Contents

Chapter 1

Processes with multiplicative Noise

1.1 Multiplicative processes

Up to now we have considered Fokker-Planck equations derived by stochastic differential equations (SDEs) of the form

$$dx = h(x)dt + A(t)\Delta W(t), \tag{1.1}$$

where the amplitude of the random force, $A(t)$, does not depend on the stochastic variable x to be integrated. Those are SDEs with, so called, *additive noise*. These equations were giving rise to a Fokker-Planck with a diffusion coefficient $D^{(2)}(x,t)$ that does not depend on x.

On the other hand there are examples in which the amplitude of the random force depends on the values of the mesoscopic variables. If this dependence can be described simply by a function of x multiplying the random force these process are called *multiplicative stochastic process* and the noise a *multiplicative noise*. A first example of a multiplicative noise originates from the interaction between the system and the surroundings that can make some parameters in the SDE fluctuating variables. This is the case, for example, of a chemical reaction in which one of the intermediate products is sensible to light exposure. If one lights the container where the reaction is occurring with a fluctuating light that tends to a white noise (i.e. with a flat power spectrum in a wide range of frequencies) one can get a case of multiplicative noise due to external fields. Another example can occur in some autocatalytic chemical reactions in which the production of a molecule of some type is enhanced by the presence of other molecules of the same type that have been produced already. Another example of stochastic process with multiplicative noise arises in economics and more particularly with the Black-Scholes theory of option pricing **??**.

A general SDE of a one-dimensional stochastic multiplicative process is of the form

$$\frac{dx}{dt} = h(x,t) + g(x,t)F(t). \tag{1.2}$$

When $g(x,t)$ depends on x the random term and $F(t)$ is the random force. Clearly for $x \to v$, $h(v,t) = -\gamma v(t)$ and , $g(v,t) = 1/m$, eq. (1.2) reduces to the Langevin equation. Equation (1.2) is in general quite difficult to solve analytically mainly because of the highly non differentiable character of a realization of $F(t)$. For a unidimensional system the distinction between additive and multiplicative noise may not be considered so crucial since, when $h(x,t) = h(x)$ and $g(x,t) = g(x)$, there always exists a change of variable such that the multiplicative noise becomes additive. The transformation to be considered is

$$x = f^{-1}(z), \quad \text{where} \quad z \equiv f(x) = \int^x \frac{dx'}{g(x')} \tag{1.3}$$

and gives an equation for the new variable z with an additive noise force where the function $h(x)$ is replaced by

$$h_1(z) = h(f^{-1}(z))/g(f^{-1}(z)). \tag{1.4}$$

In higher dimensions the change of variables given by (1.3) exists only if the multiplicative matrix $g_{\alpha\beta}(x)$ satisfies some conditions [**?**]. Before discussing the mapping to the Fokker-Planck description let us try to integrate directly a simple case of SDE with multiplicative noise.

1.1.1 An example of SDE with multiplicative noise

A simple example of a 1d SDE with multiplicative noise is given by :

$$\frac{dx}{dt} = -\gamma x(t) + \alpha x(t)F(t). \tag{1.5}$$

where γ and α are constants and $F(t)$ the random force. One can first adsorb the $-\gamma x(t)$ term by considering the transformation $x(t) = e^{-\gamma t}y(t)$. This gives the following SDE for y:

$$\frac{dy}{dt} = \alpha y(t)F(t), \tag{1.6}$$

whose formal solution is

$$y(t) = y(0)e^{\alpha \int_0^t F(t')dt'}. \tag{1.7}$$

By expanding the exponential one gets

$$\begin{aligned} y(t) &= y(0)\left[1 + \alpha \int_0^t F(t_1)dt_1 + \frac{\alpha^2}{2}\int_0^t\int_0^t F(t_1)F(t_2)dt_1dt_2 \right. \\ &+ \left. \cdots + \frac{\alpha^{2n}}{(2n)!}\int_0^t \cdots \int_0^t F(t_1)\cdots F(t_{2n})dt_1\cdots dt_{2n}\right]. \end{aligned} \tag{1.8}$$

Averaging over the noise gives

$$\begin{aligned} \langle y(t)\rangle &= y(0)\langle e^{\alpha \int_0^t F(t')dt'}\rangle \\ &= y(0)\left[1 + \alpha \int_0^t \langle F(t_1)\rangle dt_1 + \frac{\alpha^2}{2}\int_0^t\int_0^t \langle F(t_1)F(t_2)\rangle dt_1dt_2 \right. \\ &+ \left. \cdots + \frac{\alpha^{2n}}{(2n)!}\int_0^t \cdots \int_0^t \langle F(t_1)\cdots F(t_{2n})\rangle dt_1\cdots dt_{2n}\right] \end{aligned} \tag{1.9}$$

Eq. (1.9) is valid for arbitrary random noise. If in particular the noise is a Gaussian stationary process the solution can be simplified on the basis on the following properties valid for such process:

$$\langle F(t_i)\rangle = 0 \tag{1.10}$$

$$\langle F(t_1)F(t_2)\rangle = C(t_1 - t_2) \tag{1.11}$$

$$\langle F(t_1)F(t_2)\cdots F(t_{2n+1})\rangle = 0 \tag{1.12}$$

$$\langle F(t_1)F(t_2)\cdots F(t_{2n})\rangle = \sum_P C(t_{i_1} - t_{i_2})C(t_{i_3} - t_{i_4})\cdots C(t_{i_{2n-1}} - t_{i_{2n}}). \tag{1.13}$$

The last property can be further simplified if one notices that the interchange of two times in the correlation function C and the permutation of the n correlation functions in the product do not lead to different results. Hence there are $(2n)!/(2^n n!)$ different possibilities for the permutation of the $2n$ times t_i and this gives

$$\int_0^t \cdots \int_0^t \langle F(t_1)\cdots F(t_{2n})\rangle dt_1\cdots dt_{2n} = \frac{(2n)!}{(2^n n!)}\left[\int_0^t\int_0^t C(t_1 - t_2)dt_1dt_2\right]^n. \tag{1.14}$$

In this case it is possible to perform exactly the sum of the power series giving

$$\langle y(t)\rangle = y(0)\left\langle \exp\left[\alpha\int_0^t F(t')dt'\right]\right\rangle = y(0)\exp\left[\frac{\alpha^2}{2}\int_0^t\int_0^t C(t_1-t_2)dt_1dt_2\right]. \tag{1.15}$$

Back to the original variable x this gives

$$\langle x(t)\rangle = \langle e^{-\gamma t}y(t)\rangle = x(0)e^{-\gamma t}e^{\frac{\alpha^2}{2}\int_0^t\int_0^t C(t_1-t_2)dt_1dt_2}. \tag{1.16}$$

If in addition the noise is δ-correlated (white noise), with $C(t_1-t_2) = \sigma^2\delta(t_1-t_2)$, one has

$$\int_0^t\int_0^t C(t_1-t_2)dt_1dt_2 = \sigma^2 t, \tag{1.17}$$

and eq. (1.16) simplifies to

$$\boxed{\langle x(t)\rangle = x(0)e^{(\frac{\alpha^2\sigma^2}{2}-\gamma)t}} \tag{1.18}$$

Differentiating the above relation with respect to time one gets, for the average value of x, the deterministic differential equation

$$\frac{d}{dt}\langle x(t)\rangle = \langle x(t)\rangle\left(\frac{\alpha^2\sigma^2}{2}-\gamma\right) \tag{1.19}$$

with initial condition $\langle x(0)\rangle = x_0$. In the small time regime eq. (1.18) becomes

$$\langle x\rangle \sim x_0\left(1+(\frac{\alpha^2\sigma^2}{2}-\gamma)t+\cdots\right) \tag{1.20}$$

giving

$$\frac{d}{dt}\langle x(t)\rangle \sim x_0 + x_0(\frac{\alpha^2\sigma^2}{2}-\gamma)t \tag{1.21}$$

Notice that, in addition to the usual drift (dissipation) term, there is new a drift term that originates from the stochastic force.

