram has $E_e = 2$ and $P_e = 0$, hence $D = 1$. Applying the Feynman rules

we have

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \frac{-ig^{\mu\nu}}{k^2} \gamma^\nu,$$

and we can see that there are three powers of k in the dominator, so that the diagram diverges linearly, as expected.

▷ Photon self-energy or vacuum polarisation

$$i\Pi_{\mu\nu}(k) := \text{Diagram with a fermion loop and two external photon lines. The loop momenta are labeled } p \text{ (top), } p-k \text{ (bottom), and } k \text{ (left and right).}$$

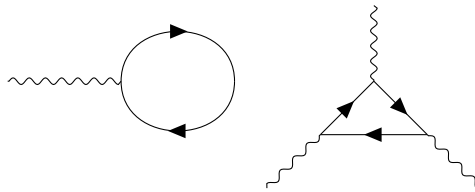
In this case $D = 2$; indeed, from the Feynman rules we have

$$i\Pi_{\mu\nu}(k) = -(-ie)^2 \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left(\gamma_\mu \frac{i}{\not{p} - m} \gamma_\nu \frac{i}{\not{p} - \not{k} - m} \right), \quad (12.12)$$

where the minus sign is due to the fermion loop. We then have that the two-point function up to one loop reads

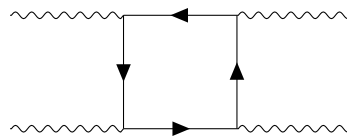
$$\begin{aligned} i\tilde{D}'_{\mu\nu}(k) &= -i\frac{g_{\mu\nu}}{k^2} + \left(\frac{-ig_{\mu\alpha}}{k^2} \right) i\Pi^{\alpha\beta}(k) \left(\frac{-ig_{\beta\nu}}{k^2} \right) \\ &= \text{Diagram with a photon line} + \text{Diagram with a photon line and a fermion loop.} \end{aligned}$$

▷ Tadpole and 3-photons vertex



However, by Furry theorem we know that the contribution related to these diagrams vanish.

▷ Box diagram



Here one has that $E_e = 0$, $P_e = 4$ hence

$$D = 4 - \frac{3}{2}E_e - P_e = 0 ,$$

so that we expect a logarithmic divergence. However, thanks to gauge invariance, this diagram is convergent. Actually, if it were divergent, then it would be necessary to add a new term to the Lagrangian density with 4 gauge fields. Since there are no gauge invariant terms with such a property, it follows that gauge invariance implies the convergence of this diagram.

Up to now we have seen that only the self-energies are “genuinely” divergent. More precisely, like in ϕ_4^4 , they are examples of *primitively divergent* graphs, i.e. graphs containing the sources of the divergences that, once regularised all the other Feynman graphs are finite. Another primitively divergent graph of QED is the vertex graph

$$-ie\Lambda_\mu(p, q, p + q) := \begin{array}{c} \mu \\ | \\ \text{---} q \\ | \\ \text{---} k \quad \text{---} k - q \\ / \quad \backslash \\ p \quad \quad p + q \\ \backslash \quad / \\ \text{---} k + p \end{array} ,$$

which has $D = 0$. Applying the Feynman rules we have

$$-ie\Lambda_\mu(p, q, p + q) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\rho\sigma}}{(k + p)^2} \gamma^\rho \frac{i}{\not{k} - \not{q} - m} \gamma_\mu \frac{i}{\not{k} - m} \gamma^\sigma .$$

12.4 Renormalisation

In the previous section we have managed to isolate the QED primitively divergent graphs. We now compute them using dimensional regularisation in Minkowski space.

Let us write down the QED Lagrangian density d dimension. Dimensional analysis shows that

$$[\psi] = L^{(1-d)/2} , \quad [A_\mu] = L^{(2-d)/2} ,$$

so that in d dimension $e_{old} := e$ has dimension $L^{(d-4)/2}$. We then have that the Lagrangian density in d dimension, in the Feynman gauge, and without the external source, is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(\not{\partial} + i\mu^{2-\omega}e_{new}\not{A})\psi - m\bar{\psi}\psi - \frac{1}{2}(\partial_\mu A^\mu)^2 ,$$

where $\omega = d/2$ and μ is the 't Hooft mass parameter, already introduced in the case of

Vacuum polarisation Let us consider the generalisation to d dimension of the vacuum polarisation diagram (12.12)

$$\begin{aligned}
 \text{Diagram} \rightarrow \Pi_{\mu\nu}(k) &= ie^2\mu^{2\epsilon} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \text{Tr} \left(\gamma_\mu \frac{1}{\not{p} - m} \gamma_\nu \frac{1}{\not{p} - \not{k} - m} \right) \\
 &= ie^2\mu^{2\epsilon} \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{\text{Tr} [\gamma_\mu (\not{p} + m) \gamma_\nu (\not{p} - \not{k} + m)]}{(p^2 - m^2)[(p - k)^2 - m^2]} .
 \end{aligned}$$

As before we use the Feynman parametrisation, then substitute $p \mapsto p' = p - kz$, so that

$$\Pi_{\mu\nu}(k) = ie^2\mu^{2\epsilon} \int_0^1 dz \int \frac{d^{2\omega}p'}{(2\pi)^{2\omega}} \frac{\text{Tr} [\gamma_\mu (\not{p}' + \not{k}z + m) \gamma_\nu (\not{p}' - \not{k}(1-z) + m)]}{[p'^2 - m^2 + k^2z(1-z)]^2} . \quad (12.14)$$

Now we have to deal with the trace. The generalisation of gamma matrices in a generic number of dimensions 2ω is not a difficult task.⁷ Given an arbitrary well-behaved function of the number of dimensions 2ω , $f(2\omega)$, such that $f(4) = 4$, we can define

$$\begin{cases}
 \text{Tr} \mathbb{I} = f(2\omega) , \\
 \text{Tr} (\gamma_\mu \gamma_\nu) = f(2\omega) g_{\mu\nu} , \\
 \text{Tr} (\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma) = f(2\omega) (g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\lambda} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\lambda}) , \\
 \text{Tr} (\text{odd } \# \text{ of } \gamma\text{'s}) = 0 .
 \end{cases} \quad (12.15)$$

Notice also that it is not restrictive to impose $f(2\omega) = 4, \forall \omega$. Due to the last identity in (12.14), and since even this time we can drop the term proportional to \not{p}' in the numerator N of the integrand in (12.14), we are left with

$$N = [p'^\kappa p'^\lambda - k^\kappa k^\lambda z(1-z)] \text{Tr} (\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\mu) + m^2 \text{Tr} (\gamma_\mu \gamma_\nu) ,$$

and after the calculations we end up with

$$N = f(2\omega) \{ 2p'_\mu p'_\nu - 2z(1-z)(k_\mu k_\nu - k^2 g_{\mu\nu}) - g_{\mu\nu} [p'^2 - m^2 + k^2 z(1-z)] \} .$$

⁷ We have to mention that the generalisation of γ^5 is all but trivial, because we have defined it as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$: we have at our disposal only other four gammas, therefore the extension to $d > 4$ is problematic. Nevertheless, QED only deals with $\gamma^\mu, \mu = 0, \dots, 3$, hence we do not worry about γ^5 .

Hence

$$\Pi_{\mu\nu}(k) = ie^2 \mu^{2\epsilon} f(2\omega) \int_0^1 dz \int \frac{d^{2\omega} p}{(2\pi)^{2\omega}} \left\{ \frac{2p_\mu p_\nu}{[p^2 - m^2 + k^2 z(1-z)]^2} - \frac{2z(1-z)[k_\mu k_\nu - g_{\mu\nu} k^2]}{[p^2 - m^2 + k^2 z(1-z)]^2} - \frac{g_{\mu\nu}}{[p^2 - m^2 + k^2 z(1-z)]} \right\}. \quad (12.16)$$

Using the identities (12.13) and

$$\int d^{2\omega} p \frac{p_\mu p_\nu}{(p^2 + 2pq - m^2)^\alpha} = (-1)^\omega \frac{i\pi^\omega}{\Gamma(\alpha)} \frac{1}{(-q^2 - m^2)^{\alpha-\omega}} \times \left[q_\mu q_\nu \Gamma(\alpha - \omega) + \frac{1}{2} g_{\mu\nu} (-q^2 - m^2) \Gamma(\alpha - 1 - \omega) \right], \quad (12.17)$$

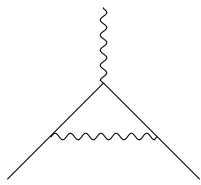
with $q = 0$, we see that the first and third terms in the right-hand side of (12.16) cancel. Note that the divergence is now logarithmic, and not quadratic. In particular, after integrating we get by (12.13)

$$\Pi_{\mu\nu}(k) = \frac{e^2}{2\pi^2} (k_\mu k_\nu - g_{\mu\nu} k^2) \left\{ \frac{1}{6\epsilon} - \frac{\gamma_E}{6} - \int_0^1 dz z(1-z) \times \log \left[\frac{-m^2 + k^2 z(1-z)}{4\pi\mu^2} \right] + \mathcal{O}(\epsilon) \right\},$$

Integrating and keeping the lowest order terms in $k^2 \rightarrow 0$ coming from the finite part, we get

$$\begin{aligned} \Pi_{\mu\nu}(k) &= \frac{e^2}{12\pi^2} (k_\mu k_\nu - g_{\mu\nu} k^2) \left(\frac{1}{\epsilon} + \frac{k^2}{5m^2} + \dots \right) \\ &= \frac{e^2}{12\pi^2 \epsilon} (k_\mu k_\nu - g_{\mu\nu} k^2) + \text{finite}. \end{aligned} \quad (12.18)$$

Vertex function The last step is to regularise the QED vertex



$$\begin{aligned} &\rightarrow -ie\mu^\epsilon \Lambda_\mu(p, q, p') \\ &= (-ie\mu^\epsilon)^3 \int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} \gamma_\nu \frac{i}{\not{p}' - \not{k} - m} \gamma_\mu \frac{i}{\not{p} - \not{k} - m} \gamma_\rho \frac{-ig^{\nu\rho}}{k^2} \\ &= -(e\mu^\epsilon)^3 \int \frac{d^{2\omega} k}{(2\pi)^{2\omega}} \frac{\gamma_\nu (\not{p}' - \not{k} m) (\not{p} - \not{k} + m) \gamma^\nu}{k^2 [(p-k)^2 - m^2] [(p'-k)^2 - m^2]}, \end{aligned}$$

where $p' = p + q$. Using the two parameters Feynman formula

$$\frac{1}{abc} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[a(1-x-y) + bx - cy]^3},$$

and making the substitution $k \mapsto k' = k - px - p'y$, we get

$$\begin{aligned} \Lambda_\mu(p, q, p') &= -2ie^2\mu^{2\epsilon} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \\ &\times \frac{\gamma_\nu[\not{p}'(1-y) - \not{p}x - \not{k} + m]\gamma_\mu[\not{p}(1-x) - \not{p}'y - \not{k} + m]\gamma^\nu}{[k^2 - m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2pp'xy]^3}. \end{aligned}$$

Such an expression is the sum of an ultraviolet divergent part, $\Lambda_\mu^{(1)}$, and of an ultraviolet convergent one $\Lambda_\mu^{(2)}$. In particular, the part of the numerator which is quadratic in k is divergent, while the rest is ultraviolet convergent; therefore we write

$$\Lambda_\mu = \Lambda_\mu^{(1)} + \Lambda_\mu^{(2)}.$$

By (12.17) it follows that the ultraviolet divergent part $\Lambda_\mu^{(1)}$ can be integrated to

$$\begin{aligned} \Lambda_\mu^{(1)}(p, q, p') &= \frac{e^2}{2} \frac{\mu^{2\epsilon}}{(4\pi)^\omega} \Gamma(\epsilon) \int_0^1 dx \int_0^{1-x} dy \\ &\times \frac{\gamma_\nu \gamma_\rho \gamma_\mu \gamma^\rho \gamma^\nu}{[-m^2(x+y) + p^2x(1-x) + p'^2y(1-y) - 2pp'xy]^\epsilon}, \end{aligned}$$

and using the identity

$$\gamma_\nu \gamma_\rho \gamma_\mu \gamma^\rho \gamma^\nu = (2-d)^2 \gamma_\mu \approx 4(1-\epsilon) \gamma_\mu,$$

we get

$$\Lambda_\mu^{(1)}(p, q, p') = \frac{e^2}{16\pi^2\epsilon} \gamma_\mu + \text{finite}.$$

For the ultraviolet convergent part we can already set $d = 4$. Then, integrating over k , which is easily performed because there is no k in the numerator of the integrand, we get by (12.13)

$$\begin{aligned} \Lambda_\mu^{(2)}(p, q, p') &= -\frac{e^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \\ &\times \frac{\gamma_\nu[\not{p}'(1-y) - \not{p}x + m]\gamma_\mu[\not{p}(1-x) - \not{p}'y + m]\gamma^\nu}{m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2pp'xy}. \end{aligned} \quad (12.19)$$

Summarising, we have seen that there are three ultraviolet divergent terms and one ultraviolet convergent term. Now we are ready to see how we can renormalise the

theory.

