# Introduction to QED: Suggested Exercises

## 3 Dirac equation and Dirac field theory

1. Dirac matrices:

   (a) Derive the properties of the $\gamma$ matrices starting from those of $\alpha_i, \beta$;

   (b) Show that it is possible to connect Dirac and Weyl $\gamma$ matrices representations by mean of a unitary transformation $U^\dagger \gamma_D U = \gamma_W$. Write $U$ explicitly;

   (c) Show that if the Dirac field transforms as $\psi'(x') = S(\Lambda)\psi(x)$ then the Dirac conjugate $\bar{\psi}$ transforms as $\bar{\psi}'(x') = \bar{\psi} S^{-1}(\Lambda)$ with $S^{-1}(\Lambda) = \gamma_0 S^\dagger(\Lambda)\gamma_0$;

   (d) By using the explicit expression of the Lorentz transformation matrix in spinorial representation $S(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right]$, show that $\gamma^\mu$ under a Lorentz transformation transforms as $S^{-1}(\Lambda)\,\gamma^\mu\, S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$;

   (e) Show that $\gamma_5$ under a Lorentz transformation transforms as $S^{-1}(\Lambda)\,\gamma_5\, S(\Lambda) = \det\Lambda\,\gamma_5$;

   (f) Show that $\Sigma^{\mu\nu}$ satisfies the $so(1,3)$ commutation rules;

   (g) Find the explicit expressions for the spin and boost matrices in the spinorial representation:

   $$\Sigma_i = \frac{1}{2}\epsilon_{ijk}\Sigma^{jk} \quad , \quad K^i = \Sigma^{0i}$$

   (h) Prove that $\Sigma_i$ satisfy the following spin relation:

   $$[\Sigma_i, \Sigma_j] = i\epsilon_{ijk}\Sigma_k \quad , \quad |\vec{\Sigma}|^2 = -\frac{3}{4}\mathbb{1}$$

2. Dirac spinors:

   (a) Find the general form of the spinors $u_r(k), v_r(k)$ using the Weyl representation for the Dirac matrices;

   (b) Knowing the general expressions of the spinors $u_r(k), v_r(k)$ show the relations:

   $$\Lambda_+(k) = \frac{1}{2m}\sum_r u_r(k)\bar{u}_r(k) \quad , \quad \Lambda_-(k) = -\frac{1}{2m}\sum_r v_r(k)\bar{v}_r(k)$$

(c) Prove the following identities:

$$u_\alpha^\dagger(k)u_\beta(k) = 2\,\omega_k\delta_{\alpha\beta} \quad , \quad v_\alpha^\dagger(k)v_\beta(k) = 2\,\omega_k\delta_{\alpha\beta} \quad , \quad u_\alpha^\dagger(k)v_\beta(-k) = v_\alpha^\dagger(-k)u_\beta(k) = 0$$

(d) Show that $P^2$ and $W^2$ (with $P^\mu$ the 4-momentum and $W^\mu = -1/2\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$) are the Casimir operators for the Poincare algebra;

(e) From the general expression of the Pauli-Lubanski pseudo-vector $W^\mu$, show that

$$\frac{W^\mu n_\mu}{m} = \frac{1}{2m}\gamma^5 \not{n}\not{p} = -\frac{1}{2m}\gamma^5 \not{p}\not{n}$$

(f) Knowing that the spinors $u_r(m), v_r(m)$ in the mass rest frame are eigenstates of $\Sigma_3$, show that in the frame where $\vec{k} = (0, 0, k)$ the spinors $u_r(k), v_r(k)$ are eigenstates of the helicity operator $\vec{\Sigma} \cdot \vec{k}/|k|$:

$$2\frac{\vec{\Sigma} \cdot \vec{k}}{|k|}u_r(k) = (-1)^{r+1}u_r(k) \quad , \quad 2\frac{\vec{\Sigma} \cdot \vec{k}}{|k|}v_r(k) = (-1)^{r+1}v_r(k)$$

3. Quantization of Dirac field:

(a) Starting from the Dirac Lagrangian derive the Eulero-Lagrange equations for $\psi(x)$;

(b) From the canonical anti-commutation rules for the Dirac fields $\psi, \bar{\psi}$ derive the anti-commutation rules for the creation/annihilation operators $c_r(k), d_r(k)$;

(c) Derive the expressions for the operators $c_r^{(\dagger)}(k), d_r^{(\dagger)}(k)$ in terms of $\psi, \bar{\psi}$;