Exercise. Show that for a white noise force term, the moments of $x(t)$ are given by

$$\langle x(t)^n\rangle = x_0^n\exp\left[-nt(\gamma-n\frac{\alpha^2\sigma^2}{2})\right] \tag{1.22}$$

while for the centered moments $\langle(x(t)-x(0))^n\rangle$,

$$\langle(x(t)-x(0))^n\rangle = x_0^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\exp\left[-kt(\gamma-k\frac{\alpha^2\sigma^2}{2})\right]. \tag{1.23}$$

1.1.2 Expansion in cumulants

In the more general case of non-Gaussian noise relations (1.10)-(1.13) are not valid any more and an explicit calculation of the power series sum is not possible. One must then rely, for example, on the cumulants expansion approach. For a single random variable ξ the cumulants are defined by means of the generating function

$$\left\langle e^{-it\xi}\right\rangle = \exp\left\{\sum_{n=1}^{\infty}\frac{(-it)^n}{n!}\langle\langle\xi^n\rangle\rangle\right\}. \tag{1.24}$$

By substituting in eq. (1.9) the random exponent $-it\xi(t)$ with the random variable $\alpha \int_0^t F(t')dt'$ one gets

$$\begin{aligned}
\langle y(t) \rangle &= y(0)\langle e^{\alpha \int_0^t F(t')dt'} \rangle \\
&= y(0) \exp \left\{ \left[1 + \alpha \int_0^t \langle F(t_1) \rangle dt_1 + \frac{\alpha^2}{2} \int_0^t \int_0^t \langle \langle F(t_1)F(t_2) \rangle \rangle dt_1 dt_2 \right. \right. \qquad (1.25) \\
&+ \left. \left. \cdots + \frac{\alpha^{2n}}{(2n)!} \int_0^t \cdots \int_0^t \langle \langle F(t_1) \cdots F(t_{2n}) \rangle \rangle dt_1 \cdots dt_{2n} \right] \right\} \qquad (1.26)
\end{aligned}$$

Note that, unlike equation (1.9), the expansion in α appears in the exponent. The connection between the moments and the cumulants is given by the relations

$$\begin{aligned}
\langle 1 \rangle &= \langle \langle 1 \rangle \rangle \\
\langle 12 \rangle &= \langle \langle 1 \rangle \rangle \langle \langle 2 \rangle \rangle + \langle \langle 12 \rangle \rangle \\
\langle 123 \rangle &= \langle \langle 1 \rangle \rangle \langle \langle 2 \rangle \rangle \langle \langle 3 \rangle \rangle + \langle \langle 1 \rangle \rangle \langle \langle 23 \rangle \rangle + \langle \langle 2 \rangle \rangle \langle \langle 13 \rangle \rangle + \langle \langle 3 \rangle \rangle \langle \langle 12 \rangle \rangle + \langle \langle 123 \rangle \rangle \\
& \cdots \qquad (1.27)
\end{aligned}$$

Note that if $F(t)$ is Gaussian all the cumulants beyond $n = 2$ are zero. The reason to consider an the expansion in cumulants instead then in moments is the following. Suppose that $\xi(t)$ has a short correlation time τ_c. This means that the random variables $\xi(t_1)$ and $\xi(t_2)$ are statistically independent when $|t_1 - t_2| >> \tau_c$. In this case the moments $\langle \xi(t_1)\xi(t_2) \rangle$ factorize into $\langle \xi(t_1) \rangle \langle \xi(t_2) \rangle$ but the cumulant $\langle \langle \xi(t_1)\xi(t_2) \rangle \rangle$ vanishes. The consequence is that each integral in (1.26) virtually vanishes unless $t_1, t_2, \cdots t_n$ are close together within a domain of order τ_c. Hence the main contribution of the n order integral arises from a domain of order $t\tau_c^{n-1}$. Accordingly the n-th term in the exponent is of order

$$(\alpha t)(\alpha \tau_c)^{n-1} \qquad (1.28)$$

Thus, eq. (1.26) is an expansion in power of $(\alpha \tau_c)$ each term being roughly linear in t. This is the main advantage of the cumulant expansion with respect to the expansion in eq. (1.9) that is an expansion in power of αt and its validity is therefore limited to small t.

Note. In the formal integration of the stochastic equation (1.5) a general Gaussian noise has been considered. This could be a process whose correlation function decays to zero with a finite time τ_c. If this is so the noise is not white any more, the stochastic process is not Markovian and an exact correspondence to a Fokker-Planck equation is lost.

1.1.3 Problems with the definition of Stochastic integrals in multiplicative processes.

The main problem one has to face in dealing with SDEs with multiplicative noise is that these equations (such as eq. (1.2)) have no meaning unless an interpretation of the multiplicative term is provided. To be more precise let us integrate (as done before for additive noise) the stochastic equation (1.2) within an interval Δt in which the variable $x(t)$ does not vary significantly

$$\int_t^{t+\Delta t} ds \frac{dx}{ds} = \int_t^{t+\Delta t} h(x(s), s)ds + \int_t^{t+\Delta t} g(x(s), s)F(s)ds \qquad (1.29)$$

Since $x(s)$ is almost constant within Δt one gets

$$x(t + \Delta t) - x(t) = h(x(t), t)\Delta t + g(x(t), t) \int_t^{t+\Delta t} F(s)ds \qquad (1.30)$$

For Δt small enough (but still much bigger then the time scale during which collisions occur) we can take the limit and write the stochastic equation in differential form

$$dx(t) = h(x(t), t)dt + g(x(t), t)dW \qquad (1.31)$$

where $dW = \dot{W}dt = F(t)dt$ is the increment of the Wiener process $W(t)$. Equation (1.31), with the initial condition $x(t_0) = x_0$ and $t > t_0$ is the typical stochastic Cauchy problem written in mathematical form. Its integral form (formal solution) is given by

$$x(t) = x_0 + \int_{t_0}^{t} h(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dW(s). \tag{1.32}$$

The main problem consists in giving a correct interpretation of the stochastic integral

$$\int_{t_0}^{t} g(x(s), s)dW(s) \tag{1.33}$$

and in general to any integrals performed with respect to the Wiener process. This is not, however, a straightforward task to perform, since the trajectories $t \to W(t, \xi)$ of the Wiener process are not functions of bounded variation *on any interval* and a simple Stielties integral cannot be defined (a function of bounded variation is a real-valued function whose total variation is bounded). It turns out that different interpretations can be given depending on the value of t at which $x(t)$ is calculated within the stochastic integral. Indeed, we will see in the next section, that, if one uses the beginning of the time step, i.e. $x = x(t)$, the Itô definition is obtained whereas if the middle point of dt is considered i.e. $x = x(t + dt/2)$ one ends up with the Stratonovich interpretation.

