Fisica Teorica B: Exercise Set N.5

6 Relativistic free Dirac field theory

1. Classical Free Dirac Field Theory:

   (a) From the Dirac Lagrangian

   $$\mathcal{L} = \frac{i}{2}\left[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi\right] - M\bar{\psi}\psi$$

   derive the Eulero-Lagrange equations of motions for ψ and $\bar{\psi}$;

   (b) Show that the following Lagrangian

   $$\mathcal{L}' = i\bar{\psi}\gamma^\mu(\partial_\mu\psi) - M\bar{\psi}\psi \equiv \bar{\psi}\left(i\partial\!\!\!/ - M\right)\psi$$

   is equivalent to $\mathcal{L}$;

   (c) Derive the canonical energy-momentum tensor $\tilde{T}^{\mu\nu}$ and show explicitly that it is a conserved current, i.e. $\partial_\mu\tilde{T}^{\mu\nu} = 0$. Show that also $\partial_\nu\tilde{T}^{\mu\nu} = 0$;

   (d) Derive the expressions of the conserved charges P^μ. Show in particular that the Hamiltonian obtained starting from $\mathcal{L}$ and $\mathcal{L}'$ are equivalent;

   (e) Derive the expressions of the conserved charges $J_{\rho\sigma}$ associated to the invariance under Lorentz infinitesimal transformations. In particular convince yourself that the total angular momentum is associated to a spin 1/2 particle;

   (f) Show that the Dirac Lagrangian is invariant under a global internal $U(1)$ symmetry. Derive the associated conserved charge;

   (g) Derive the Poisson parenthesis between the Dirac field and the conjugate field:

   $$\begin{aligned}\{\psi_\alpha(x), \psi_\beta(y)\}_{E.T.} &= 0 = \{\pi_\alpha(x), \pi_\beta(y)\}_{E.T.}\\ \{\psi_\alpha(x), \pi_\beta(y)\}_{E.T.} &= i\delta^3(\vec{x}-\vec{y})\end{aligned}$$

   Moreover shows that the Hamilton equations of motions can be written as:

   $$\dot{\psi}(\vec{x},t) = \{\psi(\vec{x},t), H\} \quad , \quad \dot{\pi}(\vec{x},t) = \{\pi(\vec{x},t), H\}$$

2. Quantization of free Dirac field (with anti-commutators) :

   (a) Derive the expression of $c_r(k)$ ($c_r^\dagger(k)$) and $d_r(k)$ ($d_r^\dagger(k)$) in terms of ψ and $\bar{\psi}$;

   (b) From the canonical anti-commutation rules for the Dirac fields $\psi, \bar{\psi}$ derive the anti-commutation rules for the creation/annihilation operators $c_r(k), d_r(k)$;

   (c) Show that despite of the anti-commuting rules between creation/annihilation operators the operators $N_r^{(c)}(k) = c_r^\dagger(k) c_r(k)$ and $N_r^{(d)}(k) = d_r^\dagger(k) d_r(k)$ can still be interpreted as density-number operators;

   (d) Derive the expressions for the conserved charges P^μ and $Q_{U(1)}$ in terms of $c_r(k), d_r(k)$. Show that with the appropriate definition of normal ordering (consistently with anti-commutator rules) the Hamiltonian H is positive definite (while Q is not);

   (e) Show that the evolution equations for $\psi(x), \pi(x)$ satisfy the usual Heisemberg relations: i.e. for classical and quantistic fields one has respectively:

$$\dot{\psi}(\vec{x}, t) = -i\,[\psi(\vec{x}, t), H] \quad , \quad \dot{\pi}(\vec{x}, t) = -i\,[\pi(\vec{x}, t), H]$$

   (f) Calculate the anti-commutators (for general space-time points) and show that:

$$\{\psi(x), \psi(y)\} = \left\{\bar{\psi}(x), \bar{\psi}(y)\right\} = 0 \quad , \quad \left\{\psi(x), \bar{\psi}(y)\right\} = (i\partial\!\!\!/ + m) D(x - y)$$