12.4.1 1-loop renormalisation

Let us recall the ultraviolet divergent expressions we found

$$\begin{aligned}\Sigma(p) &= \frac{e^2}{16\pi^2\epsilon}(-\not{p} + 4m) + \text{finite} , \\ \Pi_{\mu\nu}(k) &= \frac{e^2}{12\pi^2\epsilon}(k_\mu k_\nu - g_{\mu\nu}k^2) + \text{finite} , \\ \Lambda_\mu^{(1)}(p, q, p') &= \frac{e^2}{16\pi^2\epsilon}\gamma_\mu + \text{finite} .\end{aligned}$$

The above expressions show that the divergent parts of Σ and Λ satisfy the Ward identity (12.11). As anticipated, preserving gauge invariance is a general key property of dimensional regularisation.

Let us see which counterterms we should add to \mathcal{L} to make the above quantities finite. Consider (12.6) in momentum space

$$\Gamma^{(2)}(p) = S_F'^{-1}(p) .$$

Using the relation between the exact fermionic propagator and the self-energy

$$S_F'^{-1}(p) = S_F^{-1}(p) - \Sigma(p) ,$$

we have

$$\begin{aligned}\Gamma^{(2)}(p) &= S_F^{-1}(p) - \Sigma(p) \\ &= \not{p} - m - \frac{e^2}{16\pi^2\epsilon}(-\not{p} + 4m) \\ &= \not{p} \left(1 + \frac{e^2}{16\pi^2\epsilon}\right) - m \left(1 + \frac{e^2}{4\pi^2\epsilon}\right) .\end{aligned}$$

Notice that the finite corrections have been neglected. Since the coefficients of \not{p} and m are not equal, we need two different counterterms: one for the overall magnitude of the propagator, which contributes to the electron wave function normalisations, then another for the electron mass. It is then clear that in this case the counterterms to add at the Lagrangian density concern only the purely fermionic part, that is

$$\mathcal{L}_1 := i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi .$$

We then have

$$\begin{aligned}
 (\mathcal{L}_1)_B &= \mathcal{L}_1 + (\mathcal{L}_1)_{CT} \\
 &= (i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi) + (iB\bar{\psi}\not{\partial}\psi - A\bar{\psi}\psi) \\
 &= i(1+B)\bar{\psi}\not{\partial}\psi - (m+A)\bar{\psi}\psi,
 \end{aligned}$$

where A and B are chosen so that we get a finite propagator (to order e^2). Diagrammatically

Therefore,

$$\frac{e^2}{16\pi^2\epsilon}(-\not{p} + 4m) + A - B\not{p} = \text{finite},$$

and ignoring the finite terms we get

$$\begin{aligned}
 A &= -\frac{me^2}{4\pi^2\epsilon}, \\
 B &= -\frac{e^2}{16\pi^2\epsilon}.
 \end{aligned}$$

We can define the quantity

$$Z_2 = 1 + B = 1 - \frac{e^2}{16\pi^2\epsilon},$$

so that, if we define the bare wave function as

$$\psi_B = Z_2^{1/2}\psi,$$

we can rewrite $(\mathcal{L}_1)_B$ in the form

$$(\mathcal{L}_1)_B = i\bar{\psi}_B\not{\partial}\psi_B - m_B\bar{\psi}_B\psi_B,$$

where we have defined the bare mass as

$$\begin{aligned}
 m_B &= Z_2^{-1}(m + A) \\
 &= m\left(1 - \frac{3e^2}{16\pi^2\epsilon}\right) = m + \delta m.
 \end{aligned}$$

Now let us turn to the vacuum polarisation tensor: it gives rise to a modification of the

photon propagator

$$\begin{aligned}\tilde{D}'_{\mu\nu}(k) &= \tilde{D}_{\mu\nu}(k) - \tilde{D}_{\mu\alpha}\Pi^{\alpha\beta}(k)\tilde{D}_{\beta\nu}(k) + \dots \\ &= -\frac{g_{\mu\nu}}{k^2}\left(1 - \frac{e^2}{12\pi^2\epsilon} - \frac{e^2}{60\pi^2}\frac{k^2}{m^2}\right) - \frac{e^2}{12\pi^2\epsilon}\frac{1}{k^2}\frac{k_\mu k_\nu}{k^2} + \dots\end{aligned}\quad (12.20)$$

We notice that

- ▷ the first term in (12.20) contains both an infinite part and a finite part,
- ▷ even if we have chosen $\alpha = 1$, i.e. the Feynman gauge, the resultant propagator (12.20) is *not* in the Feynman gauge, because of the presence of the term $k_\mu k_\nu$.

Observables are gauge invariant, hence they are not affected by gauge terms. Nevertheless, the infinite term in $D'_{\mu\nu}$ must be removed by the addition of counterterms in the Lagrangian

$$\begin{aligned}(\mathcal{L}_2)_B &= \mathcal{L}_2 + (\mathcal{L}_2)_{CT} \\ &= \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2\right] + \left[-\frac{C}{4}F_{\mu\nu}F^{\mu\nu} - \frac{E}{2}(\partial_\mu A^\mu)^2\right] \\ &= -\left(\frac{1+C}{4}\right)F_{\mu\nu}F^{\mu\nu} + \text{gauge terms} \\ &= -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + \text{gauge terms},\end{aligned}$$

where

$$Z_3 = 1 - \frac{e^2}{12\pi^2\epsilon},$$

so that we have obtained a finite propagator to order e^2 . We have seen before that the electron self-energy gives rise to a bare mass which is different from the physical one. Luckily enough this does not happen for the photon, which is to say that its mass remains zero also after the renormalisation. Indeed, Eq.(12.18) shows that the structure of the vacuum-polarisation tensor is

$$\Pi^{\alpha\beta}(k) = (k^\alpha k^\beta - g^{\alpha\beta}k^2)\Pi(k^2). \quad (12.21)$$

On the other hand, as shown in the analysis subsequent to (11.14), this is just the most general covariant structure which is consistent with the requirement of gauge invariance

$$k_\alpha \Pi^{\alpha\beta}(k) = 0.$$

In other words, the fact that $k^\alpha k^\beta - g^{\alpha\beta}k^2$ has no inverse is just a consequence of gauge invariance.

Substituting (12.21) in (12.20) we get, to one loop,

$$\begin{aligned}\tilde{D}'_{\mu\nu}(k) &= \tilde{D}_{\mu\nu}(k) - \tilde{D}_{\mu\alpha}(k)(k^\alpha k^\beta - g^{\alpha\beta} k^2)\Pi(k^2)\tilde{D}_{\beta\nu}(k) \\ &= \frac{1}{k^2} \left[-g_{\mu\nu}(1 - \Pi(k^2)) - \frac{k_\mu k_\nu}{k^2} \Pi(k^2) \right] \\ &\approx \frac{1}{k^2(1 + \Pi(k^2))} \left(-g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \Pi(k^2) \right).\end{aligned}$$

The divergence is carried by $\Pi(k^2)$. By dimensional regularisation

$$\Pi(k^2) = \frac{e^2}{12\pi^2} \left(\frac{1}{\epsilon} + \frac{k^2}{5m^2} \right) = \frac{e^2}{12\pi^2\epsilon} + \Pi_f(k^2),$$

where $\Pi_f(k^2) \xrightarrow{k^2 \rightarrow 0} 0$ and for any generic k is finite. We may write the complete propagator as

$$\begin{aligned}\tilde{D}'_{\mu\nu}(k) &= -\frac{g_{\mu\nu}}{k^2(1 + \Pi(k^2))} + \text{gauge terms} \\ &= -\frac{g_{\mu\nu}}{k^2(1 + \frac{e^2}{12\pi^2\epsilon} + \Pi_f(k^2))} + \text{gauge terms} \\ &= -Z_3 \frac{g_{\mu\nu}}{k^2(1 + \Pi_f(k^2))} + \text{gauge terms}.\end{aligned}$$

This indicates that the bare gauge field is

$$A_B^\mu = Z_3^{1/2} A^\mu,$$

so that, the photon mass remains zero after renormalisation. Furthermore, we succeeded in getting rid of the infinite terms in the propagator, while the finite corrections are still there, and thus give rise to physical effects. Ignoring the gauge terms the renormalised propagator is

$$\tilde{D}'_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2} \left(1 - \frac{e^2}{60\pi^2} \frac{k^2}{m^2} + \mathcal{O}(k^4) \right).$$

The correction, known as *Uehling term*, modifies the Coulomb potential between two charges e at distance r apart, and in the coordinate space we have

$$V(r) = -\frac{e^2}{4\pi r} - \frac{e^4}{60\pi^2 m^2} \delta^{(3)}(r).$$

This extra terms modifies the hydrogen atom energy levels, giving a significant contribution to the Lamb shift, which splits the degeneracy of the $2S_{1/2}$ and $2P_{1/2}$ levels. The agreement between theory and experiment is great, hence we have a nice confirmation of QED.

As far as the vertex function is concerned, in particular its divergent part $\Lambda_\mu^{(1)}$, we have to add to the Lagrangian the counterterm

$$(\mathcal{L}_3)_{CT} = -De\mu^\epsilon \bar{\psi} \not{A} \psi ,$$

with

$$D = -\frac{e^2}{16\pi^2\epsilon} ,$$

so that

$$\begin{aligned} (\mathcal{L}_3)_B &= -(1+D)e\mu^\epsilon A^\mu \bar{\psi} \gamma_\mu \psi \\ &= -Z_1 e\mu^\epsilon A^\mu \bar{\psi} \gamma_\mu \psi , \end{aligned}$$

where

$$Z_1 = 1 - \frac{e^2}{16\pi^2\epsilon} .$$

Summarising, we have the following bare Lagrangian density

$$\begin{aligned} \mathcal{L}_B &= iZ_2 \bar{\psi} \gamma^\mu \partial_\mu \psi - (m+A) \bar{\psi} \psi - Z_1 e\mu^\epsilon A^\mu \bar{\psi} \gamma_\mu \psi \\ &\quad - \frac{Z_3}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \text{gauge terms} , \end{aligned}$$

where

$$\begin{aligned} Z_1 = Z_2 &= 1 - \frac{e^2}{16\pi^2\epsilon} , \\ Z_3 &= 1 - \frac{e^2}{12\pi^2\epsilon} , \\ A &= -\frac{me^2}{4\pi^2\epsilon} . \end{aligned}$$

As already seen in the case of ϕ_4^4 , even in this case, starting with this Lagrangian density, one gets, at one-loop, finite self-energy and vertex, with e , m corresponding to the finite experimental physical values.

