Fisica Teorica B: Exercise Set N.7

6 QED S-matrix expansion

1. S-matrix expansion in coordinate space:

   (a) Derive all possible terms originating from the QED interaction at first order in perturbation expansion;

   (b) Expand the second order S-matrix term using Wick theorem and relative corollary. Identify all possible terms with 0, 1, 2 and 3 propagators, and draw a "typical" diagram of that kind;

   (c) Write all possible terms originating at second order with one fermion propagator. Draw all the possible diagrams and identify all the possible physical ($2 \to 2$) processes;

   (d) Write all possible terms originating at second order with one photon propagator. Draw all the possible diagrams and identify all the possible physical ($2 \to 2$) processes;

   (e) Write all possible terms originating at second order with two fermion propagators. Draw all the possible diagrams and identify all the possible physical ($1 \to 1$) processes;

   (f) Write all possible terms originating at second order with one fermion and one photon propagator. Draw all the possible diagrams and identify all the possible physical ($1 \to 1$) processes;

   (g) Expand the third order S-matrix term using Wick theorem and relative corollary. Identify all possible terms with 0, 1, 2, 3 and 4 propagators propagators, and draw a "typical" diagram of that kind;

   (h) Write all possible terms originating at third order with two fermion and one photon propagator. Draw all the possible diagrams and identify all the possible processes;

2. S-matrix expansion in momentum space:

(a) Derive the transition amplitude $S_{fi} = \langle f|S|i\rangle$ at the lowest order in perturbation expansion for the following initial and final states:

$$|\,i\,\rangle = |e^-(p)\,e^+(q)\rangle \qquad , \qquad |\,f\,\rangle = |\gamma(k)\rangle$$

Show that this process is kinematically forbidden in QED;

(b) Calculate the Feynman amplitude $\mathcal{M}_{fi}$ at third order in perturbation expansion for the following initial and final states:

$$|\,i\,\rangle = |e^-(p)\,e^+(q)\rangle \qquad , \qquad |\,f\,\rangle = |\gamma(k)\rangle$$

It could be useful to do before point (h) of the previous exercise.

(c) Calculate the Feynman amplitude $\mathcal{M}_{fi}$ at second order in perturbation expansion for the following initial and final states:

1) $|\,i\,\rangle = |e^-(p)\,e^+(q)\rangle \quad , \quad |\,f\,\rangle = |e^-(p')\,e^+(q')\rangle$

2) $|\,i\,\rangle = |e^+(p)\,\gamma(q)\rangle \quad , \quad |\,f\,\rangle = |e^+(p')\,\gamma(q')\rangle$

3) $|\,i\,\rangle = |e^-(p)\,e^-(q)\rangle \quad , \quad |\,f\,\rangle = |e^-(p')\,e^-(q')\rangle$

In all the processes in which more than one Feynman diagram contribute, explicitly discuss the relative sign between the contributing diagrams;

(d) Calculate the Feynman amplitude $\mathcal{M}_{fi}$ at second order in perturbation expansion for the following initial and final states:

$$|\,i\,\rangle = |\,e^+(q)\rangle \qquad , \qquad |\,f\,\rangle = |\,e^+(q')\rangle$$

7 Cross section and Decay Rate

1. Cross sections in QED and beyond

(a) Show that in the center of mass frame the momentum p_1' of one of the outgoing particle can be written in terms of the final masses m_1', m_2' and $s = (p_1^2 + p^2)^2$ reads:

$$|p_1'| = \frac{1}{2\sqrt{s}}\left[s^2 + (m'^2_1 - m'^2_2)^2 - 2s(m'^2_1 + m'^2_2)\right]^{1/2}$$

(b) Using the well known γ matrix identity $\gamma_0\gamma_\mu\gamma_0 = \gamma_\mu^\dagger$, show that

$$\Big(\bar{u}(p)\gamma^\mu u(k)\Big)^* = \bar{u}(k)\gamma^\mu u(p)$$

(c) Calculate the unpolarized differential cross section for the following processes:
   1) $e^-(p)e^+(q) \to \mu^-(p')\mu^+(q')$
   2) $e^-(p)\mu^+(q) \to e^-(p')\mu^+(q')$
   3) $e^-(p)\mu^-(q) \to e^-(p')\mu^-(q')$

(d) Calculate the unpolarized differential cross section for the following processes:
   1) $e^-(p)e^+(q) \to e^-(p')e^+(q')$
   2) $e^-(p)e^-(q) \to e^-(p')e^-(q')$

   In all the processes in which more than one Feynman diagram contribute, explicitly discuss the relative sign between the contributing diagrams;

(e) Consider the following interaction between fermions and a real scalar field:

$$\mathcal{L}_{INT} = \varphi\, \bar{\psi}_\ell (a_\ell + i\, b_\ell\, \gamma_5)\psi_\ell$$

   with a_ℓ and b_ℓ real (small) parameters.

   i. Show that $\mathcal{L}_{INT}$ is hermitian;
   ii. Derive explicitly the Feynman rule for the interaction described by $\mathcal{L}_{INT}$. For example consider $|\, i\,\rangle = |e^-(p)\rangle$ and $|\, f\,\rangle = |s(q)\, e^-(k)\rangle$ and explicitly calculate the amplitude $\langle f|S|i\rangle$ at first order;
   iii. Calculate the unpolarized cross section for the process $e^-(p)e^+(q) \to \mu^-(p')\mu^+(q')$, mediated by the Yukawa (scalar) interaction. For simplicity take $a_e = a_\mu$ and $b_e = b_\mu$;
   iv. Compare the angular distribution with the similar process calculated in QED (calculated in point (c));

2. Decay rates of elementary processes

(a) Show that in the center of rest frame of the decaying particle the momentum p'_1 of one of the outgoing particle can be written in terms of the masses m'_1, m'_2 and M reads:

$$|p'_1| = \frac{1}{2M}\left[M^4 + (m'^2_1 - m'^2_2)^2 - 2M^2(m'^2_1 + m'^2_2)\right]^{1/2}$$

(b) Consider the following interaction between fermions and a real scalar field:

$$\mathcal{L}_{INT} = \varphi\, \bar{\psi}_\ell (a_\ell + i\, b_\ell\, \gamma_5)\psi_\ell$$

   with a_ℓ and b_ℓ real (small) parameters. Calculate the unpolarized decay rate $s(p) \to e^-(p')e^+(q')$;

(c) Calculate the unpolarized cross section for a massive vector boson decay $V(p) \to e^-(p')e^+(q')$ keeping the fermion mass;