Fisica Teorica B: Exercises N.3

4 Least Action Principle and Noether Theorem

1. Least Action Principle (finite d.o.f):

   (a) The Lagrangian $L(q,\dot{q})$ is not uniquely defined. Show that $L'(q,\dot{q}) = L(q,\dot{q}) + \frac{\partial}{\partial_t} f(q)$ (with f function solely of q) and $L(q,\dot{q})$ are physically equivalent, i.e. provide the same equations of motion;

   (b) Show that Hamilton equations can be derived from the Least Action Principle;

   (c) Verify the following Poisson bracket relations:

$$\dot{q}(t) = \{q(t), H\} \quad , \quad \dot{p}(t) = \{p(t), H\}$$
$$\{q_i(t), p_j(t)\} = \delta_{ij} \quad , \quad \{q_i(t), q_j(t)\} = \{p_i(t), p_j(t)\} = 0$$

2. Least Action Principle (field):

   (a) The Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$ is not uniquely defined. Show that $\mathcal{L}(\phi, \partial_\mu\phi)$ and $\mathcal{L}'(\phi, \partial_\mu\phi) = \mathcal{L}(\phi, \partial_\mu\phi) + \partial_\mu\mathcal{K}^\mu(\phi)$ (with $\mathcal{K}^\mu$ function solely of the field ϕ and not of derivatives) are physically equivalent, i.e. provide the same equations of motion;

   (b) Prove the Hamilton equations in all the following forms:

$$\dot{\phi}(\vec{x},t) = \frac{\delta H}{\delta\pi(\vec{x},t)} \quad , \quad \dot{\pi}(\vec{x},t) = -\frac{\delta H}{\delta\phi(\vec{x},t)}$$
$$\dot{\phi}(\vec{x},t) = \{\phi(\vec{x},t), H\} \quad , \quad \dot{\pi}(\vec{x},t) = \{\pi(\vec{x},t), H\}$$

   (c) Prove the following Poissons (Equal Time) parenthesis:

$$\begin{aligned}
\{\phi(x), \pi(y)\}_{E.T.} \equiv \{\phi(\vec{x},t), \pi(\vec{y},t)\} &= \delta^3(x-y) \\
\{\phi(x), \phi(y)\}_{E.T.} \equiv \{\phi(\vec{x},t), \phi(\vec{y},t)\} &= 0 \\
\{\pi(x), \pi(y)\}_{E.T.} \equiv \{\pi(\vec{x},t), \pi(\vec{y},t)\} &= 0
\end{aligned}$$

3. Noether Theorem:

   (a) Re–derive the general expressions of the Noether currents and conserved charges (for internal or spacetime symmetries);

(b) Show that the Noether charges are the generators of the infinitesimal canonical symmetry transformations, i.e. show the following Poisson parenthesis relations:

$$\begin{aligned}\delta_0\phi(\vec{x},t) &= -\{\phi(\vec{x},t), \epsilon^\mu P_\mu\} = -\epsilon^\mu \partial_\mu \phi(\vec{x},t) \\ \delta_0\phi(\vec{x},t) &= -\{\phi(\vec{x},t), \omega^{\mu\nu} J_{\mu\nu}\} = \omega^{\mu\nu}\left[(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Omega_{\mu\nu}\right]\phi(\vec{x},t)\end{aligned}$$

(c) Verify that P_μ are constant of motion by explicitly calculating $\{P_\mu, H\} = 0$;

4. Canonical Energy-Momentum tensor.
The Canonical Energy-Momentum tensor $\tilde{T}_{\mu\nu}$ is not automatically symmetric.

(a) Show that it is always possible to define a tensor

$$\Theta^{\mu\nu} = \tilde{T}^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$$

with $A^{\rho\mu\nu} = -A^{\mu\rho\nu}$ that still satisfies the continuity equation $\partial_\mu\Theta^{\mu\nu} = 0$;

(b) Show that it is always possible to define a tensor

$$M^\mu_{\rho\sigma} = J^\mu_{\rho\sigma} - \partial^\lambda\left(x_\rho A_{\sigma\mu\lambda} - x_\sigma A_{\rho\mu\lambda}\right)$$

that still satisfies the continuity equation $\partial_\mu M^\mu_{\rho\sigma} = 0$;

(c) By remembering that $J^\mu_{\rho\sigma} = (x_\rho\tilde{T}^\mu_\sigma - x_\sigma\tilde{T}^\mu_\rho) + \Omega^\mu_{\rho\sigma}$, show that it is always possible to find a way to write $M^\mu_{\rho\sigma}$ in the simpler form:

$$M^\mu_{\rho\sigma} = \left(x_\rho\Theta^\mu_\sigma - x_\sigma\Theta^\mu_\rho\right)$$

i.e. find the expression of $A_{\mu\rho\sigma}$ in terms of $\Omega^\mu_{\rho\sigma}$ that make it happens;

(d) Show that assuming the previous form of $M^\mu_{\rho\sigma}$ one can prove that the tensor $\Theta^{\mu\nu}$ is symmetric.