1.1.4 Itô and Stratonovich stochastic integrals

In this section we will consider Markovian processes by assuming SDEs with a multiplicative white noise. By formally integrating eq. (1.31) within a time interval Δt one obtaines the integral version of (1.31):

$$x(t + \Delta t) - x(t) = \int_{t}^{t+\Delta t} h(x(s), s)ds + \int_{t}^{t+\Delta t} g(x(s), s)dW(s) \tag{1.34}$$

where $W(s)$ is a Wiener process. The second integral is a stochastic process that depends on a path $\omega(s)$ (realization of the Wiener process) since g is a function of a stochastic process $X(t)$. As mentioned in the previous section this integral is a quite delicate task to perform because the function $t \to \omega(t)$ is not a function of bounded variation and a Riemann-Stielties is not defined. This wouldn't be a problem if the integrand were not a function of x. To see that let us compute first the following integral

$$I(\omega) = \int_{0}^{T} f(t)dW(t) \tag{1.35}$$

over the interval $[0, T]$. We have kept the dependence on ω to remind ourselves that the integral above is a random variable. The function $f(t)$ is in this case *deterministic* (it does not depend on the noise). Following the Riemann procedure we consider a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of the interval $[0, T]$ and the increments $\Delta t_j = t_{j+1} - t_j$ and $\Delta W_j = W(t_{j+1}) - W(t_j)$. One can then define the integral as the limit, for $\max_j \Delta t_j \equiv \Delta t \to 0$, of the sum

$$S_n(\omega) = \sum_{j} f(\bar{t}_j)\Delta W_j(\omega) \tag{1.36}$$

where $\bar{t}_j \in [t_j, t_{j+1}]$ is arbitrary. Clearly the sum S_n is a random variable whose value depends on the given realization $x(t)$. Since S_n is linear in ΔW_j (Gaussian process) and f is deterministic, the finite sum S_n is Gaussian and this property remains in the limit. It is then sufficient to determine the first and the second moment of S_n to get the statistics of S_n. Clearly $\mathbb{E}[S_n] = 0$ since $\mathbb{E}[\Delta W_j] = 0 \quad \forall j$ for a Wiener process. On the other hand, the probabilistic independence of the increments gives

$$\mathbb{E}[\Delta W_j \Delta W_i] = \delta_{ij} Var[\Delta W_j] = \delta_{ij}\Delta t_j. \tag{1.37}$$

Hence

$$Var[S_n] \equiv \mathbb{E}[S_n^2] = \sum_{ij} f(\bar{t}_j)f(\bar{t}_i)\mathbb{E}[\Delta W_j \Delta W_i] = \sum_j f(\bar{t}_j)^2 \Delta t_j \to \int_0^T f(t)^2 dt. \tag{1.38}$$

If $f \in L^2$ the last integral is a well defined (finite) number and it is reasonable to say that the sum (1.35) is a random variable following a normal distribution with average zero and variance $\int_0^T f(t)^2 dt$. Note that in the definition we have used the fact that W is a Gaussian process i.e. *the definition of the integral has a probabilistic meaning.*

Note. The case in which $f(t)$ is deterministic is just a more general case of the additive noise discussed before in which σ^2 is replaced by $f(t)$.

1.1.5 How to define a stochastic integral ?

The situation turns out to be quite different if the integrand itself is a stochastic process or a function of a stochastic process (i.e. a stochastic process itself). The problem in its generality can be set as follows: given a stochastic process $\{Y(t)\}_{t\geq 0}$, how we can define in a proper way the following integral

$$\int_0^t X(t)dW(t) \qquad ? \tag{1.39}$$

Let us face this problem gradually by first considering as stochastic process $Y(t)$ the Wiener process itself. The stochastic integral to be defined will be

$$I(\omega) = \int_0^t W(t)dW(t). \tag{1.40}$$

Since the Wiener process, as a function of t (Wiener paths), is *nowhere* differentiable the usual Riemann-Stieltjes definition of the integral (1.40) i.e.

$$\int_0^t W(t)dW(t) = \lim_{n\to\infty} \sum_{j=0}^{n-1} W(\bar{t}_j)\left[W(t_{j+1}) - W(t_j)\right] \tag{1.41}$$

depend on the choice of the points $\bar{t}_j$. Indeed let us consider the following three different choices

- Choosing for $\bar{t}_j$ the *left* end point of each subinterval $[t_j, t_{j+1}]$ we have the *forward Euler discretization* of the Riemann-Stieltjes integral

$$I_n^L(\omega) = \sum_{j=0}^{n-1} W(t_j;\omega)\left[W(t_{j+1};\omega) - W(t_j;\omega)\right] \tag{1.42}$$

- choosing for $\bar{t}_j$ the *right* end point of each subinterval $[t_j, t_{j+1}]$ we have the *backward Euler discretization* of the Riemann-Stieltjes integral

$$I_n^R(\omega) = \sum_{j=0}^{n-1} W(t_{j+1};\omega)\left[W(t_{j+1};\omega) - W(t_j;\omega)\right] \tag{1.43}$$

- Finally if we choose the *trapezoidal method* we have

$$I_n^T(\omega) = \sum_{j=0}^{n-1} \left[\frac{W(t_{j+1};\omega) + W(t_j;\omega)}{2}\right]\left[W(t_{j+1};\omega) - W(t_j;\omega)\right] \tag{1.44}$$

Note that we have put the paths explicitly in the integration to stress the non differentiability of the curves. Computing the expected value of $I_n^L(\omega)$, using the fact that $W(t_j)$ and $W(t_{j+1})-W(t_j)$ are independent with zero expected value, we get

$$\mathbb{E}[I_n^L(\omega)] = \sum_{j=0}^{n-1} \mathbb{E}\left[W(t_j;\omega)(W(t_{j+1};\omega) - W(t_j;\omega)\right] \tag{1.45}$$

$$= \sum_{j=0}^{n-1} \mathbb{E}\left[W(t_j;\omega)\right] \mathbb{E}\left[W(t_{j+1};\omega) - W(t_j;\omega)\right] = 0, \tag{1.46}$$

In the case of backward Euler discretization we have instead

$$\mathbb{E}[I_n^R(\omega)] = \sum_{j=0}^{n-1} \mathbb{E}\left[W(t_{j+1};\omega)(W(t_{j+1};\omega) - W(t_j;\omega)\right] \tag{1.47}$$

$$= \sum_{j=0}^{n-1} \mathbb{E}\left[W(t_j;\omega)\Delta W_j\right] + \mathbb{E}\left[\Delta W_j^2\right] =$$

$$= \sum_{j=0}^{n-1} \Delta t_i = T \neq 0, \tag{1.48}$$

If we use instead the trapezoidal rule we have

$$\mathbb{E}[I_n^T(\omega)] = \sum_{j=0}^{n-1} \mathbb{E}\left[\frac{W(t_{j+1};\omega) + W(t_j;\omega)}{2}\Delta W_j\right]$$

$$= \sum_{j=0}^{n-1} \mathbb{E}\left[W(t_j)\Delta W_j\right] + \mathbb{E}\left[(\Delta W_j)^2/2\right]$$

$$= \sum_{j=0}^{n-1} \frac{\Delta t_j}{2} = \frac{T}{2} \neq 0. \tag{1.49}$$

From this simple calculation one can immediately discover that there are several different definitions of the stochastic integral depending on the point chosen in each subinterval and each of these procedure will give rise to a different limit.

