Fisica Teorica B: Exercise Set N.2

3 Dirac Relativistic Equation and γ matrices

1. Show that imposing hermiticity and consistency with the Klein-Gordon equation one obtains the following constraints on the α_i and β matrices:

$$\alpha_i^\dagger = \alpha_i \quad , \quad \beta^\dagger = \beta$$
$$\{\alpha_i, \alpha_j\} = 2\delta_{ij} \quad , \quad \{\alpha_i, \beta\} = 0 \quad , \quad \beta^2 = \mathbb{1}$$

2. From the above properties, show that α_i, β are traceless and they must be $N \times N$ matrices with N any even number;

3. Show that it is not possible to find 4 independent anticommuting matrices α_i, β for $N = 2$;

4. Define the Dirac matrices γ as $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$. Then

   (a) Derive the properties of the γ^μ matrices starting from those of α_i, β. In particular show that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \qquad \text{and} \qquad \gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$$

   (b) Being C a nonsingular $N \times N$ matrix, show that it is always possible to define a new set of gamma matrices $\widetilde{\gamma}^\mu = C^{-1}\gamma^\mu C$ that satisfies the conditions of point a);

   (c) The Pauli representation is defined by the following set of γ matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad , \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

   while the Weyl representation is defined by:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad , \quad \gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

   Show that it is possible to connect Pauli and Weyl γ matrices representations by mean of a unitary transformation $C^\dagger\gamma_P C = \gamma_W$. Write C explicitly;

5. Spinorial representation:

   (a) By using the explicit expression of the Lorentz transformation matrix in spinorial representation $S(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right]$, with $\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$, show that γ^μ transforms under a Lorentz transformation as $S^{-1}(\Lambda)\,\gamma^\mu\,S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$. You may use the infinitesimal form of the Lorentz transformations and prove it at first order;

   (b) Show that γ_5 under a Lorentz transformation transforms as $S^{-1}(\Lambda)\,\gamma_5\,S(\Lambda) = \det\Lambda\,\gamma_5$. Use the following expression for γ^5:

$$\gamma^5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

   (c) Show that $\Sigma^{\mu\nu}$ satisfies the $so(1,3)$ commutation rules;

   (d) Find the explicit expressions for the spin and boost matrices in the spinorial representation (you may use either the Pauli or Weyl representation for the Dirac matrices):

$$\Sigma_i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk} \quad , \quad K^i = \Sigma^{0i}$$

   (e) Prove that Σ_i satisfy the following spin relations:

$$[\Sigma_i, \Sigma_j] = i\epsilon_{ijk}\Sigma_k \quad , \quad |\vec{\Sigma}|^2 = -\frac{3}{4}\mathbb{1}$$

   (f) Verify that the spin projection along an axis is not a good quantum number by calculating the commutator with the Dirac wave equation Hamiltonian $[H, \Sigma_3]$. Verify instead that the helicity is a good quantum number, i.e. $[H, \vec{\Sigma}\cdot\vec{p}] = 0$;

   (g) From the general expression of the Pauli-Lubanski pseudo-vector W^μ, show that

$$\frac{W^\mu n^{(p)}_\mu}{M} = -\frac{1}{2M}\gamma^5 \not{n}\not{p} \quad , \quad n^{(p)} = \left(\frac{|\vec{p}|}{M}, \frac{\omega_p}{M}\frac{\vec{p}}{|\vec{p}|}\right)$$

6. Dirac wave equation:

   (a) Write the Dirac equation for the Dirac conjugate spinor $\bar{\psi}(x)$;

   (b) Show that if the Dirac spinor transforms as $\psi'(x') = S(\Lambda)\psi(x)$ then the Dirac conjugate $\bar{\psi}$ transforms as $\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}(\Lambda)$. Prove that $S^{-1}(\Lambda) = \gamma_0 S^\dagger(\Lambda)\gamma_0$;

   (c) Derive the continuity equation associated to the Dirac equation and the associated conserved charge;

(d) Derive the equation of motion for the chiral spinors $\psi_{L,R}(x)$:

$$\psi_L(x) = \left(\frac{1-\gamma_5}{2}\right)\psi(x) \quad , \quad \psi_R(x) = \left(\frac{1+\gamma_5}{2}\right)\psi(x)$$

and show that they decouple in the $M = 0$ limit;

7. Spinors in momentum space

(a) Find the general form of the spinors $u_r(k), v_r(k)$ using the Weyl representation for the Dirac matrices and compare it with the results obtained in class in the Pauli representation;