The bare Lagrangian density, can be also expressed in the same functional form of the original one, in which the initial quantities are replaced by the bare ones. To this end, we set

$$e_B = e\mu^\epsilon \frac{Z_1}{Z_2 Z_3^{1/2}} = e\mu^\epsilon Z_3^{-1/2} .$$

So that

$$\mathcal{L}_B = i\bar{\psi}_B \not{\partial} \psi_B - m_B \bar{\psi}_B \psi_B - e_B \bar{\psi}_B \not{A} \psi_B - \frac{1}{4} (\partial_\mu A_{B\nu} - \partial_\nu A_{B\mu})^2 , \quad (12.22)$$

where

$$\begin{aligned} e_B &= e\mu^\epsilon Z_3^{-1/2} , \\ m_B &= Z_2^{-1}(m + A) , \\ \psi_B &= Z_2^{1/2}\psi , \\ A_B^\mu &= Z_3^{1/2}A^\mu . \end{aligned}$$

Note that we did not consider the gauge fixing term in \mathcal{L}_B . As explained in considering the Ward-Takahashi identities, such a term does not need counterterms, that is the gauge fixing term does renormalise. The fact that we can absorb the infinities in the bare quantities, keeping the Lagrangian in the same form of the original one means that, *to all order*, QED is renormalisable.

12.4.2 $Z_1 = Z_2$

We saw that at one-loop we have $Z_1 = Z_2$. Let us show that this is not a numerical coincidence. To this end it is worth writing the bare Lagrangian density in terms of the renormalised quantities, that is in terms of the finite quantities e , m , ψ and A_μ . We have

$$\mathcal{L}_B = -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + Z_2[\bar{\psi}(i\cancel{\partial} - e(Z_1/Z_2)\cancel{A})\psi - Z_0m\bar{\psi}\psi] , \quad (12.23)$$

where

$$Z_0 := 1 + \frac{A}{m} .$$

Let us first give an argument for the finiteness of the ratio Z_1/Z_2 . Note that \mathcal{L}_B is invariant under the gauge transformations

$$\psi(x) \longrightarrow \psi'(x) = e^{-ie(Z_1/Z_2)\Lambda(x)}\psi(x) ,$$

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\Lambda(x) .$$

Since such transformations concern finite quantities, it follows that the ratio Z_1/Z_2 must be finite, that is

$$Z_1 = Z_2 + \text{finite terms} .$$

It turns out that in many renormalisation scheme we in fact have $Z_1 = Z_2$.

A more stringent argument to show that Z_1/Z_2 is finite follows from the Ward-Takahashi identities. Applying the same construction leading to (12.7), but now using the Lagrangian density (12.23), we get

$$q^\mu\Gamma_\mu(p, q, p+q) = \frac{Z_1}{Z_2}(S_F'^{-1}(p+q) - S_F'^{-1}(p)) . \quad (12.24)$$

Since such an identity now involves finite quantities, it follows that Z_1/Z_2 must be finite. Let us now start with \mathcal{L}_B expressed in terms of bare parameters and fields, that is the one given in (12.22). We can make the natural renormalisation scheme

$$\Gamma_\mu(p, q, p+q) \longrightarrow Z_1^{-1} \gamma_\mu, \text{ as } q \rightarrow 0.$$

On the other hand Z_2 is the residue of the pole in $S'_F(p)$

$$S'_F(p) \sim \frac{Z_2}{\not{p} - m}.$$

Expanding (12.7) near $q = 0$ with p near the mass shell, we get

$$Z_1^{-2} \not{q} = Z_2^{-2} \not{q},$$

that is

$$Z_1 = Z_2.$$

Asymptotic behaviour

Combining the last expression of the bare charge we gave, and the definition of Z_3 , we have

$$e_B = e\mu^\epsilon \left(1 + \frac{e^2}{24\pi^2\epsilon} \right).$$

We can now find

$$\beta(e) = \mu \frac{\partial e}{\partial \mu},$$

in the same way as done, around Eq.(9.74), in the case of $\lambda(\mu)$ in ϕ_4^4 . We consider e_B as independent variable, so that deriving with respect to μ we get

$$0 = \epsilon\mu^\epsilon \left(e + \frac{e^3}{24\pi^2\epsilon} \right) + \mu^\epsilon \beta(e) \left(1 + \frac{e^2}{8\pi^2\epsilon} \right).$$

Since $\beta(e)$ is analytic in the limit $\epsilon \rightarrow 0$, we can set, at order ϵ

$$\beta(e) = A + B\epsilon.$$

Therefore,

$$(e + B)\epsilon + \left(A + \frac{Be^2}{8\pi^2} + \frac{e^3}{24\pi^2} \right) \epsilon^0 = 0,$$

where we did not consider the $Ae^2/(8\pi^2\epsilon)$ term because it would match with higher loop contributions to e_B . Therefore, we have

$$A = \frac{e^3}{12\pi^2}, \quad B = -e,$$

so that, in the limit $\epsilon \rightarrow 0$,

$$\beta(e) = \frac{e^3}{12\pi^2},$$

that is

$$e^2(\mu) = \frac{e^2(\mu_0)}{1 - \frac{e^2(\mu_0)}{6\pi^2} \log \frac{\mu}{\mu_0}},$$

from which we see that the coupling increases with the scale, as expected. We can also see the presence of the so-called *Landau pole* at

$$\mu = \mu_0 \exp\left(\frac{6\pi^2}{e^2(\mu_0)}\right),$$

which can be seen as the limit of validity of a perturbative description of QED.

12.4.3 Anomalous magnetic moment of the electron

A consequence of the presence of the finite contribution $\Lambda_\mu^{(2)}$ to the vertex function, is that a Dirac particle has an anomalous magnetic moment. Basically we have to study the interaction of a Dirac particle with an external electromagnetic field $A_\mu(x)$. The simplest way to reproduce such a system is by means of the *minimal coupling*, i.e. the substitution

$$\partial_\mu \mapsto \partial_\mu + ieA_\mu,$$

where e denotes the charge of the particle, and it is taken to be negative: $e \equiv -|e|$. The Dirac equation now reads⁸

$$(i\cancel{D} - e\cancel{A} - m)\psi(x) = 0. \quad (12.25)$$

We can rewrite equation (12.25) in a ‘‘Schrödinger-like’’ form as

$$\begin{aligned} i\frac{\partial\psi}{\partial t} &= \left[\boldsymbol{\alpha} \cdot \left(-i\nabla - e\mathbf{A} \right) + m\beta + eA^0 \right] \psi \\ &= (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi + (-e\boldsymbol{\alpha} \cdot \mathbf{A} + eA^0)\psi. \end{aligned} \quad (12.26)$$

To study the physical implications of this equation we can consider the non-relativistic limit. Therefore, we write

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

⁸ Notice that the equation is invariant under the gauge transformation

$$\begin{cases} \psi(x) & \mapsto e^{ie\alpha(x)}\psi(x), \\ A_\mu(x) & \mapsto A_\mu(x) - \partial_\mu\alpha(x). \end{cases}$$

and use the representation

$$\beta = \begin{pmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{0} & -\mathbb{I} \end{pmatrix}, \quad \alpha = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix}.$$

Equation (12.26) leads to

$$\begin{aligned} i\frac{\partial\phi}{\partial t} &= \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\chi + eA^0\phi + m\phi, \\ i\frac{\partial\chi}{\partial t} &= \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\phi + eA^0\chi - m\chi, \end{aligned}$$

where we have introduced the canonical momentum $\boldsymbol{\pi} := \mathbf{p} - e\mathbf{A}$. In the non-relativistic limit the driving term is the large energy m . We introduce the slowly varying fields $\Phi(t)$ and $X(t)$

$$\begin{cases} \phi = e^{-imt}\Phi, \\ X = e^{-imt}X, \end{cases}$$

which satisfy

$$\begin{aligned} i\frac{\partial\Phi}{\partial t} &= \boldsymbol{\sigma} \cdot \boldsymbol{\pi}X + eA^0\Phi, \\ i\frac{\partial X}{\partial t} &= \boldsymbol{\sigma} \cdot \boldsymbol{\pi}\Phi + eA^0X - 2mX. \end{aligned}$$

Assuming $eA^0 \ll 2m$, the second equation above is (approximately) solved as

$$X \approx \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\pi}}{2m}\Phi,$$

while the other one is the *Pauli equation*

$$i\frac{\partial\Phi}{\partial t} = \left[\frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m} + eA^0 \right] \Phi.$$

One has that

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2 = \sigma_i \sigma_j \pi^i \pi^j = \pi^2 + \frac{1}{4}[\sigma_i, \sigma_j][\pi^i, \pi^j] = \pi^2 - e\boldsymbol{\sigma} \cdot \mathbf{B},$$

thus the Pauli equation can be rewritten as

$$i\frac{\partial\Phi}{\partial t} = \left[\frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - \frac{e}{2m}\boldsymbol{\sigma} \cdot \mathbf{B} + eA^0 \right] \Phi.$$

In particular, we can see that the only spin dependence is through the magnetic interaction $\boldsymbol{\sigma} \cdot \mathbf{B}$, and restoring the \hbar and c factor

$$\mathcal{H}_m = -\frac{e\hbar}{2mc}\boldsymbol{\sigma} \cdot \mathbf{B} .$$

The canonical form of the Hamiltonian of interaction between spin and magnetic field is $\mathcal{H}_c = -\boldsymbol{\mu} \cdot \mathbf{B}$, we get that the magnetic moment $\boldsymbol{\mu}$ is defined as

$$\boldsymbol{\mu} := \frac{e}{mc} \frac{\hbar\boldsymbol{\sigma}}{2} = 2\left(\frac{e}{2mc}\right)\mathbf{S} ,$$

where the spin operator $\mathbf{S} \equiv \hbar\boldsymbol{\sigma}/2$. Remembering that the *gyromagnetic ratio* g is defined through the relation $\boldsymbol{\mu} = g\mathbf{S}/(2mc)$, we see that Dirac equation predicts

$$g = 2 .$$

It is instructive to derive the same result in the following way. First, consider the positive energy solution of the Dirac equation given by the plane wave

$$\psi^{(+)} = e^{-ipx}u(p) .$$

This gives

$$\not{p}u(p) = mu(p) , \tag{12.27}$$

and

$$\bar{u}(p')\not{p}' = m\bar{u}(p') . \tag{12.28}$$

Let us rewrite (12.27) in the form

$$\begin{aligned} \gamma_\mu u(p) &= \frac{1}{m}\gamma_\mu\gamma_\nu p^\nu u(p) \\ &= \frac{1}{m}(g_{\mu\nu} - i\sigma_{\mu\nu})p^\nu u(p) \\ &= \frac{1}{m}(p_\mu - i\sigma_{\mu\nu}p^\nu)u(p) , \end{aligned} \tag{12.29}$$

where we used the identity

$$\gamma_\mu\gamma_\nu = g_{\mu\nu} - i\sigma_{\mu\nu} .$$

Similarly, (12.28) is equivalent to

$$\bar{u}(p')\gamma_\mu = \frac{1}{m}\bar{u}(p')(p'_\mu + i\sigma_{\mu\nu}p'^\nu) . \tag{12.30}$$

The identities (12.29) and (12.30) imply the Gordon identity

$$\begin{aligned}\bar{u}(p')\gamma_\mu u(p) &= \frac{1}{2}\bar{u}(p')(\gamma_\mu u(p)) + \frac{1}{2}(\bar{u}(p')\gamma_\mu)u(p) , \\ &= \frac{1}{2m}\bar{u}(p')[(p+p')_\mu + i\sigma_{\mu\nu}q^\nu]u(p) ,\end{aligned}\quad (12.31)$$

where $q = p - p'$. This means that the electromagnetic current $\bar{u}(p')\gamma_\mu u(p)$ describes a particle with $g = 2$. We then should consider the effect on g once one considers the renormalised current

$$\bar{u}(p')\Gamma_\mu u(p) = \bar{u}(p')(\gamma_\mu + \Lambda_\mu)u(p) = \bar{u}(p')(\gamma_\mu + \Lambda_\mu^{(2)})u(p) , \quad (12.32)$$

where, as done for all the ultraviolet divergent parts, we posed the finite part of $\Lambda_\mu^{(1)}$ to zero.