Exercise. Show that for a fixed number τ with $0 \leq \tau \leq 1$ one can choose a point $\bar{t}_j = t_j + \tau(t_{j+1} - t_j)$ such that

$$\mathbb{E}\left[\sum_{j=0}^{n-1} W(\bar{t}_j;\omega)\left[W(t_{j+1};\omega) - W(t_j;\omega)\right]\right] = \tau T. \tag{1.50}$$

The above considerations suggest that in order to give a proper definition of (1.40) one has to choose $\bar{t}_j$ in a consistent way i.e. such that

$$\bar{t}_j = t_j + \tau(t_{j+1} - t_j), \qquad \text{for some fixed} \quad \tau \quad \text{such that} \quad 0 \leq \tau \leq 1. \tag{1.51}$$

In practice there are *infinitely* many different notions of the stochastic integral (1.40) each that could a different stochastic integral. The following two are usually considered:

- $\tau = 0$ i.e. the left-end point is used. This gives the so-called *Itô integral* which has zero expectation.

- $\tau = 1/2$, i.e. the middle point is used. This give rise to the so-called *Stratonovich integral* which has expectation $T/2$.

Sometimes it is also used the definition based on the backward Euler discretization ($\tau = 1$). In this case the stochastic integral is known as the *Haggi-Klimontovich* integral. In mathematics and economics it is usually considered the Itô integral (see Appendix for a more rigorous definition of the Itô integral) whereas in physics it is often used the Stratonovich integral because it does not imply changes in the fundamental theorem of calculus.

1.1.6 Itô and Stratonovich stochastic integrals

In general, if $W(s)$ is the Wiener process starting at 0 defines on a probability space Ω and t is a fixed positive number, for a stochastic process

$$g : [0,t] \times \Omega \to \mathbb{R}, \tag{1.52}$$

we would like to make sense of the *stochastic integral* (from now on we omit the dependence on ω to simplify the notations)

$$\boxed{\int_0^t g(W(s), s) dW(s).} \tag{1.53}$$

As before we partition the interval $[0,t]$ into subintervals $0 = t_1 < \cdots < t_n$. We then have

$$I_I = \lim_{\Delta t \to 0} \sum_{j=0}^{n-1} g(W(t_j), t_j)[W(t_{j+1}) - W(t_j)], \tag{1.54}$$

for the Itô integral and

$$I_S = \lim_{\Delta t \to 0} \sum_{j=0}^{n-1} g\left(\frac{W(t_j) + W(t_{j+1})}{2}, \frac{t_j + t_{j+1}}{2}\right)[W(t_{j+1}) - W(t_j)], \tag{1.55}$$

for the Stratonovich one. Note that $\Delta t = \max(t_{j+1} - t_j)$. It is important to stress again that if g does not depend on the stochastic process (i.e. it is not a stochastic process itself) both definitions agree.

Note. It is importante to notice that Itô's definition uses the random variables $g(W(t_j))$ in the sum. These R.v. are independent on the increments $W(t_{j+1}) - W(t_j)$. This is a crucial characteristic of the Itô interpretation which turns out to be important in some modeling applications. Suppose, for example, that $W(t_j)$ is replaced by a stochastic variable $X(t_j)$ representing the number of stocks held at time t_j and that $W(t_j)$ is the price of the stock at time t. Then the earning made from time t_j to time t_{j+1} would be $X(t_j)(W(t_{j+1}) - W(t_j))$ that is what one would like to have. This is not true in the Stratonovich interpretation since the argument of g depend on the values of W at times t_i and t_{i+1}.

Now let us use the Itô and Stratonovich interpretations to evaluate the following useful average:

$$I = \left\langle \int_0^\tau W(s) dW(s) \right\rangle. \tag{1.56}$$

Itô

$$\begin{aligned} I_I &= \lim_{\Delta s \to 0} \left\langle \sum_{j=0}^{n-1} W(s_j)[W(s_{j+1}) - W(s_j)] \right\rangle \\ &= \lim_{\Delta s \to 0} \sum_{j=0}^{n-1} [\langle W(s_j) W(s_{j+1}) \rangle - \langle W(s_j) W(s_j) \rangle] \\ &= \lim_{\Delta s \to 0} \sum_{j=0}^{n-1} (\sigma^2 s_j - \sigma^2 s_j) \\ &= 0. \end{aligned} \tag{1.57}$$

Stratonovich

$$
\begin{aligned}
I_S &= \lim_{\Delta s \to 0} \left\langle \sum_{j=0}^{n-1} \left[\frac{W(s_j) + W(s_{j+1})}{2} \right] [W(s_{j+1}) - W(s_j)] \right\rangle \\
&= \lim_{\Delta s \to 0} \frac{1}{2} \sum_{j=0}^{n-1} [\langle W(s_j) W(s_{j+1}) \rangle + \langle W(s_{j+1}) W(s_{j+1}) \rangle \\
&- \langle W(s_j) W(s_j) \rangle - \langle W(s_{j+1}) W(s_j) \rangle] \\
&= \lim_{\Delta s \to 0} \frac{1}{2} \sum_{j=0}^{n-1} (\sigma^2 s_j + \sigma^2 s_{j+1} - \sigma^2 s_j - \sigma^2 s_j) \\
&= \frac{\sigma^2}{2} \sum_{j=0}^{n-1} (s_{j+1} - s_j) \\
&= \frac{\sigma^2}{2} \tau. \qquad (1.58)
\end{aligned}
$$

1.1.7 Statistical properties of the Itô's integral

If $Y(t)$ is a generic stochastic process that is Itô integrable i.e. that $\int_0^t Y(s) dW(s)$ exists. We say that $Y \in \mathcal{L}$. We have the following properties

Linearity If $Y, Z \in \mathcal{L}$ then

$$
\int_0^t (aY(s) + bZ(s))\, dW(s) = a \int_0^t Y(s) dW(s) + b \int_0^t Z(s) dW(s) \qquad (1.59)
$$

where $a, b \in \mathbb{R}$ are constants.

Additivity If $0 \geq \tau \geq t$ then

$$
\int_0^t Y(s) dW(s) = \int_0^\tau Y(s) dW(s) + \int_\tau^t Y(s) dW(s); \qquad (1.60)
$$

Zero mean

$$
\mathbb{E} \left\{ \int_0^t Y(t) dW(t) \right\} = 0 \qquad (1.61)
$$

This is because for all partions $0 = t_0 < t_1 < \cdots < t_n = t$, the Itô's integration rules gives:

$$
\begin{aligned}
\mathbb{E} \left\{ \int_0^t Y(t) dW(t) \right\} &= \mathbb{E} \left\{ \sum_{j=0}^{n-1} Y(t_j) [W(t_{j+1}) - W(t_j)] \right\} \\
&= \sum_{j=0}^{n-1} \mathbb{E} \left\{ Y(t_j) [W(t_{j+1}) - W(t_j)] \right\} \\
&= \sum_{j=0}^{n-1} \mathbb{E} \left\{ Y(t_j) \right\} \mathbb{E} \left\{ [W(t_{j+1}) - W(t_j)] \right\}, \qquad (1.62)
\end{aligned}
$$

where the last line holds because $Y(t_j)$ and $[W(t_{j+1}) - W(t_j)]$ are indipendent. Hence, since $\mathbb{E}\{\Delta W\} = 0$ we have the results.