(b) Prove the following identities:

$$u_\alpha^\dagger(k)u_\beta(k) = 2\,\omega_k\delta_{\alpha\beta} \quad , \quad v_\alpha^\dagger(k)v_\beta(k) = 2\,\omega_k\delta_{\alpha\beta} \quad , \quad u_\alpha^\dagger(k)v_\beta(-k) = v_\alpha^\dagger(-k)u_\beta(k) = 0$$

(c) Knowing the general expressions of the spinors $u_r(k), v_r(k)$ show the relations:

$$\Lambda_+(k) = \frac{1}{2M}\sum_r u_r(k)\bar{u}_r(k) \quad , \quad \Lambda_-(k) = -\frac{1}{2M}\sum_r v_r(k)\bar{v}_r(k)$$

(d) Knowing the expression of the general solution of the Dirac equation (and its conjugate):

$$\psi(x) = \frac{1}{(2\pi)^3}\int\frac{d^3k}{\sqrt{2\omega_k}}\sum_r\left[c_r(k)u_r(k)\,e^{-ik\cdot x} + d_r^*(k)v_r(k)\,e^{ik\cdot x}\right]_{k_0=\omega_k}$$

derive the expressions for $c_r^{(*)}(k), d_r^{(*)}(k)$ in terms of $\psi(x), \bar{\psi}(x)$;

(e) Knowing that the spinors $u_r(M), v_r(M)$ in the mass rest frame are eigenstates of Σ_3, show that in the frame where $\vec{k} = (0,0,k)$ the spinors $u_r(k), v_r(k)$ are eigenstates of the helicity operator $\vec{\Sigma}\cdot\vec{k}/|k|$:

$$2\frac{\vec{\Sigma}\cdot\vec{k}}{|k|}u_r(k) = (-1)^{r+1}u_r(k) \quad , \quad 2\frac{\vec{\Sigma}\cdot\vec{k}}{|k|}v_r(k) = (-1)^{r+1}v_r(k)$$

(f) Determine the Dirac spinor solutions in the mass rest frame with the spin oriented along a generic direction $\vec{n}$;

8. Dirac bilinears:

(a) By knowing the transformation properties of $\psi(x), \bar{\psi}(x), \gamma_\mu$ and γ_5 under a Lorentz (and spinorial) transformation, determine the Lorentz transformation properties of the following bilinears:

$$\bar{\psi}(x)\psi(x) \quad , \quad \bar{\psi}(x)\gamma_5\psi(x) \quad , \quad \bar{\psi}(x)\gamma_\mu\psi(x) \quad , \quad i\bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) \quad , \quad \bar{\psi}(x)\sigma_{\mu\nu}\psi(x)$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

(b) Without using any explicit Dirac matrices representation show that:

$$\bar{u}(p)\sigma^{\mu\nu}(p+k)_\nu u(k) = i\,(p-k)^\mu\,\bar{u}(p)\,u(k)$$

(c) Without using any explicit Dirac matrices representation show the Gordon identities:

$$\bar{u}(p)\gamma^\mu u(k) = +\frac{1}{2M}\bar{u}(p)\left[(p+k)^\mu + i\sigma^{\mu\nu}(p-k)_\nu\right]u(k)$$
$$\bar{v}(p)\gamma^\mu v(k) = -\frac{1}{2M}\bar{v}(p)\left[(p+k)^\mu + i\sigma^{\mu\nu}(p-k)_\nu\right]v(k)$$

(d) Consider the current $J_\mu(p_1,p_2) = \bar{u}(p_2)\not{p}_1\gamma_\mu\not{p}_2 u(p_1)$. Show that J_μ can be written as:

$$J_\mu = \bar{u}(p_2)\left[F_1(q^2,M)\,\gamma_\mu + F_2(q^2,M)\,\sigma_{\mu\nu}q^\nu\right]u(p_1)$$

with $q^\mu = p_2^\mu - p_1^\mu$. Determine the functions $F_1(q^2,M), F_2(q^2,M)$;

(e) Consider the current $J_\mu(p_1,p_2) = \bar{u}(p_2)p^\rho q^\nu \sigma_{\mu\rho}\gamma_\nu u(p_1)$, with $p^\mu = p_2^\mu + p_1^\mu$ and $q^\mu = p_2^\mu - p_1^\mu$. Show that J_μ can be written as:

$$J_\mu = \bar{u}(p_2)\left[F_1(q^2,M)\,\gamma_\mu + F_2(q^2,M)\,q_\mu + F_3(q^2,M)\,\sigma_{\mu\nu}q^\nu\right]u(p_1)$$

and determine the functions $F_1(q^2,M), F_2(q^2,M), F_3(q^2,M)$;