Fisica Teorica B: Exercise Set N.4

5 Relativistic Free Scalar Field Theory

1. Real scalar field.

   (a) Given the Lagrangian density of the free real scalar field theory, obtain the Hamiltonian density in terms of the independent "variables" (π, ϕ);

   (b) Using the explicit expression for the Hamiltonian verify the Hamilton equations in terms of the Poisson parenthesis:

   $$\dot{\phi}(\vec{x}, t) = \{\phi(\vec{x}, t), H\} \qquad , \qquad \dot{\pi}(\vec{x}, t) = \{\pi(\vec{x}, t), H\}$$

   (c) Write the explicit expression of the conserved currents associated to the Poincaré invariance and show that they satisfy the continuity equations: $\partial_\mu J^\mu_{(a)} = 0$;

   (d) Write the explicit expression of the conserved charges associated to the Poincaré invariance;

   (e) Given the general solution of the free real KG equation:

   $$\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2\omega_k}} \left[a(k)\, e^{-ik\cdot x} + a^*(k)\, e^{ik\cdot x}\right] \qquad , \qquad k = (\omega_k, \vec{k})$$

   calculate the expressions for conserved 4-momentum P^μ in terms of $a(k), a^*(k)$;

2. Complex scalar field.

   (a) Given the Lagrangian density of the free complex scalar field theory, obtain the Hamiltonian density in terms of the independent "variables" (π, ϕ) and (π^*, ϕ^*);

   (b) Using the explicit expression for the Hamiltonian verify the Hamilton equations in terms of the Poisson parenthesis:

   $$\dot{\phi}(\vec{x}, t) = \{\phi(\vec{x}, t), H\} \qquad , \qquad \dot{\pi}(\vec{x}, t) = \{\pi(\vec{x}, t), H\}$$
   $$\dot{\phi}(\vec{x}, t)^* = \{\phi(\vec{x}, t)^*, H\} \qquad , \qquad \dot{\pi}(\vec{x}, t)^* = \{\pi(\vec{x}, t)^*, H\}$$

   (c) Write the explicit expression of the conserved currents associated to the Poincaré invariance and show that they satisfy the continuity equations: $\partial_\mu J^\mu_{(a)} = 0$;

(d) Write the explicit expression of the conserved charges associated to the Poincaré invariance;

(e) Given the general solution of the free complex KG equation:

$$\phi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{2\omega_k}} \left[a(k)\, e^{-ik\cdot x} + b^*(k)\, e^{ik\cdot x}\right] \qquad , \qquad k = (\omega_k, \vec{k})$$

calculate the expressions for the conserved 4-momentum P^μ and the conserved $U(1)$ charge $Q_{U(1)}$ in terms of $a(k), b(k), a^*(k), b^*(k)$;

3. Equivalence between a complex scalar field and two real fields of same mass m.
Consider the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{1}{2}m^2\phi_2^2 - \frac{1}{16}\lambda(\phi_1^2+\phi_2^2)^2$$

for the case of two real scalar fields, ϕ_1 and ϕ_2.

(a) Define a two dimensional (real) vector field $\Phi = (\phi_1, \phi_2)^T$ and write the Lagrangian density in terms of Φ;

(b) Find the internal symmetry and the associated conserved currents;

(c) Explain why this theory is equivalent to the one with a single complex scalar field. In particular write the corresponding complex scalar Lagrangian density and derive the corresponding conserved charges;

4. The previous problem can be generalized to higher dimensional internal groups. As an example consider the case of a two dimensional complex scalar field $\Phi = (\phi_1, \phi_2)^T$ with $\phi_{1,2}$ complex scalar fields;

(a) Write the free Lagrangian density;

(b) Find the internal symmetry group, the associated conserved currents and show that the four conserved charges can be written as

$$Q_{(\mu)} = iq \int d^3x \left[\Phi^\dagger \sigma_\mu \Pi^\dagger - \Pi\, \sigma_\mu \Phi\right]$$

with $\sigma_\mu \equiv (1, \vec{\sigma})$ and $\Pi = \partial_0\Phi^\dagger, \Pi^\dagger = \partial_0\Phi$ the conjugate momenta;

(c) Establish a relation with a theory with only real scalar fields (i.e. how many? which internal symmetry? ...);

5. Consider the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

   (a) Calculate the Noether current for the dilatation transformation (with α a real constant):

$$\delta x^\mu = \alpha x^\mu \qquad , \qquad \delta\phi = -\alpha\phi$$

   (b) Show that the dilatation current is conserved only if $m = 0$;

6. Quantization of a real scalar field.

   (a) Verify that the evolution of ϕ and π satisfies the relations:

$$\dot{\phi}(\vec{x}, t) = -i\,[\phi(\vec{x}, t), H] \qquad , \qquad \dot{\pi}(\vec{x}, t) = -i\,[\pi(\vec{x}, t), H]$$

   (b) Derive the expressions for $a(k), a^\dagger(k)$ in terms of ϕ, π;

   (c) Verify that imposing the following commutation relations:

$$\left[a(k), a^\dagger(p)\right] = \delta^3(\vec{k} - \vec{p}) \qquad , \qquad [a(k), a(p)] = \left[a^\dagger(k), a^\dagger(p)\right] = 0$$

   one obtains the canonical commutation relations for the operators ϕ, π;

   (d) Verify that the operator P^μ is conserved;

   (e) Prove that $2\omega_k\delta^3(\vec{k} - \vec{p})$ is invariant under Lorentz transformations;

7. Quantization of a complex scalar field.

   (a) Derive the expressions for $a(k), a^\dagger(k), b(k), b^\dagger(k)$ in terms of $\phi^{(\dagger)}, \pi^{(\dagger)}$;

   (b) Derive the expression of the conserved charge $Q_{U(1)}$ in terms of $a(k), b(k)$ operators;

8. Covariant Commutators.

   (a) Show that for real and complex scalar field one has, respectively:

$$[\phi(x), \phi(y)] = D(x - y) \qquad , \qquad \left[\phi(x), \phi^\dagger(y)\right] = D(x - y)$$

   (b) Show explicitly that $D(x - y)$ is invariant under (proper) Lorentz transformations and that it vanishes on a space-like interval (i.e. for example $D(0, \vec{x}) = 0$);

(c) For real and complex scalar fields derive the expression for the following covariant commutator $[\phi(x), \pi(y)]$;

(d) Verify the following properties of $D(x-y)$:

$$
\begin{aligned}
a) &\qquad D(-x) = -D(x) \\
b) &\qquad (\Box + m^2)D(x) = 0 \\
c) &\qquad \partial_0 D(x)|_{x_0=0} = -\delta^3(x) \\
d) &\qquad \partial_i D(x)|_{x_0=0} = 0
\end{aligned}
$$

(e) Given the following observable (for a complex scalar field) $\mathcal{O}(x) = \phi^\dagger(x)\phi(x)$, verify that (micro)causality condition is satisfied, i.e. one has that:

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0 \qquad \text{if} \qquad (x-y)^2 < 0$$

(f) Using the previous example, show that (micro)causality condition is not satisfied if one quantizes the (compelex) scalar field using anticommutation relations;

(g) Verify that for a real scalar field the momentum operator satisfies (micro)causality condition, i.e. that one has:

$$[\mathcal{P}_i(x), \mathcal{P}_j(y)] = 0 \qquad \text{if} \qquad (x-y)^2 < 0$$