Itô isometry

$$
\mathbb{E} \left\{ \left[\int_0^t Y(t) dW(t) \right]^2 \right\} = \mathbb{E} \left\{ \int_0^t Y^2(t) dW(t) \right\} \qquad (1.63)
$$

To see that let us compute it:

$$\begin{aligned}
\mathbb{E}\left\{\left[\int_0^t Y(t)dW(t)\right]^2\right\} &= \mathbb{E}\left\{\left[\sum_{j=0}^{n-1} Y(t_j)\left[W(t_{j+1})-W(t_j)\right]\right]^2\right\} \\
&= \sum_{j=0}^{n-1}\mathbb{E}\left\{\left[Y(t_j)\left[W(t_{j+1})-W(t_j)\right]\right]^2\right\} \\
&+ 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\mathbb{E}\left\{Y(t_j)Y(t_i)\left[W(t_{i+1})-W(t_i)\right]\left[W(t_{j+1})-W(t_j)\right]\right\} \\
&= \sum_{j=0}^{n-1}\mathbb{E}\left\{Y^2(t_j)\right\}\mathbb{E}\left\{\left[W(t_{j+1})-W(t_j)\right]^2\right\} \\
&+ 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\mathbb{E}\left\{Y(t_j)Y(t_i)\left[W(t_{i+1})-W(t_i)\right]\right\}\mathbb{E}\left\{\left[W(t_{j+1})-W(t_j)\right]\right\} \\
&= \sum_{j=0}^{n-1}\mathbb{E}\left\{Y_j^2\right\}\mathbb{E}\left\{\left[W(t_{j+1})-W(t_j)\right]^2\right\} \qquad (1.64)
\end{aligned}$$

On the other hand, by the property of the Wiener process

$$\mathbb{E}\left\{\left[W(t_{j+1})-W(t_j)\right]^2\right\} = t_{j+1}-t_j = \Delta t_j \qquad (1.65)$$

By taking the limit as in the definition of a Riemann integral we have the result.

Covariance By using the same approach than above one can show that

$$\begin{aligned}
Cov\left\{\int_0^t Y(s)dW(s), \int_0^t Z(s)dW(s)\right\} &= \mathbb{E}\left\{\left[\int_0^t Y(s)dW(s)\right]\left[\int_0^t Z(s)dW(s)\right]\right\} \\
&\int_0^t \mathbb{E}\left\{Y(s)Z(s)\right\}ds. \qquad (1.66)
\end{aligned}$$

Martingale property The integral

$$\int_0^t Y(s)dW(s) \qquad (1.67)$$

is a martingale. To see that let $\tau \leq t$ then by the second property we have first

$$\mathbb{E}\left\{\int_0^t Y(s)dW(s)\right\} = \mathbb{E}\left\{\int_0^\tau Y(s)dW(s) + \int_\tau^t Y(s)dW(s)\right\} \qquad (1.68)$$

On the other hand

$$\mathbb{E}\left\{\int_0^\tau Y(s)dW(s) + \int_\tau^t Y(s)dW(s)\right\} = \mathbb{E}\left\{\int_0^\tau Y(s)dW(s)\right\}+\mathbb{E}\left\{\int_\tau^t Y(s)dW(s)\right\} \qquad (1.69)$$

By the independence of the increments of $W(t)$ and by property (3) we have

$$\mathbb{E}\left\{\int_\tau^t Y(s)dW(s)\right\} = \mathbb{E}\left\{\int_\tau^t Y(s)dW(s)\right\} = 0 \qquad (1.70)$$

Hence

$$\mathbb{E}\left\{\int_0^t Y(s)dW(s)\right\} = \mathbb{E}\left\{\int_0^\tau Y(s)dW(s)\right\} = \int_0^\tau Y(s)dW(s) \qquad (1.71)$$

1.1.8 From Langevin to Fokker-Planck

In order to obtain the Fokker-Planck equation associated to the SDE (1.31) we have to compute the Kramers-Moyal expansion coefficients

$$D^{(m)}(x,t) = \frac{1}{m!} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle [x(t+\Delta t) - x]^m \rangle \,|_{x(t)=x} \tag{1.72}$$

Note that $x(t+\Delta t)$ is a solution of the equation (1.31) which at time t has the fixed value $x(t) = x$ (initial condition). The integral form is then

$$x(t+\Delta t) = x(t) + \int_t^{t+\Delta t} h(x(s),s)ds + \int_t^{t+\Delta t} g(x(s),s)dW(s). \tag{1.73}$$

For simplicity we neglect the explicit time dependence of the function h and g (see [**?**] for the more general case). Then eq. (1.73) becomes

$$\begin{aligned} x(t+\Delta t) - x &= \int_t^{t+\Delta t} h(x(s))ds + \int_t^{t+\Delta t} g(x(s))dW(s). && (1.74) \\ &= \int_0^{\Delta t} h(x(t+\tau))d\tau + \int_0^{\Delta t} g(x(t+\tau))dW(\tau+t) && (1.75) \end{aligned}$$

where the change of variable $\tau = s - t$ has been considered. Since at the end of the calculation one has to take the limit $\Delta t \to 0$ it is reasonable to expand the functions h and g for $x(\tau+t)$ close to $x(t) \equiv x$, namely:

$$\begin{aligned} h(x(t+\tau)) &= h(x(t)) + \frac{\partial h}{\partial x(\tau)} \,|_{x(\tau)=x(t)=x} (x(t+\tau) - x) + O\left((x(t+\tau)-x)^2\right) && (1.76) \\ g(x(t+\tau)) &= g(x(t)) + \frac{\partial g}{\partial x(\tau)} \,|_{x(\tau)=x(t)} (x(t+\tau) - x) + O\left((x(t+\tau)-x)^2\right) && (1.77) \end{aligned}$$

Let us call for simplicity

$$\frac{\partial h}{\partial x(\tau)} \,|_{x(\tau)=x(t)} \equiv h'(x); \qquad \frac{\partial g}{\partial x(\tau)} \,|_{x(\tau)=x(t)=x} \equiv g'(x). \tag{1.78}$$

The integral equation (1.75) becomes

$$\begin{aligned} x(t+\Delta t) - x &= \int_0^{\Delta t} h(x)d\tau + \int_0^{\Delta t} h'(x)\,(x(t+\tau) - x)\,d\tau \\ &+ \int_0^{\Delta t} g(x)dW(\tau) + \int_0^{\Delta t} g'(x)\,(x(t+\tau)-x)\,dW(\tau) \\ &+ O\left((x(t+\tau)-x)^2\right) && (1.79) \end{aligned}$$

