Introduction to QED: First partial exam - 30/04/2014

All results obtained in class must be adequately discussed and motivated.

1 Real Scalar Field

Let $\phi(x)$ be the operator describing a real scalar field of mass m:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \left(a(\vec{k})\, e^{-ik\cdot x} \; + \; a^\dagger(\vec{k})\, e^{ik\cdot x} \right)_{k_0=\omega_k} \; .$$

with $\omega_k = \sqrt{|\vec{k}|^2 + m^2}$ and $a(\vec{k}), a^\dagger(\vec{k})$ the annihilation and creation operators respectively.

1. Derive the expression for the conjugate operator $\pi(x)$;
2. Derive the expressions for $a(\vec{k})$ and $a^\dagger(\vec{k})$ in terms of $\phi(x)$ and $\pi(x)$.
3. Show that by assuming the canonical (equal time) commutation relations

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = i\, \delta^3(\vec{x}-\vec{y}) \qquad , \qquad [\phi(\vec{x},t), \phi(\vec{y},t)] = 0 = [\pi(\vec{x},t), \pi(\vec{y},t)]$$

   one obtains the following commutators for the creation and annihilation operators

$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right] = \delta^3(\vec{k}-\vec{k}') \qquad , \qquad \left[a(\vec{k}), a(\vec{k}')\right] = 0 = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right]$$

   (HELP: at a certain point of the derivation you may restric to the $x_0 = y_0$ case and use the equal time commutation relations ...)

4. Given the free scalar field Lagrangian density, derive the expression of the Hamiltonian density in terms of $\phi(x)$ and $\pi(x)$.
5. Using the known expression of the fields in terms of the creation and annihilation operators, show that the Hamiltonian can be written as:

$$:\!H\!: = \int d^3k\, \omega_k\, a^\dagger(\vec{k})\, a(\vec{k})$$

6. By using the explicit expression of the real scalar field calculate the covariant (i.e. generic time) commutator $D(x-y) = [\phi(x), \phi(y)]$ and show that it satisfies the homogeneous Klein-Gordon equation.
7. **Optional**: Consider the function $D_R(x-y) = \theta(x^0-y^0)[\phi(x), \phi(y)]$. Show that the function $D_R(z)$ is a Green function, i.e. satisfies the following differential equation:

$$(\Box_z + m^2) D_R(z) = -i\, \delta^4(z)\, .$$

2 Dirac Spinor Field

Let $\psi(x)$ be the operator describing a Dirac spinorial field of mass m.

1. Using the general form of the Noether conserved current, derive the following expressions for the conserved charges associated to the Poincarè invariance of the free Dirac Lagrangian :

$$\begin{aligned} P_\mu &= \int d^3x\, \psi^\dagger(x)\mathcal{P}_\mu\, \psi(x) \\ J_{\mu\nu} &= \int d^3x\, \psi^\dagger(x)\, \mathcal{J}_{\mu\nu}\, \psi(x) \end{aligned}$$

with

$$\mathcal{P}_\mu = i\partial^\mu \qquad , \qquad \mathcal{J}_{\mu\nu} = (x_\mu \mathcal{P}_\nu - x_\nu \mathcal{P}_\mu) + \frac{1}{2}\sigma_{\mu\nu} \qquad \left(\sigma_{\mu\nu} \equiv \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]\right)$$

2. Calculate the commutator $[J_{\mu\nu}, \psi(x)]$ and comment this result. (HELP: Writing explicitly the spinorial indeces may be helpful, though not necessary.)

3. Prove at least three of the following identities:

$$\begin{aligned} \gamma^\mu\gamma_\mu &= 4 \qquad , \qquad \gamma^\mu\gamma^\rho\gamma_\mu = -2\gamma^\rho \qquad , \qquad \gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = 4\eta^{\rho\sigma} \\ \sigma^{\mu\nu}\sigma_{\mu\nu} &= 12 \qquad , \qquad \sigma^{\mu\rho}\sigma_{\mu\sigma} = 2(\gamma^\rho\gamma_\sigma + \eta^\rho_\sigma) \end{aligned}$$

4. The helicity projectors $\Pi(\pm n_k)$ are defined as:

$$\Pi(\pm n_k) = \frac{1 \pm \gamma_5 \not{n}_k}{2} \qquad \text{with} \qquad n_k^\mu = \left(\frac{|\vec{k}|}{m}, \frac{\omega_k}{m}\frac{\vec{k}}{|\vec{k}|}\right)$$

show that $\Pi(\pm n_k)$ are projectors and are orthogonal.

5. Prove that for the positive energy solution spinor $u(k)$ one has:

$$\frac{1}{2}\gamma_5 \not{n}_k\, u_r(k) = +\frac{\vec{\Sigma}\cdot\vec{k}}{|\vec{k}|} u_r(k)$$

and that consequently $\Pi(+n_k)$ projects over positive (negative) helicity states for r=1 (2).

6. Show that this is related to the Pauli-Lubanski tensor, i.e.:

$$\Pi(+n_k)\, u_r(k) = \frac{1}{2}\left(1 - \frac{\mathbf{W}\cdot n_k}{2m}\right) u_r(k)$$

where

$$\mathbf{W}^\mu \psi(x) = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{J}_{\nu\rho}\mathcal{P}_\sigma\, \psi(x)$$

To do this you can use the results obtained in class.