Fisica Teorica B: Exercise Set N.6

6 Interacting field theory

1. Yukawa interactions:

   (a) Show that the Yukawa interaction between a real scalar field φ and a fermion ψ, given by the Lagrangian density:

   $$\mathcal{L}_Y = \varphi\,\bar{\psi}\,(y_s + iy_p\gamma_5)\,\psi$$

   is renomrmalizable (i.e. $[y_{s,p}] = M^\alpha$ with $\alpha \geq 0$);

   (b) Show that $\mathcal{L}_Y$ is hermitian (i.e. $\mathcal{L}_Y = \mathcal{L}_Y^\dagger$);

   (c) Consider the Yukawa interaction between a complex scalar field ϕ and a fermion ψ:

   $$\mathcal{L}_Y = \phi\,\bar{\psi}\,(y_s + iy_p\gamma_5)\,\psi \; + \; h.c$$

   where $h.c.$ stands for hermitian conjugate. Write explicitly the hermitian conjugate;

2. Gauge interactions:

   (a) Show that the QED Lagrangian density:

   $$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\not{D} - m\right)\psi$$

   with $D_\mu = \partial_\mu - iqA_\mu$ is invariant under the local $U(1)$ gauge transformation:

   $$\begin{cases} A'_\mu(x) & = A_\mu(x) + \partial_\mu\alpha(x) \\ \psi'(x) & = e^{iq\alpha(x)}\psi(x) \end{cases}$$

   (b) Show that the QED Lagrangian is invariant under a (proper) Lorentz transformation;

   (c) Defining $\psi_{L,R} = P_{L,R}\psi$, show that the QED Lagrangian density is invariant under a $L \leftrightarrow R$ transformation (i.e. is a "vector" like theory);

   (d) Applying the "minimal coupling" ansatz ($\partial_\mu \to D_\mu$ derive the Lagrangian density for the Scalar QED theory, i.e. the Lagrangian describing the gauge interaction between a complex scalar field and the E.M field A_μ. Show that this Lagrangian density is indeed gauge invariant under the local $U(1)$ gauge transformation:

$$\begin{cases} A'_\mu(x) & = A_\mu(x) + \partial_\mu \alpha(x) \\ \phi'(x) & = e^{iq\alpha(x)}\phi(x) \end{cases}$$

(e) Write explicitly the SQED Lagrangian density and identify all possible interactions between the scalar and the E.M field;

3. Interaction Picture and S-matrix:

   (a) Derive the equation of motion for the states and operators in the Interaction picture;

   (b) Derive the equation of motion for the time evolution operator:

$$U_I(t,t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}$$

   (c) Show that the differential equation for the time evolution operator can be formally solved as:

$$U_I(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} d\tau\, H^I_{INT}(\tau)\, U_I(\tau,t_0)$$

   with the initial condition fixed in order to have $U_I(t_0,t_0) = \mathbb{1}$;

   (d) Convince yourselves that in general $\left[H^I_{INT}(t), H^I_{INT}(t')\right] \neq 0$;

   (e) Show that the previous integral equation has the following perturbative solution:

$$U_I(t,t_0) = \sum_n \frac{(-i)^n}{n!}\int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{n-1}} d\tau_n\, T\left(H^I_{INT}(\tau_1)H^I_{INT}(\tau_2)\cdots H^I_{INT}(\tau_n)\right)$$

   in terms of T-products.

4. T-products, Wick's theorem and propagators:

   (a) Verify the following relations for real and complex scalar fields:

$$\begin{aligned} T\left(\varphi(x)\varphi(y)\right) &= N\left(\varphi(x)\varphi(y)\right) + \langle 0|T\left(\varphi(x)\varphi(y)\right)|0\rangle && \text{(real scalar)} \\ T\left(\phi(x)\phi^\dagger(y)\right) &= N\left(\phi(x)\phi^\dagger(y)\right) + \langle 0|T\left(\phi(x)\phi^\dagger(y)\right)|0\rangle && \text{(complex scalar)} \\ T\left(\phi(x)\phi(y)\right) &= N\left(\phi(x)\phi(y)\right) && \text{(complex scalar)} \end{aligned}$$

   (b) Verify explicitly that for a charged scalar field $D_F(x-y) \neq D_F(y-x)$;

(c) Verify the following relations for Dirac spinorial fields:

$$\begin{aligned} T\left(\psi(x)\bar{\psi}(y)\right) &= N\left(\psi(x)\bar{\psi}(y)\right) + \langle 0|T\left(\psi(x)\bar{\psi}(y)\right)|0\rangle \\ T\left(\psi(x)\psi(y)\right) &= N\left(\psi(x)\psi(y)\right) \end{aligned}$$

(d) Prove that the fermionic Feynamn propagator satisfies the following relation:

$$S_F(x-y) \equiv \theta(x_0-y_0)S_+(x-y) - \theta(y_0-x_0)S_-(x-y) = \left(i\partial\!\!\!/ + m\right) D_F(x-y)$$

(e) Calculate the following T-product:

$$T\Big(\bar{\psi}(x_1)\psi(x_2)\bar{\psi}(x_3)\psi(x_4)\Big)$$

(f) Given $\mathcal{H}^I_{INT}(x) = N\left(A^\mu A_\mu \phi^\dagger \phi\right)_x$, calculate the following T-product:

$$T\Big(\mathcal{H}^I_{INT}(x)\mathcal{H}^I_{INT}(y)\Big)$$

using Wick's theorem and Wick's theorem corollary;

(g) Verify the equivalence of the following propagator expressions:

$$D_F(x-y) = \int_{C_F} \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2-m^2} = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon} \qquad (\text{with } \epsilon > 0)$$

(h) Verify that the scalar Feynman propagator $D_F(x-y)$ is a Green function for the Klein-Gordon equation, i.e.

$$(\Box + m^2)D_F(x-y) = -i\delta^4(x-y)$$

(i) Verify that the fermionic Feynman propagator $S_F(x-y)$ is a Green function for the Dirac equation, i.e.

$$(i\partial\!\!\!/ - m)S_F(x-y) = -i\delta^4(x-y)$$

You can prove it in two ways: directly using the expression for the fermionic Feynman propagator and indirectly by using the result in (h) and the relation that connect scalar and fermionic Feynman propagator.

(j) Verify that the (massless) vector Feynman propagator in the generic ξ gauge is a Green function for the vector massless theory (with the generic gauge fixing term):

$$\left(\eta^{\mu\nu}\Box + \frac{1-\xi}{\xi}\partial^\mu\partial^\nu\right) D_F^{\nu\rho}(x-y) = i\eta^{\mu\rho}\delta^4(x-y)$$

where the (massless) Feynman propagator in the generic ξ gauge reads:

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)}\frac{-i}{k^2+i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$