Equation (1.79) can be solved iteratively i.e. the solution for $x(t+\Delta t) - x$ at order m is obtained by plugging the expression for $\xi(t+\tau) - x$ obtained at order $m-1$ in the right hand side of the equation. At order 1 we put the solution at order 0 i.e. $x(t+\tau) = x(t) \equiv x$. This gives:

$$x^{(1)}(t+\Delta t) - x = h(x)\Delta t + g(x) \int_0^{\Delta t} dW(\tau) \tag{1.80}$$

where

$$\int_0^{\Delta t} dW(\tau) = W(\Delta t) - W(0) = W(\Delta t) \tag{1.81}$$

since $W(0) = 0$ for a Wiener process. At second order we plug $x^{(1)}(t+\tau)-x = h(x)\tau+g(x)\Delta W(\tau)$ back to the left hand side obtaining:

$$\begin{aligned} x^{(2)}(t+\Delta t) - x &= h(x)\Delta t + g(x)W(\Delta t) + h'(x)h(x)\int_0^{\Delta t} \tau d\tau + h'(x)g(x)\int_0^{\Delta t} W(\tau) \\ &+ g'(x)h(x)\int_0^{\Delta t} \tau W(\tau) + g(x)g'(x)\int_0^{\Delta t} W(\tau)dW(\tau). \end{aligned} \tag{1.82}$$

Since we are going to divide by Δt and let $\Delta t \to 0$ the term proportional to $h(x)h'(x)$ can be dropped since if of order Δt^2. Moreover if we now average over the noise the term proportional to $h'(x)g(x)$ is zero since $\langle \Delta W(\tau)\rangle = 0$. The integral proportional to $g'(x)h(x)$ is a special case of the "deterministic" integral discussed before with $f(\tau) = \tau$. This gives $\left\langle \int_0^{\Delta t} \tau \Delta W(\tau)\right\rangle = 0$. Hence by averaging over the noise we finally get

$$\left\langle x^{(2)}(t+\Delta t) - x\right\rangle = h(x)\Delta t + g'(x)g(x)\left\langle \int_0^{\Delta t} W(\tau)dW(\tau)\right\rangle + O\left((x(t+\Delta t)-x)^2\right) \tag{1.83}$$

On the other hand from eqs. (1.57) and (1.58) one has

$$\left\langle \int_0^{\Delta t} W(\tau)dW(\tau)\right\rangle = 0 \quad \text{Itô} \tag{1.84}$$

and

$$\left\langle \int_0^{\Delta t} W(\tau)dW(\tau)\right\rangle = \frac{\sigma^2}{2}\Delta t \quad \text{Stratonovich} \tag{1.85}$$

Since the drift coefficient is defined as

$$D^{(1)}(x) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle x(t+\Delta t) - x\right\rangle|_{x(t)=x} \tag{1.86}$$

we finally get

$$\boxed{D_I^{(1)}(x) = h(x) \quad \text{Itô}} \tag{1.87}$$

and

$$\boxed{D_S^{(1)}(x) = h(x) + \frac{\sigma^2}{2}\frac{\partial g}{\partial x}(x)g(x) \quad \text{Stratonovich.}} \tag{1.88}$$

For the diffusion coefficient $D^{(2)}(x)$ we have to compute

$$D^{(2)}(x) = \frac{1}{2}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle (x(t+\Delta t) - x)^2\right\rangle|_{x(t)=x}. \tag{1.89}$$

Since in (1.89) the solution $x(t+\Delta t)-x$ is squared, at order Δt it is sufficient to use the first order approximation $x^{(1)}(t+\Delta t) - x$ in (1.80). In this approximation, however, there are no stochastic integrals and the two interpretation must coincide. Indeed we have

$$\begin{aligned} \left\langle (x^{(1)}(t+\Delta t) - x)^2\right\rangle|_{x(t)=x} &= \left\langle (h(x)\Delta t + g(x)W(\Delta t))^2\right\rangle & (1.90)\\ &= h^2(x)\Delta t^2 + 2g(x)h(x)\Delta t\left\langle W(\Delta t)\right\rangle & (1.91)\\ &+ g^2(x)\left\langle W(\Delta t)^2\right\rangle. & (1.92) \end{aligned}$$

By keeping just the terms up to first order in Δt and using $\langle W(s)\rangle = 0$, $\langle W(s)W(s)\rangle = \sigma^2\Delta t$, and eq. (1.89) one gets

$$\boxed{D^{(2)}(x) = \frac{\sigma^2}{2}g^2(x).} \tag{1.93}$$

Note that the expression of the diffusion coefficient coincides in the Itô and Stratonovich cases. Referring back to the general expression of the Fokker-Planck equation, i.e.

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[D^{(1)}(x)p(x,t)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[D^{(2)}(x)p(x,t)\right], \tag{1.94}$$

we see that the Itô interpretation of the Langevin equation

$$\frac{dx}{dt} = h(x) + g(x)F(t) \tag{1.95}$$

leads to the Fokker-Planck equation in the Itô form

$$\boxed{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[h(x)p(x,t)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[g(x)^2 p(x,t)\right], \quad \text{Itô}} \tag{1.96}$$

while the Stratonovich gives the more familiar (to physicists) Fokker-Planck equation

$$\boxed{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[(h(x) + \frac{\sigma^2}{2}g'(x)g(x))p(x,t)\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[g(x)^2 p(x,t)\right]} \tag{1.97}$$

or in the equivalent form

$$\boxed{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[h(x)p(x,t)\right] + \frac{\sigma^2}{2}\frac{\partial}{\partial x}\left[g(x)\frac{\partial}{\partial x}\left(g(x)p(x,t)\right)\right] \quad \text{Stratonovich.}} \tag{1.98}$$

Exercise. Show that if $h = h(x,t)$ and $g = g(x,t)$ i.e. also an explicit time dependence is included, the Fokker-Planck equation generalizes simply to

$$\boxed{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[h(x,t)p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[g(x,t)^2 p(x,t)\right] \quad \text{Itô,}} \tag{1.99}$$

and

$$\boxed{\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}\left[h(x,t)p(x,t)\right] + \frac{\partial}{\partial x}\left[g(x)\frac{\partial}{\partial x}\left(g(x,t)p(x,t)\right)\right] \quad \text{Stratonovich.}} \tag{1.100}$$

Note. By going backwards one can say that to the Fokker-Planck equation, for example in the Itô picture (eq. (1.99)), corresponds the Langevin equation

$$-\frac{dx}{dt} = D^{(1)}(x,t) + \sqrt{D^{(2)}(x,t)}F(t) \tag{1.101}$$

where $F(t)$ is a Gaussian delta-correlated noise.

Remark. If $g(x,t)$ does not depend on x we have

$$D^{(1)}(x,t) = h(x,t), \qquad D^{(2)}(x,t) = D_2(t) = g^2(t) \tag{1.102}$$

and the Itô and the Stratonovich pictures coincide. This would correspond to a stochastic differential equation with additive noise and noise amplitude that depends on time.

Note. If $D^{(4)}(x,t)$ is not zero (as it is for a Gaussian noise) the Pawula's theorem says that an infinite set of $D^{(k)}(x,t)$ is necessary in the Kramers- Moyal expansion. In terms of the Langevin equation (1.101), if $D^{(4)}(x,t) \neq 0$ no Gaussian noise is present.

Example. Suppose one can estimate the drift and diffusion coefficients of a stochastic process from some experimental data. If those coefficients were of the form

$$D^{(1)}(x,t) = Ax \tag{1.103}$$
$$D^{(2)}(x,t) = Be^{-t/2} + Cx^{,} \tag{1.104}$$

it would mean that the corresponding Langevin equation (1.101) describes, for $t >> 1$, a stochastic process with *multiplicative noise* (quadratic noise Ornstein-Uhlenbeck process [**?**]),

$$\frac{dx}{dt} = -Ax(t) + x(t)f(t) \tag{1.105}$$

while for small t the additive noise term is dominant.

