Introduction to QED: Suggested Exercises

2 Relativistic Scalar Field Theory

1. Real scalar field.

   (a) Using the explicit expression for the Hamiltonian:

   $$H = \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

   verify the Hamilton equations:

   $$\dot{\phi}(\vec{x},t) = \{\phi(\vec{x},t), H\} \qquad , \qquad \dot{\pi}(\vec{x},t) = \{\pi(\vec{x},t), H\}$$

   (b) Write the explicit expression of the conserved currents associated to the Poincaré invariance and show that they satisfy the continuity equations: $\partial_\mu J^\mu_{(a)} = 0$;

   (c) Given the general solution of the free real KG equation:

   $$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega_k}} \left[a(k)\, e^{-ik\cdot x} + a^*(k)\, e^{ik\cdot x}\right] \qquad , \qquad k = (\omega_k, \vec{k})$$

   calculate the expressions for conserved 4-momentum P_μ in terms of $a(k), a^*(k)$;

2. Complex scalar field.

   (a) Using the explicit expression for the Hamiltonian:

   $$H = \int d^3x \left[\pi^*\pi + (\nabla\phi)^*(\nabla\phi) + m^2\phi^*\phi\right)$$

   verify the Hamilton equations:

   $$\dot{\phi}(\vec{x},t) = \{\phi(\vec{x},t), H\} \qquad , \qquad \dot{\pi}(\vec{x},t) = \{\pi(\vec{x},t), H\}$$
   $$\dot{\phi}(\vec{x},t)^* = \{\phi(\vec{x},t)^*, H\} \qquad , \qquad \dot{\pi}(\vec{x},t)^* = \{\pi(\vec{x},t)^*, H\}$$

   (b) Write the explicit expression of the conserved currents associated to the Poincaré invariance and show that they satisfy the continuity equations: $\partial_\mu J^\mu_{(a)} = 0$;

(c) Given the general solution of the free complex KG equation:

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\sqrt{2\omega_k}}\left[a(k)\,e^{-ik\cdot x} + b^*(k)\,e^{ik\cdot x}\right] \qquad , \qquad k = (\omega_k, \vec{k})$$

calculate explicitly the Hamiltonian, the Momentum and the conserved $U(1)$ charge in terms of $a(k), b(k), a^*(k), b^*(k)$;

3. Equivalence between a complex scalar field and two real fields.
   Consider the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{1}{2}m^2\phi_2^2 - \frac{1}{16}\lambda(\phi_1^2+\phi_2^2)^2$$

for the case of two real scalar fields, ϕ_1 and ϕ_2.

   (a) Define a two dimensional vector field $\Phi = (\phi_1, \phi_2)^T$ and write the Lagrangian density in terms of Φ;

   (b) Find the internal symmetry and the associated conserved currents;

   (c) Explain why this theory is equivalent to the one with a single complex scalar field and write the corresponding complex scalar Lagrangian density;

4. The previous problem can be generalized to higher dimensional internal groups. As an example consider the case of a two dimensional complex scalar field $\Phi = (\phi_1, \phi_2)^T$ with $\phi_{1,2}$ complex scalar fields;

   (a) Write the general Lagrangian density including all possible couplings with mass dimension ≥ 0;

   (b) Find the internal symmetry, the associated conserved currents and show that the four conserved charges can be written as

$$Q_{(\mu)} = iq\int d^3x\left[\Phi^\dagger\sigma_\mu\Pi^\dagger - \Pi\,\sigma_\mu\Phi\right]$$

   with $\sigma_\mu \equiv (1, \vec{\sigma})$ and $\Pi = \partial_0\Phi^\dagger, \Pi^\dagger = \partial_0\Phi$ the conjugate momenta;

   (c) Establish a relation with a theory with only real scalar fields (i.e. how many? which internal symmetry? ...);

5. Consider the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

(a) Calculate the Noether current for the dilatation transformation (with α a real constant):

$$\delta x^\mu = \alpha x^\mu \qquad , \qquad \delta\phi = -\alpha\phi$$

(b) Show that the dilatation current is conserved only if $m = 0$ (while $\lambda \neq 0$ can be kept);

6. Quantization of a real scalar field.

(a) Verify that the evolution of ϕ and π satisfies the relations:

$$\dot{\phi}(\vec{x}, t) = -i\,[\phi(\vec{x}, t), H] \qquad , \qquad \dot{\pi}(\vec{x}, t) = -i\,[\pi(\vec{x}, t), H]$$

(b) Derive the expressions for $a(k), a^\dagger(k)$ in terms of ϕ, π;

(c) Verify that imposing the following commutation relations:

$$\left[a(k), a^\dagger(p)\right] = \delta^3(\vec{k} - \vec{p}) \qquad , \qquad [a(k), a(p)] = \left[a^\dagger(k), a^\dagger(p)\right] = 0$$

one obtains the canonical commutation relations for the operators ϕ, π;

(d) Verify that the operator P^μ is conserved;

(e) Prove that $2\omega_k\delta^3(\vec{k} - \vec{p})$ is invariant under Lorentz transformations;

7. Quantization of a complex scalar field.

(a) Derive the expressions for $a(k), a^\dagger(k), b(k), b^\dagger(k)$ in terms of $\phi^{(\dagger)}, \pi^{(\dagger)}$;

(b) Derive the expression of the conserved charge $Q_{U(1)}$ in terms of $a(k), b(k)$ operators;

8. Covariant Commutators.

(a) Show that for real and complex scalar field one has, respectively:

$$[\phi(x), \phi(y)] = D(x - y) \qquad , \qquad \left[\phi(x), \phi^\dagger(y)\right] = D(x - y)$$

(b) Show explicitly that $D(x - y)$ is invariant under (proper) Lorentz transformations and that it vanishes on a space-like interval (i.e. for example $D(0, \vec{x}) = 0$);

(c) For real and complex scalar fields derive the expression for the following covariant commutator $[\phi(x), \pi(y)]$;

(d) Verify the following properties of $D(x-y)$:

$$
\begin{aligned}
&a) && D(-x) = -D(x) \\
&b) && (\Box + m^2)D(x) = 0 \\
&c) && \partial_0 D(x)|_{x_0=0} = -\delta^3(x) \\
&d) && \partial_i D(x)|_{x_0=0} = 0
\end{aligned}
$$

(e) Given the following observable (for a complex scalar field) $\mathcal{O}(x) = \phi^\dagger(x)\phi(x)$, verify that (micro)causality condition is satisfied, i.e. one has that:

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0 \qquad \text{if} \qquad (x-y)^2 < 0$$

(f) Using the previous example, show that (micro)causality condition is not satisfied if one quantizes the (compelex) scalar field using anticommutation relations;

(g) Verify that for a real scalar field the momentum operator satisfies (micro)causality condition, i.e. that one has:

$$[\mathcal{P}_i(x), \mathcal{P}_j(y)] = 0 \qquad \text{if} \qquad (x-y)^2 < 0$$