Let us now compute $\Lambda_\mu^{(2)}$

$$\Lambda_\mu^{(2)}(p, q, p') = -\frac{e^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{\gamma_\nu[\not{p}'(1-y) - \not{p}x + m]\gamma_\mu[\not{p}(1-x) - \not{p}'y + m]\gamma^\nu}{m^2(x+y) - p^2x(1-x) - p'^2y(1-y) + 2pp'xy} .$$

Since our interest is to compute (12.32), we can use some identities the fact that $\Lambda_\mu^{(2)}$ is sandwiched between $\bar{u}(p')$ and $u(p)$ to simplify the calculations. Let us first consider the various terms in the numerator in the integrand of $\Lambda_\mu^{(2)}$. In doing this we will perform the substitution, justified by (12.27) and (12.28),

$$\not{p}' \rightarrow m ,$$

when \not{p}' is on the left of the expression, and

$$\not{p} \rightarrow m ,$$

when \not{p} is on the right of the expression.

(i) The first term is⁹

$$\begin{aligned}\gamma_\nu \not{p}' \gamma_\mu \not{p} \gamma^\nu &= (2p'_\nu - \not{p}'\gamma_\nu)\gamma_\mu(2p^\nu - \gamma^\nu \not{p}) = (2p'_\nu - m\gamma_\nu)\gamma_\mu(2p^\nu - \gamma^\nu m) \\ &= 4p'_\nu p^\nu \gamma_\mu - 2p'_\nu \gamma_\mu \gamma^\nu m - 2m \not{p} \gamma^\mu - 2m^2 \gamma_\mu \\ &= (4p'_\nu p^\nu + 2m^2)\gamma_\mu - 4m(p'_\mu + p_\mu) ,\end{aligned}$$

where we used the identities

$$\not{p}\gamma_\mu = 2p_\mu - \gamma_\mu \not{p} , \quad (12.33)$$

⁹ In the following calculation we omit the factors $(1-x)$, $(1-y)$, x and y .

and

$$\gamma_\nu \gamma_\mu \gamma^\nu = -2\gamma_\mu . \quad (12.34)$$

(ii) The second term reads

$$-\gamma_\nu \not{p}' \gamma_\mu \not{p}' \gamma^\nu = -\gamma_\nu \gamma_\alpha \gamma_\mu \gamma_\beta \gamma^\nu p^{\alpha'} p^{\beta'} = 2\not{p}' \gamma_\mu \not{p}' = 2m\gamma_\mu \not{p}' = 2m(2p'_\mu - m\gamma_\mu) ,$$

where we used (12.33) and the identity

$$\gamma_\nu \gamma_\alpha \gamma_\mu \gamma_\beta \gamma^\nu = -2\gamma_\beta \gamma_\mu \gamma_\alpha . \quad (12.35)$$

(iii) The third term is

$$m\gamma_\nu \not{p}' \gamma_\mu \gamma^\nu = 4mp'_\mu ,$$

where we used the identity

$$\gamma_\nu \gamma_\alpha \gamma_\mu \gamma^\nu = 4g_{\alpha\mu} . \quad (12.36)$$

(iv) By (12.35) the fourth term reads

$$-\gamma_\nu \not{p}' \gamma_\mu \not{p}' \gamma^\nu = 2m(2p_\mu - m\gamma_\mu) ,$$

(v) The fifth term can again be computed using (12.35)

$$\gamma_\nu \not{p}' \gamma_\mu \not{p}' \gamma^\nu = -2m^2 \gamma_\mu .$$

(vi) The sixth, seventh and eighth terms can be computed by (12.36)

$$-m\gamma_\nu \not{p}' \gamma_\mu \gamma^\nu = -4mp_\mu .$$

(vii)

$$m\gamma_\nu \gamma_\mu \not{p}' \gamma^\nu = 4mp_\mu .$$

(viii)

$$-m\gamma_\nu \gamma_\mu \not{p}' \gamma^\nu = -4mp'_\mu .$$

(ix) Finally, the ninth and last term follows by (12.34)

$$m^2 \gamma_\nu \gamma_\mu \gamma^\nu = -2m^2 \gamma_\mu .$$

Summarising, we proved the following relation

$$\begin{aligned} & \bar{u}(p')\gamma_\nu[\not{p}'(1-y) - \not{p}x + m]\gamma_\mu[\not{p}(1-x) - \not{p}'y + m]\gamma^\nu u(p) \\ &= 4m\bar{u}(p')[(x - xy - y^2)p'_\mu + (y - xy - x^2)]p_\mu u(p) \\ &+ \bar{u}(p')\{4p'p(1-x)(1-y) + 2m^2[2(x^2 + y^2) - 3(x+y)]\}\gamma_\mu u(p) . \end{aligned}$$

We are interested in the $\sigma_{\mu\nu}$ term, so we omit the γ_μ term, that should be regularised because it is infrared divergent. On mass shell we have $p^2 = p'^2 = m^2$ and $q^2 = 0$. Since $q^2 = (p - p')^2$, we also have $pp' = 0$, and the denominator of the integrand reduces to $m^2(x + y)^2$. Finally, neglecting all terms in γ_μ , we get

$$\begin{aligned} \bar{u}(p')\Lambda_\mu^{(2)}u(p) &= -\frac{e^2}{4\pi^2 m} \int_0^1 dx \int_0^{1-x} dy \frac{(y - xy - x^2)\bar{u}(p')p_\mu u(p) + (x - xy - y^2)\bar{u}(p')p'_\mu u(p)}{(x+y)^2} \\ &= -\frac{e^2}{16\pi^2 m} \bar{u}(p')(p + p')_\mu u(p) . \end{aligned}$$

By the Gordon identity we get

$$\bar{u}(p')(p + p')_\mu u(p) = \bar{u}(p')(2m\gamma_\mu - i\sigma_{\mu\nu}q^\nu)u(p) .$$

It turns out that the term in γ_μ cancels the one we neglected above, so the total vertex is

$$\bar{u}(p')\Gamma_\mu u(p) = \bar{u}(p') \left[\frac{(p + p')_\mu}{2m} + \left(1 + \frac{\alpha}{2\pi}\right) \frac{i\sigma_{\mu\nu}q^\nu}{2m} \right] u(p) ,$$

where $\alpha = e^2/(4\pi)$. The lowest order correction to the magnetic moment of the electron is $\alpha/(2\pi)$, therefore it turns out that the gyromagnetic ratio is

$$\frac{g}{2} = 1 + \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) .$$

This was first computed by Schwinger in 1948 and agreed greatly with the contemporary experimental results. The theoretical prediction have been determined analytically up to order α^3 and have been calculated up to order α^5

$$a_{\text{th}} := \frac{1}{2}(g - 2) = 1\,159\,652.181\,643(764) \times 10^{-9} ,$$

in remarkable agreement with the experimentally measured value, which is

$$a_{\text{exp}} = 1\,159\,652.180\,73(28) \times 10^{-9} .$$

Chapter 13

The Wilsonian Renormalisation Group¹

Nowadays, in Physics there are a lot of powerful theories which allow us to describe the Universe even at very high energy regimes. This is a natural outcome for the search of a unified theory of all the fundamental interactions.

Still, sometimes we do not require the most powerful tools in order to describe phenomena at standard regimes. For example, in order to study the basic structure of the Hydrogen atom energy levels we can exploit Quantum Mechanics instead of Quantum Field Theory (QFT); we will surely miss the finest details, but it will be sufficient if we are not interested in highlighting them.

In QFT, an Effective Field Theory (EFT) is the *simplest* theory which can successfully describe a given phenomenon with an *arbitrary precision*. In order to build an EFT, we have to determine

- ▷ the *relevant* degrees of freedom;
- ▷ the symmetries of the theory.

Given a theory described by a lagrangian \mathcal{L} , the Wilsonian approach consists in introducing a physical cutoff Λ . Examples of physical cutoffs in Physics are the atomic spacing ξ in a crystal lattice, or the length of the strings l_s in a string theory. The introduction of the cutoff is obtained by means of integrating out the high energy degrees of freedom of the theory in the Z functional

$$Z[J] = \int \mathcal{D}\phi e^{\frac{i}{\hbar}(S[\phi] + \int d^4x J\phi)} = \int \mathcal{D}\phi_{\Lambda > |k|} e^{\frac{i}{\hbar}(S_{eff}[\phi, \Lambda] + \int d^4x J\phi)} ,$$

¹ Enrico Marchetto

where

$$e^{\frac{i}{\hbar}S_{eff}[\phi,\Lambda]} = \int \mathcal{D}\phi_{\Lambda < |k|} e^{\frac{i}{\hbar}S[\phi]} .$$

Integrating out the high energy momenta induces a Λ dependence in the couplings of the theory: $\mathcal{L} \Rightarrow \mathcal{L}[\Lambda]$. The Wilson approach is based upon the following assumption: if we integrate out an infinitesimal shell of high energy momenta $\Lambda' < |k| < \Lambda$, we will have a transition in the space of the theories described by \mathcal{L} (i.e. the space whose elements are the theories with the same formal structure of \mathcal{L} but without necessarily the same values for the couplings) from $\mathcal{L}[\Lambda]$ to $\mathcal{L}[\Lambda']$. The descriptions of low-energy physics given by $\mathcal{L}[\Lambda]$ and $\mathcal{L}[\Lambda']$ are imposed to be equal. Thus, a *flow* is generated in such space. Our purpose will be to write and study a Renormalisation Group Equation which describes that flow. Finally, note that the theory, once the cutoff Λ has been introduced, is well-defined because all loops are finite.

13.1 Operators ranking

In the Wilsonian approach to renormalisation we can rank the *operators* composing our theory $\mathcal{L}[\Lambda]$ by means of three categories: *irrelevant*, *marginal*, *relevant*. These definitions are tied to the *mass dimensions* of the operators. In order to clarify the last statement, we introduce a generic lagrangian density, composed of a finite or an infinite set of operators \mathcal{O}_d , where d represents their mass dimensions²

$$\mathcal{L}[\Lambda] = \sum_d C_d(\Lambda) \Lambda^{4-d} \mathcal{O}_d .$$

In the expression above we have introduced the adimensional coefficients $C_d(\Lambda)$; in order to write terms of mass dimension 4, we have multiplied each term for the correct power of Λ , which has mass dimension 1. The $C_d(\Lambda)$ numbers can be large and they could be even impossible to compute. However, this is not our aim: in fact, *not all* $C_d(\Lambda)$ *contribute to the low energy Physics*.