Note. Suppose to consider the simple stochastic differential equation with multiplicative sum

$$\Delta X = g(X(t))\Delta W(t) \tag{1.106}$$

If we integrate it formally between $[0, t]$ and take the average we have (suppose $X(0) = 0$)

$$\mathbb{E}\{X(t)\} = \mathbb{E}\left\{\int_0^t g(X(s))dW(s)\right\} \tag{1.107}$$

On the other hand the Itô interpretation of the integral sets the above average to zero. This is the **non-anticipating** character of the Itô interpretation. In other words in this interpretation the variable $x(t)$ depends on the noise $\eta(t')$ only for $t' < t$, and so is independent on the value of the noise at t (i.e. $\eta(t)$). The Itô interpretation has a number of advantages

- The drift coefficient is noise free $D^{(1)}(x) = h(x)$
- The non-anticipating character and the martingale property is crucial for rigorous proofs.
- In Itô interpretation the conditional mean and variance of a process are calculated at time t. This conforms better with economic intuition that such quantities are calcolated by the economic agents at time t on the basis of the information available to them at that time.

On the other hand the fact that the Stratonovitch integral preserves the rules of standard calculus has often been viewed as an advantage. Rigorous proof are however made difficult by the anticipating nature of this interpretation.

1.2 Itô formula

Let $X(t)$ be a stochastic process solution of the following SDE

$$\Delta X(t) = h(X(t), t)\Delta t + g(X(t), t)\Delta W(t), \qquad t > 0 \tag{1.108}$$

and let $f(t, X(t))$ be a *twice differentiable* function on $[0, T] \times \mathbb{R}$. Then the stochastic process

$$f(t, X(t)), \tag{1.109}$$

is the solution of the following SDE

$$\begin{aligned} \Delta f(t, X(t)) &= \left[\partial_t f(X(t), t) + h(X(t), t)\partial_x f(X(t), t) + \frac{g^2(X(t), t)}{2}\partial_{xx} f(X(t), t)\right]\Delta t \\ &+ g(X(t), t)\partial_x f(X(t), t)\Delta W(t). \end{aligned} \tag{1.110}$$

To see this result intuitively let us Taylor expand f up to second order

$$df = \partial_t f dt + \partial_x f dX + \frac{1}{2}\partial_{xx} f\,(dX)^2\,. \tag{1.111}$$

On the other hand,

$$\begin{aligned} (dX)^2 &= (h\Delta t + g\Delta W)(h\Delta t + g\Delta W) \\ &= h\Delta t^2 + hg\Delta t\Delta W + hg\Delta W\Delta t + g^2\Delta W^2 \\ &\sim g^2\Delta t \end{aligned} \tag{1.112}$$

where we have used $dtdW \sim dt^{3/2}$ and $dWdW \sim dt$. If we now insert the last relation in eq. (1.111) we get the result (1.110).

ccc

Example. Let $X(t) = W(t)$ and $f(x) = \frac{x^2}{2}$. Then

$$d\left(\frac{W^2(s)}{2}\right) = W(s)dW(s) + \frac{1}{2}(dW(s))^2 = W(s)dW(s) + \frac{ds}{2} \tag{1.113}$$

Exercise. Let $X(t) = W(t)$ and $f(x) = x^4$. Verify that

$$d\left(W^4(s)\right) = 6W^2(s)ds + 4W^3(s)dW(s) \tag{1.114}$$

Note. Note that the Itô formula can be seen as a Taylor expansion with a multiplicative table TODO Indeed we have

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(h(x)dt + g(x)dW) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(h(x)dt + g(x)dW)^2 \end{aligned} \tag{1.115}$$

By using the rules in the table and keeping terms up to dt we get back the results presented above.

Using the Ito's formula one can compute Ito's integrals as follows:

Example. Compute the Ito's integral

$$\int_0^t W(s)^3 dW_s. \tag{1.116}$$

Let $f(x(t), t) = \frac{1}{4}x^4$ and $X(t) = W(t)$. Then

$$Y(t) = f(X(t), t) = \frac{1}{4}W(t)^4, \qquad \frac{\partial f}{\partial x} = x^3 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2} = 3x^2. \tag{1.117}$$

By Ito's formula we obtain

$$dY(t) = W(t)^3 dW_t + \frac{3}{2}W(t)^2(dW_t)^2 = W(t)^3 dW_t + \frac{3}{2}W(t)^2 dt. \tag{1.118}$$

Hence, its integral form is

$$\frac{1}{4}W(t)^4 = \frac{1}{4}W(0)^4 + \int_0^t W(s)^3 dW_s + \frac{3}{2}\int_0^t W(s)^2 ds, \tag{1.119}$$

which, since $W(0) = 0$, implies

$$\int_0^t W(s)^3 dW_s = \frac{1}{4}W(t)^4 - \frac{3}{2}\int_0^t W(s)^2 ds. \tag{1.120}$$

1.3 Fluctuating potential barriers: adiabatic elimination of variables and multiplicative noise

Suppose to consider a system in which all the sources of noise are included. In this case one can neglect any multiplicative noise coming from external fluctuating fields. It is still possible, however, that additive noisy terms become multiplicative because of the adiabatic elimination of some variables. We will now illustrate this situation by a simple example. Let us consider the following model

$$\frac{dx}{dt} = v(t) \tag{1.121}$$

$$\frac{dv}{dt} = -\lambda v(t) - \frac{dU(x)}{dx} + g(x)\xi + F(t) \tag{1.122}$$

$$\frac{d\xi}{dt} = -\gamma\xi(t) + \eta(t) \tag{1.123}$$

where $F(t)$ and $\eta(t)$ are two (white noise) stochastic forces with strength σ_F^2 and σ_η^2 respectively. If $U(x)$ is, for example, a double well potential, the system (1.121)-(1.123) can be considered as a generalization of the Kramers' model introduced before. In particular if

$$U(x) = U_0(x^2 - a^2)^2 \tag{1.124}$$

with $U_0 > 0$ the deterministic force is

$$-\frac{dU}{dx} = -4U_0 a^2 x + 4U_0 x^3. \tag{1.125}$$

We further assume, for simplicity, $g(x)$ to be linear in x:

$$g(x) = Gx. \tag{1.126}$$

Note that $g(x)$ couples the equation for v with the ones for ξ. The time evolution of $v(t)$ is then governed by the presence of an effective force $\bar{F}(x,t)$ i.e. such that

$$\frac{dv}{dt} = -\lambda v(t) + \bar{F}(x,t) + F(t) \tag{1.127}$$

where

$$\bar{F}(x,t) = \left(G\xi - 4U_0 a^2\right) x + 4U_0 x^3. \tag{1.128}$$

The force $\bar{F}(x,t)$ can be considered as generated by an effective potential. Since this potential is a function of the stochastic, Ornstein-Uhlenbeck ,process $\xi(t)$ it is itself a fluctuating object. For example, in some catalytic reactions, the presence of a fluctuating potential is due to the interaction catalizzatore-ambient. Let us consider for simplicity $F(t) = 0$. In this case the crossing of the potential barrier is not allowed since the effective force is zero at the maximum of the barrier. The transition from one well to the other is then represented by the migration of the particle from one well to the top of the barrier and the system of eqs (1.121)-(1.123) reduces to:

$$\frac{dx}{dt} = v(t) \tag{1.129}$$

$$\frac{dv}{dt} = -\lambda v(t) - \frac{dU(x)}{dx} + g(x)\xi \tag{1.130}$$

$$\frac{d\xi}{dt} = -\gamma\xi + \eta(t) \tag{1.131}$$

one can perform a crude adiabatic elimination of the variables v and ξ by putting $\dot{v} = \dot{\xi} = 0$. Solving eq. (1.130)-(1.131) for ξ and v and plugging back into the first equation one gets

$$\frac{dx}{dt} = -\frac{1}{\lambda}\frac{dU}{dx} + \frac{1}{\lambda\gamma}g(x)\eta(t). \tag{1.132}$$

