Theoretical Physics: Group Theory

1 General Definitions

1. Identify the symmetries of an isoscele triangle and of an equilateral triangle.

2. The Dihedral Group D_n, for $n \geq 3$ is given by the set of transformations that leave the n–sided regular polygon invariant. The group is generated by the rotation R of angle $2\pi/n$ and the reflection r with respect to the median. Clearly $R^n = \mathbb{1}$ and $r^2 = \mathbb{1}$.

   - Verify that $r \cdot R \cdot r = R^{-1}$;
   - Show that D_n has $2n$ elements;
   - Verify explicitly the above statements for D_4 (Box) and D_5 (Pentagon);

3. Prove that the N–roots of 1 form a group, called Z_N.

   - Verify that Z_N has order N.
   - Show that $Z_2 \otimes Z_2 \neq Z_4$ and $Z_2 \otimes Z_4 \neq Z_8$;
   - Verify that $Z_2 \subset Z_4$ but $Z_2 \not\subseteq Z_5$;
   - Verify that the set of elements $\{1, i, -1\}$ is not a group;

4. Let S_n be the group of permutation of N objects and A_n the group of even permutations:

   - Which is the order of S_n and A_n ?
   - Show that for $m < n$ one has that $S_m \subset S_n$;
   - Verify that $A_n \subset S_n$;
   - Show that S_4 is not simple;
   - Show that A_4 is an invariant subgroup of S_4;

2 Groups Representations

1. Consider the group Z_2.

- Write the trivial, the 1-dimensional and the 2-dimensional representations;
- Find explicitly a 2-dimensional representation for Z_2;
- Show that the 2-dimensional representation is completely reducible;

2. Consider the group Z_3.

   - Write the trivial, the 1-dimensional and the 3-dimensional representations;
   - Show that the following matrices form a representation of Z_3:

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad D(\omega) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad D(\omega^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   - Show that this representation is completely reducible;

3. Write a generic element of $SO(2)$ in the 2-dimensional representation.

4. Write a generic element of $SU(2)$ in the 2-dimensional representation.

5. Write a generic element of $SO(3)$ in the 3-dimensional representation, and show that $SO(2) \subset SO(3)$.

6. Suppose that D_1 and D_2 are equivalent, irreducible representations of a finite group G, such that

$$D_2(g) = S D_1(g) S^{-1} \qquad \forall g \in G$$

   What you can say about and operator A that satisfies

$$A\, D_1(g) = D_2(g) A \qquad \forall g \in G\,?$$

3 Lie Groups and Algebra

1. Show that $GL(n, \mathbb{R})$ is a Lie group of dimension n^2.

2. Show that $SU(n)$ is a Lie group of dimension $n^2 - 1$.

3. Show that $U(n)$ is not isomorphic to $U(1) \times SU(n)$. In particular show that the identity of $U(n)$ can be obtained in n different ways as product of $U(1)$ and $SU(n)$ elements. Show instead that the associated Lie Algebras are isomorphic.

4. Show that $Sp(2n, \mathbb{R})$ is a group. In particular show that if $M \in Sp(2n, \mathbb{R})$ also $M^{-1} \in Sp(2n, \mathbb{R})$. Moreover show also that also $M^T \in Sp(2n, \mathbb{R})$.

5. Show that the Poisson bracket of a QM system with n d.o.f are invariant under a symplectic transformation $Sp(2n, \mathbb{R})$.

6. Show that the dimension of $Sp(2n, \mathbb{R})$ is $n(2n + 1)$.

7. Show that $Sp(2) \simeq SL(2, \mathbb{R})$. Describe the associated Algebras.

8. Verify that a generic element of $SO(1, 1)$ can be written as

$$M = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \quad .$$

9. Calculate to the third order (one more to what explicitly done in class) the product:

$$e^{tX_t} e^{sX_s}$$

being X_t and X_s two generic element of the Algebra (not necessarily commuting).

10. Find explicitly the conditions satisfied by the generators of the Lie Algebras of $U(n)$, $SU(n)$, $SO(n)$ and $SO(p, q)$.