Let us consider the following example: the (spinor) operator $\mathcal{A}_4 = \bar{\psi}\psi\bar{\psi}\psi$, for a theory with a Λ cutoff. Following the building rule for the operators in a lagrangian density given above, the dimensional correct operator is, actually, $\mathcal{A}_4 = \frac{1}{\Lambda^2}\bar{\psi}\psi\bar{\psi}\psi$. If $\Lambda \sim 10^{19}$ GeV, much bigger than the other energy scales (masses, for example) considered in our theory, then we immediately realise that the operator \mathcal{A}_4 is negligible for a theory at low energies. If this is the case, we say that \mathcal{A}_4 is irrelevant. Following the same reasoning, heuristically

- ▷ \mathcal{O}_d is irrelevant if $d > 4$;

² Some examples: if ϕ, ψ, A are respectively a scalar, a spinor and a vector field, then $\bar{\psi}\psi = \mathcal{O}_3$, $\phi^2 = \mathcal{O}_2$, $\bar{\psi}A\psi = \mathcal{O}_4$.

- ▷ \mathcal{O}_d is marginal if $d = 4$;
- ▷ \mathcal{O}_d is relevant if $d < 4$.

13.2 Changing the cutoff

We assume to have a theory with a large *physical* cutoff Λ_H . In a theory placed on an atomic lattice, Λ could be thought, as we already said, as the inter-atomic distance ξ^{-1} . However, when we perform computations at low energies E_{low} , we could also consider a lower cutoff $E_{low} \ll \Lambda_L \ll \Lambda_H$ and the physics wouldn't change, being the high energy contributions suppressed by powers of $\frac{E}{\Lambda_L}$. For example, if we are working at the electroweak energy scale $E \approx 100$ GeV, the physical cutoff is represented by the Planck energy scale $\Lambda_H \approx 10^{18}$ GeV; still, a much lower cutoff $\Lambda_L \approx 10^{10}$ GeV would do the work. Changing the cutoff, we also change our theory

$$\mathcal{L}[\Lambda_H] = \sum_d C_d(\Lambda_H) \Lambda_H^{4-d} \mathcal{O}_d \Rightarrow \mathcal{L}[\Lambda_L] = \sum_d C_d(\Lambda_L) \Lambda_L^{4-d} \mathcal{O}_d .$$

We require the physical observables computed through $\mathcal{L}[\Lambda_H]$ and $\mathcal{L}[\Lambda_L]$ to be the same (we stress: at low energy). This constraint requires the coefficients $C_d(\Lambda_L)$ to be *functions* of the original coefficients $C_d(\Lambda_H)$. The crucial point, which follows from the Wilsonian approach, is: $C_d(\Lambda_L)$ *does not depend on* $C_d(\Lambda_H)$ *if* $d > 4$. This allows us to restrict our analysis to a finite number of coefficients: the number of relevant operators of our theory will be finite.

Now, we want to study the Renormalisation Group Equation of the Wilsonian Renormalisation Group. First of all, note the difference between the Wilsonian approach and the standard approach to the renormalisation of a theory: usually we would introduce an energy scale μ similar to the energy scale of the considered physical phenomena, while in the Wilsonian approach we introduce a cutoff Λ very distant from the studied processes.

We start by considering a scalar theory with a cutoff Λ

$$Z[J, \Lambda] = \int^{\Lambda} \mathcal{D}\phi \exp \left[i \int d^4x \left(-\frac{1}{2} \phi(\square + m^2)\phi + \sum_n \frac{g_n}{n!} \phi^n + J\phi \right) \right] .$$

The Renormalisation Group Equation would simply be $\Lambda \frac{d}{d\Lambda} Z[J, \Lambda] = 0$. Doing so, we would obtain the new couplings $m', \{g'_n\}$, as functions of the original ones. Unfortunately, the computation is very difficult, so we will exploit an idea by J. Polchinski [38].

The idea is to introduce the cutoff in a smoother way by means of a suppressor $\exp\left(\frac{\square}{\Lambda^2}\right)$

$$\begin{aligned} Z[J, \Lambda] &= \int \mathcal{D}\phi \exp \left[i \int d^4x \left(-\frac{1}{2} \phi (\square + m^2) e^{\frac{\square}{\Lambda^2}} \phi + \sum_n \frac{g_n}{n!} \phi^n + J\phi \right) \right] \\ &= \int \mathcal{D}\phi \exp \left[i \int \frac{d^4p}{(2\pi)^4} \left(\frac{1}{2} \phi(p) (p^2 - m^2) e^{-\frac{p^2}{\Lambda^2}} \phi(-p) + \sum_n \frac{g_n}{n!} \phi^n + J\phi \right) \right] . \end{aligned}$$

We can now perform the derivative

$$\begin{aligned} \Lambda \frac{d}{d\Lambda} Z[J, \Lambda] &= i \int \mathcal{D}\phi \left[\int \frac{d^4p}{(2\pi)^4} \left(\phi(p) (p^2 - m^2) \frac{p^2}{\Lambda^2} \phi(-p) e^{-\frac{p^2}{\Lambda^2}} + \Lambda \frac{d}{d\Lambda} \mathcal{L}_{int}(\phi) \right) \right] e^{iS+\phi J} \\ &= 0 . \end{aligned}$$

Note that the function $f(p) = \frac{p^2}{\Lambda^2} e^{-\frac{p^2}{\Lambda^2}}$ has support only near $p^2 \sim \Lambda^2$, thus the change in \mathcal{L}_{int} will come from the same region in the momentum space. As a consequence, Λ will be the only mass parameter capable of adjusting the mass dimension of every coupling in our theory.

Let us focus on a specific example in order to explicitly show what has been said above. We consider a theory with an operator with dimension 4 (coupling g_4) and an operator with dimension 6 (coupling g_6). Building the Renormalisation Group Equation

$$\begin{aligned} \Lambda \frac{d}{d\Lambda} Z[J, \Lambda] = 0 &= i \int \mathcal{D}\phi \left[\int \frac{d^4p}{(2\pi)^4} \left(\dots + \Lambda \frac{d}{d\Lambda} \mathcal{L}_{int}(\phi) \right) \right] e^{iS+\phi J} \\ &= i \int \mathcal{D}\phi \left[\int \frac{d^4p}{(2\pi)^4} \left(\dots + \Lambda \frac{d}{d\Lambda} (g_4 \mathcal{O}_4 + g_6 \mathcal{O}_6) \right) \right] e^{iS+\phi J} \\ &= i \int \mathcal{D}\phi \left[\int \frac{d^4p}{(2\pi)^4} \left(\dots + \Lambda \frac{dg_4}{d\Lambda} \mathcal{O}_4 + \Lambda \frac{dg_6}{d\Lambda} \mathcal{O}_6 \right) \right] e^{iS+\phi J} , \end{aligned}$$

we immediately see that solving the Renormalisation Group Equation requires the solution of the coupled set of equations

$$\begin{aligned} \Lambda \frac{dg_4}{d\Lambda} &= \beta_4(g_4, \Lambda^2 g_6) , \\ \Lambda \frac{dg_6}{d\Lambda} &= \frac{1}{\Lambda^2} \beta_6(g_4, \Lambda^2 g_6) , \end{aligned}$$

where β_4 and β_6 are some general functions. Note that the factors Λ^2 have been placed in order to satisfy the correct mass dimension. Let us introduce the adimensional

parameters $\lambda_4 = g_4$ and $\lambda_6 = \Lambda^2 g_6$. The previous equations now read

$$\begin{aligned}\Lambda \frac{d\lambda_4}{d\Lambda} &= \beta_4(\lambda_4, \lambda_6) , \\ \Lambda \frac{d\lambda_6}{d\Lambda} - 2\lambda_6 &= \beta_6(\lambda_4, \lambda_6) .\end{aligned}$$

We want to solve these equations for β_4 and β_6 small. We should be very careful: in fact, if we just assume $\beta_6 \ll 1$, we would have the following solution

$$\Lambda \frac{d\lambda_6}{d\Lambda} - 2\lambda_6 \approx 0 \Rightarrow \lambda_6(\Lambda) = \lambda_6(\Lambda_H) \left(\frac{\Lambda}{\Lambda_H} \right)^2 ,$$

and, for $\Lambda \ll \Lambda_H$, λ_6 dies away. However, the correct coupling would be, in this case,

$$g_6(\Lambda) = \frac{\lambda_6(\Lambda)}{\Lambda^2} = \frac{\lambda_6(\Lambda_H)}{\Lambda_H^2} ,$$

which does not die away. We need to be more precise than the 0-th order, so we go to 1-st order and represent β_4 and β_6 as linear combination of their arguments. Introducing the small real numbers a, b, c, d

$$\Lambda \frac{d\lambda_4}{d\Lambda} = a\lambda_4 + b\lambda_6 , \tag{13.1}$$

$$\Lambda \frac{d\lambda_6}{d\Lambda} = c\lambda_4 + (2 + d)\lambda_6 . \tag{13.2}$$

This is the linear approximation of the differential equations describing the renormalisation flow. The solution is easily achievable through diagonalisation. The diagonal basis is

$$\tilde{\lambda}_4 = -\frac{c}{\Delta}\lambda_4 - \frac{2 + d - a - \Delta}{2\Delta}\lambda_6 , \quad \tilde{\lambda}_6 = \frac{c}{\Delta}\lambda_4 + \frac{2 + d - a + \Delta}{2\Delta}\lambda_6 ,$$

where $\Delta = \sqrt{4bc + (d - a + 2)^2}$. Thus, given the boundary conditions $\tilde{\lambda}_4(\Lambda_0)$ and $\tilde{\lambda}_6(\Lambda_0)$, we can simply solve the system in the diagonal basis

$$\begin{aligned}\tilde{\lambda}_4(\Lambda) &= \tilde{\lambda}_4(\Lambda_0) \left(\frac{\Lambda}{\Lambda_0} \right)^{\frac{d+2+a-\Delta}{2}} , \\ \tilde{\lambda}_6(\Lambda) &= \tilde{\lambda}_6(\Lambda_0) \left(\frac{\Lambda}{\Lambda_0} \right)^{\frac{d+2+a+\Delta}{2}} .\end{aligned}$$

Now we can recover the original basis, obtaining an exact solution for the equations

(13.1) and (13.2)

$$\begin{aligned} \lambda_4(\Lambda) = & \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a-\Delta}{2}} \left[\left(\frac{2+d-a+\Delta}{2\Delta}\right) \lambda_4(\Lambda_0) - \frac{b}{\Delta} \lambda_6(\Lambda_0) \right] \\ & + \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a+\Delta}{2}} \left[-\left(\frac{2+d-a-\Delta}{2\Delta}\right) \lambda_4(\Lambda_0) + \frac{b}{\Delta} \lambda_6(\Lambda_0) \right], \end{aligned} \quad (13.3)$$

$$\begin{aligned} \lambda_6(\Lambda) = & \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a-\Delta}{2}} \left[-\left(\frac{2+d-a-\Delta}{2\Delta}\right) \lambda_6(\Lambda_0) - \frac{c}{\Delta} \lambda_4(\Lambda_0) \right] \\ & + \left(\frac{\Lambda}{\Lambda_0}\right)^{\frac{d+2+a+\Delta}{2}} \left[\left(\frac{2+d-a+\Delta}{2\Delta}\right) \lambda_6(\Lambda_0) + \frac{c}{\Delta} \lambda_4(\Lambda_0) \right]. \end{aligned} \quad (13.4)$$

Let us investigate the behaviour of $\lambda_6(\Lambda)$ as Λ changes. In order to simplify the computations, we set, as initial condition, $\lambda_6(\Lambda_H) = 0$: the same conclusions could be obtained with any other condition. Combining the equations (13.3) and (13.4) we obtain

$$\lambda_6(\Lambda) = \frac{2c \left[\left(\frac{\Lambda}{\Lambda_H}\right)^\Delta - 1 \right]}{(2+d-a+\Delta) - (2+d-a-\Delta) \left(\frac{\Lambda}{\Lambda_H}\right)^\Delta} \lambda_4(\Lambda).$$

If $a, b, c, d \ll 2$, then $\Delta \sim 2$ and the above equation becomes

$$\lambda_6(\Lambda) = -\frac{c}{2} \left(1 - \frac{\Lambda^2}{\Lambda_H^2} \right) \lambda_4(\Lambda).$$

Let us consider λ_6 at $\Lambda = \Lambda_L \ll \Lambda_H$: for a fixed $\lambda_4(\Lambda_L)$ we have

$$\lambda_6(\Lambda) \approx -\frac{c}{2} \lambda_4(\Lambda)$$

and

$$g_6(\Lambda_L) \approx -\frac{c}{2} \frac{1}{\Lambda_L} \lambda_4(\Lambda).$$

We have explicitly shown that the *irrelevant* 6-dimensional operator \mathcal{O}_6 , associated to the coupling g_6 , at low energy is completely independent of the physical cutoff Λ_H of the theory.