(d) Show that despite of the anti-commuting rules between creation/annihilation operators the operators $N_r^{(c)}(k) = c_r^\dagger(k)c_r(k)$ and $N_r^{(d)}(k) = d_r^\dagger(k)d_r(k)$ can still be interpreted as density-number operators;

(e) Show explicitly that the canonical energy-momentum tensor $\tilde{T}^{\mu\nu}$ is a conserved current, i.e. $\partial_\mu\tilde{T}^{\mu\nu} = 0$. Show that also $\partial_\nu\tilde{T}^{\mu\nu} = 0$

(f) Derive the expressions for the conserved charges $P^\mu, Q_{U(1)}$ in terms of $c_r(k), d_r(k)$. Show that the appropriate definition of Normal Ordering $H$ is positive definite (while $Q$ is not);

(g) Show that the evolution equations for $\psi(x), \pi(x)$ satisfy the usual Hamilton relations, i.e. for classical and quantistic fields one has:

$$\dot{\psi}(\vec{x}, t) = \{\psi(\vec{x}, t), H\} \quad , \quad \dot{\pi}(\vec{x}, t) = \{\pi(\vec{x}, t), H\}$$
$$\dot{\psi}(\vec{x}, t) = -i\,[\psi(\vec{x}, t), H] \quad , \quad \dot{\pi}(\vec{x}, t) = -i\,[\pi(\vec{x}, t), H]$$

(h) Calculate the anti-commutators (at general times):

$$\{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0 \quad , \quad \{\psi(x), \bar{\psi}(y)\} = (i\partial\!\!\!/ + m)D(x-y)$$

(i) Derive the equation of motion for the chiral fields $\psi_{L,R}(x)$:

$$\psi_L(x) = \left(\frac{1-\gamma_5}{2}\right)\psi(x) \quad , \quad \psi_R(x) = \left(\frac{1+\gamma_5}{2}\right)\psi(x)$$

and show that they decouple in the $m = 0$ limit;

4. Dirac bilinears:

(a) By knowing the transformation properties of $\psi(x), \bar{\psi}(x), \gamma_\mu$ and $\gamma_5$ show that the transformation properties of the following bilinears:

$$\bar{\psi}(x)\psi(x) \quad , \quad \bar{\psi}(x)\gamma_5\psi(x) \quad , \quad \bar{\psi}(x)\gamma_\mu\psi(x) \quad , \quad \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) \quad , \quad \bar{\psi}(x)\sigma_{\mu\nu}\psi(x)$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

(b) Given the current $J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$, calculate the commutator (at general times): $[J_\mu(x), J_\nu(y)]$. Show that it vanishes for space-like intervals (i.e. $(x-y)^2 < 0$);

(c) Without using any explicit Dirac matrices representation show that:

$$\bar{u}(p)\sigma^{\mu\nu}(p+k)_\nu u(k) = i\,\bar{u}(p)\,(p-k)^\mu\,u(k)$$

(d) Without using any explicit Dirac matrices representation show the Gordon identities:

$$\bar{u}(p)\gamma^\mu u(k) = +\frac{1}{2m}\bar{u}(p)\left[(p+k)^\mu + i\sigma^{\mu\nu}(p-k)_\nu\right]u(k)$$
$$\bar{v}(p)\gamma^\mu v(k) = -\frac{1}{2m}\bar{v}(p)\left[(p+k)^\mu + i\sigma^{\mu\nu}(p-k)_\nu\right]v(k)$$

(e) Consider the current $J_\mu(p_1, p_2) = \bar{u}(p_2)\not{p}_1\gamma_\mu\not{p}_2 u(p_1)$. Show that $J_\mu$ can be written as:

$$J_\mu = \bar{u}(p_2)\left[F_1(q^2, m)\,\gamma_\mu + F_2(q^2, m)\,\sigma_{\mu\nu}q^\nu\right]u(p_1)$$

with $q^\mu = p_2^\mu - p_1^\mu$. Determine the functions $F_1(q^2, m), F_2(q^2, m)$;

(f) Consider the current $J_\mu(p_1, p_2) = \bar{u}(p_2)p^\rho q^\nu \sigma_{\mu\rho}\gamma_\nu u(p_1)$, with $p^\mu = p_2^\mu + p_1^\mu$ and $q^\mu = p_2^\mu - p_1^\mu$. Show that $J_\mu$ can be written as:

$$J_\mu = \bar{u}(p_2)\left[F_1\,\gamma_\mu + F_2\,q_\mu + F_3\,\sigma_{\mu\nu}q^\nu\right]u(p_1)$$

and determine the functions $F_1, F_2, F_3$;