11. Let σ_i be the Pauli σ–matrices, then compute the following exponents

$$M_1 = e^{i\alpha_1\sigma_1} \quad , \quad M_3 = e^{i\alpha_3\sigma_3} \quad , \quad M = e^{i\vec{\alpha}\cdot\vec{\sigma}}$$

12. Write explicitly the generators of $SO(2)$ and $SO(3)$;

13. Show that $SO(3)$ is locally isomorphic to $SU(2)$, i.e. show that the corresponding Algebras are isomorphic: $so(3) \simeq su(2)$;

14. Write the Adjoint representation for the $su(2)$ Algebra.

15. Three dimensional irreducible representation of $SU(2)$.

    - using the rules given in class, derive the $SU(2)$ generators in the three dimensional representation;
    - Verify explicitly the commutation relations between them;
    - Determine the Cartan–Killing form g^{ij} and calculate the second order $SU(2)$ Casimir;

16. Construction of the $so(n)$ Algebra. We can proceed as follows

   - Set the m-th row and n-th column element to be 1. Then by antisymmetry the n-th row and m-th column element is set to be -1. Coll $\mathcal{J}_{(mn)}$ the corresponding generator. Then call $J_{(mn)} = i\mathcal{J}_{(mn)}$ the corresponding Hermitian generator;
   - Show that
$$\left(J_{(mn)}\right)^{ij} = -i\left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right)$$
   - Convince yourself that there are only $n(n-1)/2$ independent matrices $J_{(mn)}$;
   - By looking to $n = 3$ of $n = 4$ case verify that the following commutation relations hold:
$$\left[J_{(mn)}, J_{(pq)}\right] = i\left(\delta_{mp}J_{(nq)} + \delta_{nq}J_{(mp)} - \delta_{np}J_{(mq)} - \delta_{mq}J_{(np)}\right)$$

17. Show that $SO(4)$ is locally isomorphic to $SU(2)\otimes SU(2)$, i.e. show that the corresponding Algebras are isomorphic: $so(4) \simeq su(2) \oplus su(2)$;

18. Compute the exponent of the following matrices
$$M = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix} \quad , \quad N = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix}$$

19. Find the generators of $SU(3)$ (they are called Gell–Mann matrices).

   - Mimicking the $SU(2)$ case try to write a set of 8 Hermitian and traceless 3×3 matrices;
   - Realize that there are 3 different sub–algebras (not all three independent why ?);
   - Calculate explicitly the constant structure for $SU(3)$;
   - Show that the following operator is a Casimir for $SU(3)$
$$C_1 = \sum_{i=1}^{8} \lambda_i^2 = \frac{16}{3}\mathbb{1}$$

20. Build the Adjoint representation of $su(3)$.

4 Poincarè Group and Algebra

1. The Poincarè group is defined as $\mathcal{P} = O(1,3) \rtimes \mathbb{R}^4$
   - Verify that $\mathcal{P}$ is a group;
   - Verify that $\mathbb{R}^4$ is a normal subgroup of $\mathcal{P}$;
   - Verify that $O(1,3)$ is not a normal subgroup of $\mathcal{P}$;
   - Argument why $\mathcal{P}$ is not a compact or connected group;
2. Show that translations are an Abelian subgroup of the Poincarè group, while rotations or rotations and translations do not commute.
3. Verify that the generators of the Lorentz algebra in the defining representation satisfy the following commutation relations:

$$[\mathcal{J}_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho}\mathcal{J}_{\nu\sigma} + \eta_{\nu\sigma}\mathcal{J}_{\mu\rho} - \eta_{\nu\rho}\mathcal{J}_{\mu\sigma} - \delta_{\mu\sigma}\mathcal{J}_{\nu\rho}\right)$$

   and comment the similarity with the commutation relation for $SO(4)$.
4. Derive the commutator relations between the Lorentz algebra generators in a generic representation.
5. Derive the commutator relations in terms of generators of boosts and 3-dimensional rotations.
6. Verify that the generators of Lorentz boots and rotations, in the defining representations are:

$$K_1 = -i\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad K_2 = -i\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}, \quad K_3 = -i\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}$$

$$J_1 = i\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}, \quad J_2 = i\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}, \quad J_3 = i\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

7. Derive the commutator relations between the Poincarè algebra generators in a generic representation.