For completeness purposes, we can also study the behaviour of the marginal operator coupling $g_4(\Lambda)$. If we had put $\lambda_6(\Lambda_H) = 0$, then we would have found

$$\lambda_4(\Lambda) = \frac{2b \left[1 - \left(\frac{\Lambda}{\Lambda_H}\right)^\Delta \right]}{(2+d-a-\Delta) - (2+d-a+\Delta) \left(\frac{\Lambda}{\Lambda_H}\right)^\Delta} \lambda_6(\Lambda),$$

which in the limit $a, b, c, d \ll 2$ becomes

$$\lambda_4(\Lambda) = \frac{b}{2} \left(1 - \frac{\Lambda_H^2}{\Lambda^2} \right) \lambda_6(\Lambda) .$$

Now, for $\Lambda_H \rightarrow \infty$, λ_4 diverges, so the operator is extremely sensitive to the physical cutoff and does not disappear from the low energy physics of our theory.

13.3 Magnetic moment of the electron

Now we can heuristically explore a simple application of the idea presented above.

We recall the result previously found for g , the Landé factor of the electron: the general structure of the QED vertex is given by the Gordon decomposition

$$\bar{u}(p)\Gamma^\mu u(q) = \bar{u}(p) \left(\frac{f_1}{2m}(q+p)^\mu + \frac{f_1 + f_2}{2m} i\sigma^{\mu\nu}(q-p)_\nu \right) u(q) ,$$

and it has been shown that $g = 2(f_1 + f_2)$. At 1-loop we have found $f_1 = 1$ and $f_2 = \frac{\alpha}{2\pi}$, so $g_{1-loop} = 2 + \frac{\alpha}{\pi}$. We could add to the QED lagrangian the 5-dimensional (irrelevant) operator $\mathcal{O}_5 = \frac{c_5}{4} \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}$ with the coefficient c_5 . Given the structure of the new operator and of the general QED vertex, we can verify that \mathcal{O}_5 modifies the Landé factor, which becomes $g = 2 + \frac{\alpha}{2} + c_5$. We would like to explain why experimental evidences totally rule out the presence of c_5 (or: they set it extremely low).

In order to do this, we just exploit the example studied in the previous section: if we assume that our theory stops being valid at the Planck energy scale, c_5 , as g_6 , is completely independent from the physical cutoff $\Lambda_H \sim M_{pl} \sim 10^{19}$ GeV at lower energies. Moreover, as $g_6(\Lambda)$ depended on $g_4(\Lambda)$, c_5 will depend on the value of the other operators which contribute to the computation of g . This means that c_5 will be a function of α , given that the other operators contribute to g with $\frac{\alpha}{2}$. If we lower the cutoff to $\Lambda_L \sim 10^{15}$ GeV $\ll \Lambda_H$, for example, c_5 will obtain the following Λ_H -independent structure

$$c_5(\Lambda_L) \sim \frac{f(\alpha)}{\Lambda_L} .$$

If we assume a condition of *naturalness* on $f(\alpha)$, i.e. $f(\alpha) \sim \alpha$, and we recall that the Wilsonian approach is correctly employed only if the lowered cutoff Λ_L is still much bigger than the energy scale of the experiment, we verify that the \mathcal{O}_5 contribution to g is negligible.

13.4 Polchinski's Renormalisation Equation

In the standard approach to renormalisation, we introduced previously the Callan-Symanzik equation, which had its origin in the independence of the generic correlation function from the scale parameter μ . Our purpose is to introduce an analogous tool for the Wilsonian approach, Polchinski's Renormalisation Equation [38]. In particular, imposing the independence of the generating functional $Z[J, \Lambda]$ from the cutoff Λ (the low energy Physics of the system is not dependent on the cutoff Λ), we will be able to find the effective action. For simplicity, we consider a theory with an interacting scalar field in a d -dimensional Euclidean space-time; its action is

$$S[\phi, \Lambda] = S_0[\phi, \Lambda] + S'[\phi, \Lambda] ,$$

where

$$S_0[\phi, \Lambda] = \frac{1}{2} \int d^d x d^d y (\phi(x) \Delta'^{-1}(x-y) \phi(y))$$

is the kinetic term and $\Delta'^{-1}(x-y)$ is the inverse (in the functional sense) of the exact propagator. Due to the presence of the cutoff Λ , $\Delta'(x-y)$ differs from the free propagator only in a neighborhood of $|x-y|=0$ of radius Λ^{-1} . The system we are studying is contained in a periodic d -cube of volume $V = L^d$: the periodicity implies the discreteness of the spectrum of the momentum operator. Thus, the field $\phi(x)$ is related to its Fourier transform ϕ_k by

$$\phi(x) = V^{-1/2} \sum_k e^{-ik \cdot x} \phi_k . \quad (13.5)$$

Substituting (13.5) into the expression for $S_0[\phi, \Lambda]$ and performing a change of the integration variables we get

$$S_0[\phi, \Lambda] = \frac{1}{2} \sum_k G_k^{-1}(\Lambda) \phi_k \phi_{-k} , \quad (13.6)$$

with

$$G_k^{-1}(\Lambda) = \int d^d x e^{ik \cdot x} \Delta'^{-1}(x) ,$$

where the dependence on Λ is hidden in the complex structure of $\Delta'^{-1}(x)$, as it has been said above. We already know that in the Wilsonian renormalisation scheme we integrate out high momentum modes, changing the form of the interacting action $S'[\phi, \Lambda]$. We want to describe this evolution through an explicit equation. The starting point is to impose the independence of the generating functional Z , which contains all the

information about the physics, from Λ

$$\frac{d}{d\Lambda} Z[J, \Lambda] = 0 . \tag{13.7}$$

In [38] Polchinski showed that (13.7) is true if

$$\frac{dS'}{d\Lambda} = -\frac{1}{2} \int d^d x d^d y \frac{\partial \Delta'(x-y)}{\partial \Lambda} \left(\frac{\delta^2 S'}{\delta \phi(x) \delta \phi(y)} - \frac{\delta S'}{\delta \phi(x)} \frac{\delta S'}{\delta \phi(y)} \right) \tag{13.8}$$

holds. This is Polchinski's Renormalisation Equation in the coordinate space. In the momentum space, the equation becomes

$$\frac{dS'}{d\Lambda} = -\frac{1}{2} \sum_k \frac{\partial G_k}{\partial \Lambda} \left(\frac{\partial^2 S'}{\partial \phi_k \partial \phi_{-k}} - \frac{\partial S'}{\partial \phi_k} \frac{\partial S'}{\partial \phi_{-k}} \right) . \tag{13.9}$$

The information given by this equation is crucial: in fact, now we can study the evolution of the interacting action after an infinitesimal change of the cutoff

$$S'[\phi, \Lambda_0 + \delta\Lambda] \approx S'[\phi, \Lambda_0] + \delta\Lambda \left(\frac{dS'}{d\Lambda} \right)_{\Lambda=\Lambda_0} .$$

We can simply substitute (13.8) in the last term. Now we verify that (13.9) solves (13.7). In order to simplify the calculations, we set $J = 0$. We start by calculating $dZ/d\Lambda$ in the momentum space

$$\begin{aligned} \frac{dZ}{d\Lambda} &= - \int \mathcal{D}\phi e^{-S} \frac{dS}{d\Lambda} \\ &= - \int \mathcal{D}\phi e^{-S} \left(\frac{dS_0}{d\Lambda} + \frac{dS'}{d\Lambda} \right) \\ &= - \int \mathcal{D}\phi e^{-S} \left[\frac{1}{2} \sum_k \phi_k \phi_{-k} \frac{\partial G_k^{-1}}{\partial \Lambda} - \frac{1}{2} \sum_k \frac{\partial G_k}{\partial \Lambda} \left(\frac{\partial^2 S'}{\partial \phi_k \partial \phi_{-k}} - \frac{\partial S'}{\partial \phi_k} \frac{\partial S'}{\partial \phi_{-k}} \right) \right] , \end{aligned}$$

where we applied (13.6) and (13.9). Formally $\frac{\partial G_k^{-1}}{\partial \Lambda} = -\frac{\partial G_k}{\partial \Lambda} G_k^{-2}$, so

$$\frac{dZ}{d\Lambda} = \frac{1}{2} \sum_k \frac{\partial G_k}{\partial \Lambda} \int \mathcal{D}\phi e^{-S} \left(\phi_k \phi_{-k} G_k^{-2} + \frac{\partial^2 S'}{\partial \phi_k \partial \phi_{-k}} - \frac{\partial S'}{\partial \phi_k} \frac{\partial S'}{\partial \phi_{-k}} \right) . \tag{13.10}$$

A side calculation

$$\begin{aligned} \frac{\partial^2 S'}{\partial \phi_k \partial \phi_{-k}} e^{-S} &= \frac{\partial}{\partial \phi_k} \left(\frac{\partial S'}{\partial \phi_{-k}} e^{-S} \right) + \frac{\partial S'}{\partial \phi_{-k}} \frac{\partial S}{\partial \phi_k} e^{-S} \\ &= \frac{\partial}{\partial \phi_k} \left(\frac{\partial S'}{\partial \phi_{-k}} e^{-S} \right) + \frac{\partial S'}{\partial \phi_{-k}} \frac{\partial S_0}{\partial \phi_k} e^{-S} + \frac{\partial S'}{\partial \phi_{-k}} \frac{\partial S'}{\partial \phi_k} e^{-S} , \end{aligned}$$

hence

$$\left(\frac{\partial^2 S'}{\partial \phi_k \partial \phi_{-k}} - \frac{\partial S'}{\partial \phi_k} \frac{\partial S'}{\partial \phi_{-k}} \right) e^{-S} = \frac{\partial}{\partial \phi_k} \left(\frac{\partial S'}{\partial \phi_{-k}} e^{-S} \right) + \frac{\partial S'}{\partial \phi_{-k}} \frac{\partial S_0}{\partial \phi_k} e^{-S} .$$