Equation (1.132) has the form of a stochastic differential equation with multiplicative noise where $h(x,t) = -\lambda^{-1}dU/dx$ and $g(x,t) = (\lambda\gamma)^{-1}g(x)$. The corresponding Fokker-Planck equation in the Itô picture is then

$$\boxed{\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \lambda^{-1}\frac{\partial}{\partial x}\frac{dU(x)}{dx}p(x,t|x_0,t_0) + \frac{D}{\lambda^2\gamma^2}\frac{\partial^2}{\partial x^2}g^2(x)p(x,t|x_0,t_0),} \tag{1.133}$$

whereas

$$\boxed{\frac{\partial}{\partial t}p(x,t|x_0,t_0) = \lambda^{-1}\frac{\partial}{\partial x}\frac{dU(x)}{dx}p(x,t|x_0,t_0) + \frac{D}{\lambda^2\gamma^2}\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)p(x,t|x_0,t_0)} \tag{1.134}$$

in the Stratonovich picture.

1.3.1 Examples for multiplicative noise.

In the case of multiplicative noise the stationary solution depends on the interpretation given to the stochastic integral. Indeed for Itô, since $D^{(1)}(x) = h(x)$ and $D^{(2)}(x) = \frac{\sigma^2}{2} g^2(x)$ we have

$$\boxed{p_s(x) = \frac{\mathcal{N}_0}{g^2(x)} \exp\left(\frac{2}{\sigma^2} \int_0^x \frac{h(x')}{g^2(x')} dx' \right) \qquad \text{Itô}} \tag{1.135}$$

whereas for the Stratonovich interpretation $D^{(1)}(x) = h(x) + \frac{\sigma^2}{2} g(x) \partial g(x)/\partial x$, $D^{(2)}(x) = \frac{\sigma^2}{2} g^2(x)$ and we get

$$\boxed{p_s(x) = \frac{\mathcal{N}_0}{g^2(x)} \exp\left(\frac{2}{\sigma^2} \int_0^x \frac{h(x') + \frac{\sigma^2}{2} g(x') g'(x')}{g^2(x)} dx' \right) \qquad \text{Stratonovich}} \tag{1.136}$$

Let us focuss on the Itô's expression. The peaks of the distribution $p_s(x)$ are the most likely to be observed in an experiment. These peaks are solutions of the variational problem

$$\frac{dp_s(x)}{dx} \,|_{x=x_M} = 0 \,, \quad \frac{d^2 p_s(x)}{dx^2} \,|_{x=x_M} < 0 \tag{1.137}$$

with the necessary condition

$$h(x_0) - \frac{\sigma^2}{4} \frac{d}{dx} g^2(x) \,|_{x=x_M} = 0. \tag{1.138}$$

It is then apparent a drastic difference between additive and multiplicative processes. While the most probable values of an additive process coincides with the deterministic steady state value

$$\frac{dg}{dx} = 0 \to h(x_M) = 0, \tag{1.139}$$

in a multiplicative process they do depend on the strength σ^2 of the random force, i.e.

$$x_M = x_M(\sigma^2) \tag{1.140}$$

approaching the deterministic value only in the limit $\sigma^2 \to 0$. Differences between the additive process and the multiplicative one can be found also in the notion of stability of a stochastic process. An additive process of the kind

$$\frac{dx}{dt} = h(x) + F(t) \tag{1.141}$$

has a stable stationary solution and all its moments $\langle x^n \rangle$ exist up to nth order if the associated deterministic problem has a globally stable state with respect to arbitrarily large fluctuations. This is true when $h(x)$ satisfies the constraint

$$-\frac{2}{\sigma^2} \int^x h(x') dx' > (n+1) \ln x + A \tag{1.142}$$

for x big enough. For a multiplicative process, however, the proof of the stability of the deterministic problem is not sufficient to guarantee stability when fluctuations are present. To see that let us consider a very simple multiplicative process

$$\frac{dx}{dt} = -\gamma x(t) + x(t) F(t) \tag{1.143}$$

where $F(t)$ is a Gaussian process. In this case it is easy to show that the moments are given by

$$\langle x^n(t) \rangle = \langle x^n(t_0) \rangle \exp\left[-nt(\gamma - (n\sigma^2)/2) \right]. \tag{1.144}$$

Moreover, since $D^{(1)}(x) = -\gamma x$ and $D^{(2)}(x) = \frac{\sigma^2}{2} x^2$, the stationary solution is given by

$$\begin{aligned} p_s(x) &= \frac{\mathcal{N}_0}{x^2} \exp\left(-\frac{2\gamma}{\sigma^2} \int_{x_0}^{x} \frac{1}{x'} \right) \\ &= \frac{\mathcal{N}_0}{x^2} \exp\left(-\frac{2\gamma}{\sigma^2} [\ln x - \ln x_0] \right) \\ &= \frac{\mathcal{N}_0}{x^2} \left(x^{-2\gamma/\sigma^2} x_0^{2\gamma/\sigma^2} \right) \\ &= \mathcal{N} x^{-2-2\gamma/\sigma^2} \end{aligned} \tag{1.145}$$

Exercises

1. Use Ito's formula to show that

$$\int_0^t W(s)dW_s = \frac{1}{2}W(t)^2 - \frac{1}{2}t. \tag{1.146}$$

2. If $n \in \mathbb{N}$, find a recursive formula for the integral

$$\int_0^t W(s)^n dW_s \tag{1.147}$$

3. The size of a certain population is a stochastic process $P(t)$ satisfying the SDE

$$dP(t) = AP(t)dt + BP(t)dW_t. \tag{1.148}$$

If the initial population is P_0, find the solution for $P(t)$. Compute also the expected value $\mathbb{E}[P(t)]$.

4. Use Ito's formula to compute the differential of the stochastic process $Y(t) = W(t)^3$.

5. Use Ito's formula with $f(x(t), t) = tx$ to prove the "integration by parts" formula

$$\int_0^t s dW_s = tW(t) - \int_0^t W(s)ds. \tag{1.149}$$

5. Use Ito's formula with $f(x(t), t) = t^2 x$ to prove the "integration by parts" formula

$$\int_0^t s^2 dW_s = t^2 W(t) - \int_0^t 2sW(s)ds. \tag{1.150}$$

6. Use Ito's formula to compute the differential of the stochastic process $Y(t) = 7 + 8t^2 + Ae^{W(t)}$.