Fisica Teorica B: Exercise Set N.1

1 Schrödinger Non-Relativistic Wave Equation

1. Derive the continuity equation for the Schrödinger wave equation. Show moreover that the quantity

$$P(t) \equiv \int d^3x \rho(\vec{x},t) \qquad , \qquad \rho(\vec{x},t) = \psi^*(\vec{x},t)\psi(\vec{x},t)$$

is conserved, i.e. its time derivative vanishes. Prove it assuming an hamiltonian of the form $H = P^2/2M + V(X)$. Can you prove it for a generic hermitian Hamiltonian H?

2 Klein-Gordon Relativistic Wave Equation

1. Derive the continuity equation for the Klein-Gordon wave equation. Show moreover that the quantity

$$Q(t) \equiv \int d^3x\, \rho(\vec{x},t) \qquad , \qquad \rho(\vec{x},t) = \frac{i}{2}\Big(\psi^*(\vec{x},t)\,\partial_0\psi(\vec{x},t) - \partial_0\psi(\vec{x},t)^*\,\psi(\vec{x},t)\Big)$$

is conserved, i.e. its time derivative vanishes;

2. Show that the density ρ defined in point 1 is positive (negative) when calculated on a monocromatic solution of positive (negative) energy;

3. Given the general solution of the free real KG equation:

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega_k}}\left[a(k)\,e^{-ik\cdot x} + a^*(k)\,e^{ik\cdot x}\right]_{k_0=\omega_k} \qquad , \qquad \omega_k = (|\vec{k}|^2 + M^2)^{1/2}$$

derive the expressions for $a(k)$ and $a^*(k)$ in terms of $\phi(x)$ and $\partial_0\phi(x)$;

4. Given the general solution of the free complex KG equation:

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega_k}}\left[a(k)\,e^{-ik\cdot x} + b^*(k)\,e^{ik\cdot x}\right]_{k_0=\omega_k} \qquad , \qquad \omega_k = (|\vec{k}|^2 + M^2)^{1/2}$$

derive the expressions for $a(k), b(k)$ and $a^*(k), b^*(k)$ in terms of $\phi(x)$ and $\partial_0\phi(x)$ and their conjugates;

5. Assuming the minimal coupling ansatz, $D_\mu = \partial_\mu + iqA_\mu$ write explicitly the Klein-Gordon equation coupled to an external E.M field. Show that the monocromatic positive and negative energy solutions can be identified with systems of opposite charge;

6. Derive the continuity equation for the Klein-Gordon equation minimally coupled with an external E.M field;

7. Describe the scattering of a monocromatic (positive energy) wave from a potential step of high V. Show that for $E < V - M$ one obtains an unphysical behavior (i.e. $\mathcal{R} > 1$ and $\mathcal{T} < 0$). Using the expressions for the conserved density and (3-)current try to explain this behavior in terms of particle-antiparticle creations;

8. Derive the non-relativistic limit of the Klein-Gordon equation and show that this coincides with the Schrodinger equations coupled with an external E.M. field;