Also

$$\begin{aligned} \frac{\partial S'}{\partial \phi_{-k}} \frac{\partial S_0}{\partial \phi_k} e^{-S} &= \frac{\partial(S - S_0)}{\partial \phi_{-k}} \frac{\partial S}{\partial \phi_k} e^{-S} \\ &= \frac{\partial S}{\partial \phi_{-k}} \frac{\partial S_0}{\partial \phi_k} e^{-S} - \frac{\partial S_0}{\partial \phi_{-k}} \frac{\partial S_0}{\partial \phi_k} e^{-S} \\ &= -\frac{\partial e^{-S}}{\partial \phi_{-k}} G_k^{-1} \phi_{-k} - G_k^{-2} \phi_k \phi_{-k} e^{-S} \\ &= -G_k^{-1} \frac{\partial}{\partial \phi_{-k}} (\phi_{-k} e^{-S}) + G_k^{-1} e^{-S} - G_k^{-2} \phi_k \phi_{-k} e^{-S} , \end{aligned}$$

thus

$$\begin{aligned} \left(\frac{\partial^2 S'}{\partial \phi_k \partial \phi_{-k}} - \frac{\partial S'}{\partial \phi_k} \frac{\partial S'}{\partial \phi_{-k}} \right) e^{-S} &= \\ \frac{\partial}{\partial \phi_k} \left(\frac{\partial S'}{\partial \phi_{-k}} e^{-S} \right) - G_k^{-1} \frac{\partial}{\partial \phi_{-k}} (\phi_{-k} e^{-S}) + G_k^{-1} e^{-S} - G_k^{-2} \phi_k \phi_{-k} e^{-S} . \end{aligned} \quad (13.11)$$

Substituting (13.11) in (13.10) we eventually obtain

$$\frac{dZ}{d\Lambda} = \frac{1}{2} \sum_k \frac{\partial G_k}{\partial \Lambda} \int \mathcal{D}\phi \left[\frac{\partial}{\partial \phi_k} \left(\frac{\partial S'}{\partial \phi_{-k}} e^{-S} \right) - G_k^{-1} \frac{\partial}{\partial \phi_{-k}} (\phi_{-k} e^{-S}) + G_k^{-1} e^{-S} \right] .$$

Note that the last term in the expression above is proportional to Z (G_k^{-1} does not depend on the fields and it can be absorbed in the measure $\mathcal{D}\phi$). In order to neglect it, we need to relax the condition (13.7), allowing it to hold modulo a normalisation.

Thus, we obtain

$$\begin{aligned} \frac{dZ}{d\Lambda} &= \frac{1}{2} \sum_k \frac{\partial G_k}{\partial \Lambda} \int \mathcal{D}\phi \left[\frac{\partial}{\partial \phi_k} \left(\frac{\partial S'}{\partial \phi_{-k}} e^{-S} \right) - G_k^{-1} \frac{\partial}{\partial \phi_{-k}} (\phi_{-k} e^{-S}) \right] \\ &= \frac{1}{2} \sum_k \frac{\partial G_k}{\partial \Lambda} \left[\int \mathcal{D}\phi \frac{\partial}{\partial \phi_k} \left(\frac{\partial S'}{\partial \phi_{-k}} e^{-S} \right) - G_k^{-1} \int \mathcal{D}\phi \frac{\partial}{\partial \phi_{-k}} (\phi_{-k} e^{-S}) \right] = 0 . \end{aligned}$$

The last expression is null because it is a difference of two integrated total derivatives: the argument of the total derivatives contain a factor e^{-S} , which in a reasonable physical situation goes to zero at infinity. This verifies Polchinski's equation (13.9).

Appendix A

Computation of some integrals and proof of some formulae¹

A.1 Integration in arbitrary dimension

Consider the integral

$$I_N = \int d^N l F(L) , \quad L^2 = l_\mu l^\mu , \quad (\text{A.1})$$

with $N \in \mathbb{N}_+$ and $F(L)$ an arbitrary function, depending only on the length of l_μ ($\mu = 1, \dots, N$). Introduce polar coordinates in N dimensions

$$(l_1, \dots, l_N) \Rightarrow (L, \phi, \theta_1, \dots, \theta_{N-2}) .$$

Then

$$d^N l = L^{N-1} dL d\phi \sin \theta_1 d\theta_1 \sin^2 \theta_2 d\theta_2 \dots \sin^{N-2} \theta_{N-2} d\theta_{N-2} , \quad (\text{A.2})$$

with the variables assuming values over the integrals

$$L \in [0, \infty) , \quad \phi \in [0, 2\pi) , \quad \theta_k \in [0, \pi] , \quad k = 1, \dots, N-2 . \quad (\text{A.3})$$

It is easy, starting from (A.1), to obtain

$$I_N = 2\pi \prod_{k=1}^{N-2} \int_0^\pi d\theta_k \sin^k \theta_k \int_0^\infty dL L^{N-1} F(L) . \quad (\text{A.4})$$

¹ Umberto Natale and Davide Dal Cin

The use of the well-known formula

$$\int_0^{\frac{\pi}{2}} dt (\sin t)^{2x-1} (\cos t)^{2y-1} = \frac{1}{2} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \text{Re } x > 0, \text{ Re } y > 0, \quad (\text{A.5})$$

with $y = \frac{1}{2}$, yields

$$\int_0^{\pi} dt \sin^k t = 2 \int_0^{\frac{\pi}{2}} dt \sin^k t = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+2}{2})} = \sqrt{\pi} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}. \quad (\text{A.6})$$

Putting it back in (A.4), we obtain

$$I_N = 2\pi^{\frac{N+2}{2}} \frac{\Gamma(1)\Gamma(\frac{3}{2})\Gamma(2)\cdots\Gamma(\frac{N-1}{2})}{\Gamma(\frac{3}{2})\Gamma(2)\Gamma(\frac{5}{2})\cdots\Gamma(\frac{N-1}{2})\Gamma(\frac{N}{2})} \int_0^{\infty} \frac{dx}{2} x^{\frac{N-2}{2}} F(x), \quad (\text{A.7})$$

where $x = L^2$.

A.2 Dimensional regularisation revisited

We formulated the dimensional regularisation by starting with the integral

$$I(\omega) = \int d^{2\omega} l \frac{1}{\ell^2 + m^2},$$

and then, by formally considering the integration as an operation defined for any ω , we showed that it is

(i) IR convergent for

$$2 < \omega < 3,$$

(ii) UV convergent for

$$\omega < 1.$$

We then manipulated the integrand of $I(\omega)$ to arrive to the integral (9.16) which is both IR and UV convergent in the interval

$$0 < \omega < 3,$$

except for the pole at

$$\omega = 2.$$

This is what we wanted, because we now know the precise form of the singularity in ω : a simple pole. We then showed that the integral (9.16) corresponds to (9.17). A less pedagogical but faster way to get the same result, is to make the identification of $I(\omega)$

with its expression in terms of the Γ -functions. First we do such an identification in the case when the integral is well-defined, and then simply consider the resulting expression in terms of Γ -functions as the values of such integrals for arbitrary values, both of the space-time dimension and of the order of the singularities in the momenta.

Let us show such an alternative, by considering the integral (A.7) in the case

$$F(x) = (x + a^2)^{-A}, \quad A = 2, 3, \dots, \tag{A.8}$$

so that

$$\int_0^\infty dx \frac{x^{\frac{N-2}{2}}}{(x + a^2)^A} = (a^2)^{-A+\frac{N}{2}} \int_0^\infty dy y^{\frac{N-2}{2}} (1 + y)^{-A} . \tag{A.9}$$

Comparing with the expression for the Beta function

$$B\left(\frac{N}{2}, A - \frac{N}{2}\right) = \frac{\Gamma(\frac{N}{2})\Gamma(A - \frac{N}{2})}{\Gamma(A)} = \int_0^\infty dy y^{\frac{N}{2}-1} (1 + y)^{-A} , \tag{A.10}$$

which is valid for $\text{Re } \frac{N}{2}$ and $\text{Re } (A - \frac{N}{2}) > 0$, we get

$$\int d^N l \frac{1}{(l^2 + a^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(a^2)^{A-N/2}} . \tag{A.11}$$

The outcome of the dimensional regularisation, is to use the right-hand side of (A.11) to define the integral on the left-hand side even for complex values of N . In this way, the analytic structure, including the singularities, is determined by the Γ -function. By letting $l = l' + p$, and relabeling $b^2 = a^2 + p^2$, we can write (A.11) in the form

$$\int d^N l \frac{1}{(l^2 + 2pl + b^2)^A} = \pi^{N/2} \frac{\Gamma(A - N/2)}{\Gamma(A)} \frac{1}{(b^2 - p^2)^{A-N/2}} . \tag{A.12}$$

Next, by successive differentiation of (A.12) with respect to p_μ , we get

$$\int d^N l \frac{l_\mu}{(l^2 + 2pl + a^2)^A} = \pi^{N/2} \frac{\Gamma(A - \frac{N}{2})}{\Gamma(A)} \frac{(-p_\mu)}{(a^2 - p^2)^{A-N/2}} , \tag{A.13}$$

and

$$\begin{aligned} \int d^N l \frac{l_\mu l_\nu}{(l^2 + 2pl + a^2)^A} &= \frac{\pi^{N/2}}{\Gamma(A)(a^2 - p^2)^{A-N/2}} \\ &\times \left[p_\mu p_\nu \Gamma(A - N/2) + \frac{1}{2} \delta_{\mu\nu} (a^2 - p^2) \Gamma(A - 1 - N/2) \right] . \end{aligned}$$

A.3 Some properties of the Γ function

In the following we illustrate some useful properties of the Γ -function. Let us start by reporting its Euler representation

$$\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1} . \quad (\text{A.14})$$

One may check that it satisfies the recursion relation

$$\Gamma(z+1) = \Gamma(z)z . \quad (\text{A.15})$$

Clearly the Γ has simple poles when z is a negative integer, and an essential singularity at complex infinity, because $\Gamma(\frac{1}{z})$ has a non-defined limit for $z \rightarrow 0$.

Let us show that near a pole at $-n$, $n \in \mathbb{N}$, the Γ -function has the expansion

$$\Gamma(z-n) = \frac{(-1)^n}{n!} \left(\frac{1}{z} + \psi(n+1) + \mathcal{O}(z) \right) , \quad (\text{A.16})$$

that, up to terms proportional to positive powers of z , is the expansion reported in (9.18). Note that

$$\Gamma(z-n) = \frac{1}{z-n} \Gamma(z-n+1) , \quad (\text{A.17})$$

implies

$$\Gamma(z-n) = \frac{1}{(z-n)(z-n+1)\cdots(z+n)} \Gamma(z+n+1) = \frac{1}{z} G(z, n) , \quad (\text{A.18})$$

with

$$G(z, n) := \frac{\Gamma(z+n+1)}{(z^2-n^2)\cdots(z^2-1)} = \frac{(-1)^2}{(n!)^2} \Gamma(z+n+1)(1 + \mathcal{O}(z^2)) , \quad (\text{A.19})$$

where we recognised the factorisation of the difference between squares, factored all the terms n^2 , $(n-1)^2$, ... 1, and used the geometric series.

By Taylor expanding $G(z, n)$ around $z=0$ in (A.18) we get²

$$\Gamma(z-n) = \frac{1}{z} (G(0, n) + zG'(0, n) + \mathcal{O}(z^2)) = \frac{(-1)^n}{n!} \left(\frac{1}{z} + \frac{\Gamma'(n+1)}{n!} + \mathcal{O}(z) \right) , \quad (\text{A.20})$$

that is the claimed result.

We have seen that we can define the logarithmic derivative of the Γ -function in the following way:

$$\psi(z) = \frac{d \log(\Gamma(z))}{dz} , \quad (\text{A.21})$$

² $\Gamma(n+1) = n!$

from the recursion relation of the Γ -function follows that

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \quad (\text{A.22})$$

For a positive integer n , we can iterate the recursion (A.22) to obtain :

$$\psi(n+1) = \psi(1) + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad (\text{A.23})$$

and the value of $\psi(1)$ is

$$\psi(1) = -\gamma_E,$$

where γ_E is the Euler-Mascheroni number.

There are some alternative definitions of the Γ -function. One, due to Euler, is

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1 + \frac{1}{n})^z}{1 + \frac{z}{n}}, \quad (\text{A.24})$$

an useful result is to take the log of the above formula, namely:

$$\log(\Gamma(z)) = -\log(z) + \sum_{n=1}^{\infty} \left[z \log\left(1 + \frac{1}{n}\right) - \log\left(1 + \frac{z}{n}\right) \right], \quad (\text{A.25})$$

in fact now we can prove that $\psi(1) = -\gamma_E$, we start with the definition of the Euler-Mascheroni number

$$\gamma_E = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \log\left(1 + \frac{1}{n}\right) \right]. \quad (\text{A.26})$$

Let us consider (A.25) and take the derivative

$$\psi(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left[\log\left(1 + \frac{1}{n}\right) - \frac{1}{n+z} \right]. \quad (\text{A.27})$$

This implies

$$\psi(1) = -1 + \sum_{n=1}^{\infty} \left[\log\left(1 + \frac{1}{n}\right) - \frac{1}{1+n} \right]. \quad (\text{A.28})$$

Redefining the variable in the last term and absorbing the first term, we get

$$\psi(1) = - \sum_{n=1}^{\infty} \left[\frac{1}{n} - \log\left(1 + \frac{1}{n}\right) \right], \quad (\text{A.29})$$

in which we recognise the definition (A.26).

Considering (A.26) with (A.27) we get

$$\psi'(z+1) = \frac{1}{z^2} + \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} - \frac{1}{z^2}, \quad (\text{A.30})$$

that for $z = 0$ reads

$$\psi'(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is the well-known solution of the Basilea problem obtained by Euler in 1735.

We can rewrite (A.30) for integer z as

$$\psi'(n+1) = \sum_{k=1}^{\infty} \frac{1}{(k+n)^2}. \quad (\text{A.31})$$

Then, changing variable from k to $k' = k + n$

$$\psi'(n+1) = \sum_{k=1+n}^{\infty} \frac{1}{k^2} = \sum_{k=1+n}^{\infty} \frac{1}{k^2} - \sum_{k=1}^n \frac{1}{k^2} + \sum_{k=1}^n \frac{1}{k^2},$$

and combining the first and the third terms, we get

$$\psi'(n+1) = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^n \frac{1}{k^2} = \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2},$$

which is (9.20).

A.4 Derivation of Eq.(4.4.22) from Eq.(4.4.21) in Ramond's book

In the following we show the calculations that starting from Eq.(4.4.21) in Ramond's book

$$\Sigma(p) = -\frac{1}{4\omega} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left\{ \left(l_{\mu} \frac{\partial}{\partial l_{\mu}} + q_{\mu} \frac{\partial}{\partial q_{\mu}} \right) \times \frac{1}{(l^2 + m^2)(q^2 + m^2)[(p+q-l)^2 + m^2]} \right\},$$

lead to Eq.(4.4.22), which is

$$\Sigma(p) = -\frac{1}{2\omega - 3} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{3m^2 + p^\mu (p + q - l)_\mu}{(l^2 + m^2)(q^2 + m^2)[(p + q - l)^2 + m^2]^2}. \tag{A.32}$$

Let us start by setting

$$\begin{aligned} L(l, m) &= l^2 + m^2, \\ Q(q, m) &= q^2 + m^2, \\ P(l, q, p, m) &= (p + q - l)^2 + m^2, \end{aligned}$$

in Eq.(A.32). One has

$$\begin{aligned} \Sigma(p) = & -\frac{1}{4\omega} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left\{ -\frac{2l^\mu l_\mu QP}{L^2 Q^2 P^2} \right. \\ & + \frac{2LQl^\mu (p + q - l)_\mu}{L^2 Q^2 P^2} - \frac{2q^\mu (q + p - l)_\mu QL}{L^2 Q^2 P^2} \\ & \left. - \frac{2q^\mu q_\mu L [(p + q - l)^2 + m^2]}{L^2 Q^2 P^2} \right\}. \end{aligned}$$

By separating the various addends and adding and subtracting m^2 or p^μ (as appropriate) to reconstruct the form of one of the factors in the denominator, we have

$$\begin{aligned} \Sigma(p) = & -\frac{1}{2\omega} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left[-\frac{l^2 + m^2 - m^2}{L^2 QP} \right. \\ & \left. - \frac{q^2 + m^2 - m^2}{LQ^2 P} - \frac{(q - l + p - p)^\mu (p + q - l)_\mu}{LQP^2} \right]. \end{aligned}$$

Let us separate the various terms further

$$\begin{aligned} \Sigma(p) = & -\frac{1}{2\omega} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left[-\frac{L}{L^2 QP} \right. \\ & + \frac{m^2}{L^2 QP} + \frac{m^2}{LQ^2 P} + \frac{m^2}{LQP^2} - \frac{Q}{LQ^2 P} \\ & \left. + \frac{p^\mu (p + q - l)_\mu}{LQP^2} - \frac{(q - l + p)^2 + m^2}{LQP^2} \right], \end{aligned}$$

so that, simplifying and separating the terms with the mass from the other addends, we obtain

$$\begin{aligned} \Sigma(p) = & -\frac{1}{2\omega} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left[-\frac{3}{LQP} \right. \\ & \left. + \frac{m^2}{LQP} \left(\frac{1}{L} + \frac{1}{Q} + \frac{1}{P} \right) + \frac{p^\mu (p+q-l)_\mu}{LQP^2} \right]. \end{aligned}$$

Notice that the first term in the brace is proportional to (9.38). More precisely,

$$\frac{3}{2\omega} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{1}{LQP} = \frac{3}{2\omega} \Sigma(p),$$

from which

$$\begin{aligned} \Sigma(p) = & -\frac{1}{2\omega} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left[\frac{p^\mu (p+q-l)_\mu}{LQP^2} \right. \\ & \left. + \frac{m^2}{LQP} \left(\frac{1}{L} + \frac{1}{Q} + \frac{1}{P} \right) \right] + \frac{3}{2\omega} \Sigma(p), \end{aligned}$$

that is

$$\begin{aligned} \Sigma(p) = & -\frac{1}{2\omega-3} \frac{\lambda^2}{6} (\mu^2)^{4-2\omega} \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \left[\frac{p^\mu (p+q-l)_\mu}{LQP^2} \right. \\ & \left. + \frac{m^2}{LQP} \left(\frac{1}{L} + \frac{1}{Q} + \frac{1}{P} \right) \right]. \end{aligned} \tag{A.33}$$

Consider the following sum of three double integrals

$$\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{m^2}{LQP} \left(\frac{1}{L} + \frac{1}{Q} + \frac{1}{P} \right).$$

We carry out the substitutions

$$\begin{aligned} q &\rightarrow q, & l &\rightarrow l' = p + q - l, \\ l &\rightarrow l, & q &\rightarrow q' = -p - q + l, \end{aligned} \tag{A.34}$$

in the first and second double integral, respectively. Since both transformations have jacobian 1 and the integration domain, $\mathbb{R} \times \mathbb{R}$, remains unchanged, it follows that the three integrals are identical. Therefore, we have

$$\int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{m^2}{LQP} \left(\frac{1}{L} + \frac{1}{Q} + \frac{1}{P} \right) = \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{3m^2}{LQP^2},$$

from which (A.32) follows.

Let us conclude this section by observing that Eq.(4.4.24) in Ramond's book

$$K(p) = \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{1}{(l^2 + m^2)^2 (q^2 + m^2) [(p + q - l)^2 + m^2]} ,$$

can be obtained using again the substitution $l \rightarrow p + q - l$ in

$$K(p) = \int \frac{d^{2\omega}l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega}q}{(2\pi)^{2\omega}} \frac{1}{(l^2 + m^2) (q^2 + m^2) [(p + q - l)^2 + m^2]^2} .$$

The point is just the observed invariance of the integration domain and the fact that the Jacobian is the unity. As such, the effect of the substitution is only the one of moving the square from the first round bracket in the denominator to the square bracket.

A.5 Scaling properties of the setting sun

In the following we will show that the expression (A.33) can be derived from (9.38) by exploiting the scale properties

$$\Sigma(p, m) = (m^2)^{2\omega-3} \Sigma\left(\frac{p}{m}, 1\right) . \quad (\text{A.35})$$

One has

$$\frac{\partial \Sigma(p, m)}{\partial m^2} = (2\omega - 3) (m^2)^{2\omega-4} \Sigma\left(\frac{p}{m}, 1\right) + (m^2)^{2\omega-3} \frac{\partial \Sigma\left(\frac{p}{m}, 1\right)}{\partial m^2} ,$$

that by (A.35) is equivalent to

$$\frac{\partial \Sigma(p, m)}{\partial m^2} = (2\omega - 3) (m^2)^{-1} \Sigma(p, m) + (m^2)^{2\omega-3} \frac{\partial \Sigma\left(\frac{p}{m}, 1\right)}{\partial m^2} .$$

Since

$$\frac{\partial}{\partial m^2} = \frac{\partial \frac{p^\mu}{m}}{\partial m^2} \frac{\partial}{\partial \frac{p^\mu}{m}} = -\frac{1}{2m^2} \frac{p^\mu}{m} \frac{\partial}{\partial \frac{p^\mu}{m}} = -\frac{1}{2m^2} p^\mu \frac{\partial}{\partial p^\mu} ,$$

it follows that

$$\frac{\partial \Sigma(p, m)}{\partial m^2} = (2\omega - 3) (m^2)^{-1} \Sigma(p, m) - \frac{(m^2)^{2\omega-3}}{2m^2} p^\mu \frac{\partial \Sigma\left(\frac{p}{m}, 1\right)}{\partial p^\mu} ,$$

that is

$$\Sigma(p, m) = \frac{m^2}{2\omega - 3} \frac{\partial \Sigma(p, m)}{\partial m^2} + \frac{(m^2)^{2\omega-3}}{2(2\omega - 3)} p^\mu \frac{\partial \Sigma\left(\frac{p}{m}, 1\right)}{\partial p^\mu} .$$

Using again the (A.35) to calculate the derivative in the second addend, we have

$$\Sigma(p, m) = \frac{m^2}{2\omega - 3} \frac{\partial \Sigma(p, m)}{\partial m^2} + \frac{p^\mu}{2(2\omega - 3)} \frac{\partial \Sigma(p, m)}{\partial p^\mu},$$

that reproduces (A.33) since (9.38) implies

$$\frac{\partial \Sigma(p, m)}{\partial m^2} = -\frac{\lambda^2 (\mu^2)^{4-2\omega}}{6} \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega} q}{(2\pi)^{2\omega}} \left(\frac{1}{L^2 Q P} + \frac{1}{L Q^2 P} + \frac{1}{L Q P^2} \right),$$

and

$$\frac{\partial \Sigma(p, m)}{\partial p^\mu} = -\frac{\lambda^2 (\mu^2)^{4-2\omega}}{3} \int \frac{d^{2\omega} l}{(2\pi)^{2\omega}} \int \frac{d^{2\omega} q}{(2\pi)^{2\omega}} \frac{(p + q - l)_\mu}{(l^2 + m^2)(q^2 + m^2)[(p + q - l)^2 + m^2]^2}.$$

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