BAYES and FREQUENTISM: The Return of an Old Controversy

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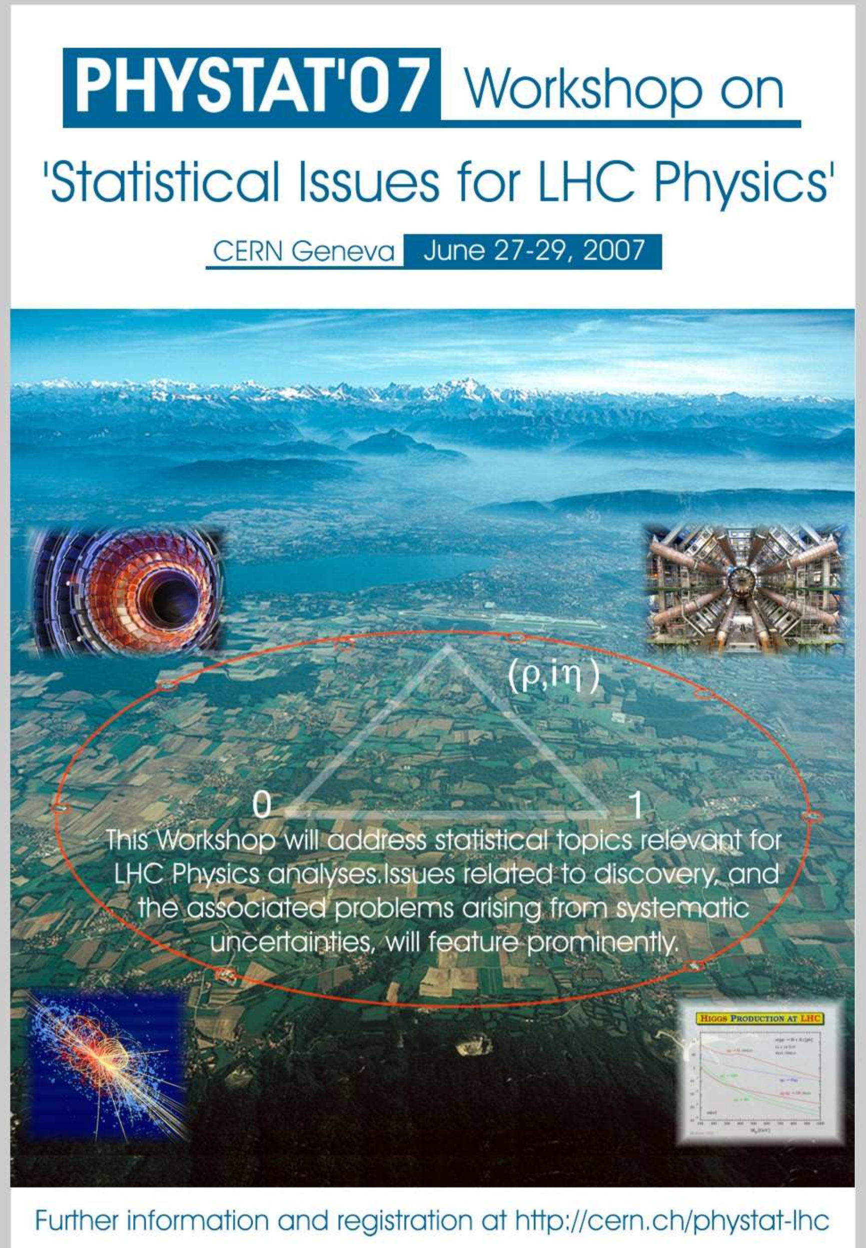
PHYSTAT'07 Workshop on
'Statistical Issues for LHC Physics'
CERN Geneva June 27-29, 2007
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This Workshop will address statistical topics relevant for LHC Physics analyses.Issues related to discovery, and the associated problems arising from systematic uncertainties, will feature prominently.
Further information and registration at http://cern.ch/phystat-lhc

Topics

- Who cares?
- What is probability?
- Bayesian approach
- Examples
- Frequentist approach
- Systematics
- Summary

It is possible to spend a lifetime analysing data without realising that there are two very different fundamental approaches to statistics:

Bayesianism and Frequentism.

How can textbooks not even mention Bayes / Frequentism?

For simplest case $(m \pm \sigma) \leftarrow Gaussian$

with no constraint on $m(true)$ then

$$m - k\sigma < m(true) < m + k\sigma$$

at some probability, for both Bayes and Frequentist

(but different interpretations)

See Bob Cousins "Why isn't every physicist a Bayesian?" Amer Jrnl Phys 63(1995)398

We need to make a statement about
Parameters, Given Data

The basic difference between the two:

Bayesian : Probability (parameter, given data)
(an anathema to a Frequentist!)

Frequentist : Probability (data, given parameter)
(a likelihood function)

PROBABILITY

MATHEMATICAL

Formal

Based on Axioms

FREQUENTIST

Ratio of frequencies as $n \rightarrow$ infinity

Repeated “identical” trials

Not applicable to **single event** or **physical constant**

BAYESIAN Degree of belief

Can be applied to single event or physical constant

(even though these have unique truth)

Varies from person to person ***

Quantified by “fair bet”

Bayesian versus Classical

Bayesian

P(A and B) = P(A;B) x P(B) = P(B;A) x P(A)

e.g. A = event contains t quark

B = event contains W boson

or A = I am in Padova

B = I am giving a lecture

P(A;B) = P(B;A) x P(A) /P(B)

Completely uncontroversial, provided….

Bayesian

$$P(A;B) = \frac{P(B;A) \text{ x } P(A)}{P(B)}$$

Bayes' Theorem

p(param | data) α p(data | param) * p(param)

$\uparrow$ posterior $\quad$ $\uparrow$ likelihood $\quad$ $\uparrow$ prior

Problems:	p(param)	Has particular value
		"Degree of belief"
	Prior	What functional form?
	Coverage	

P(parameter)   Has specific value

"Degree of Belief"

Credible interval

Prior: What functional form?

Uninformative prior: flat?

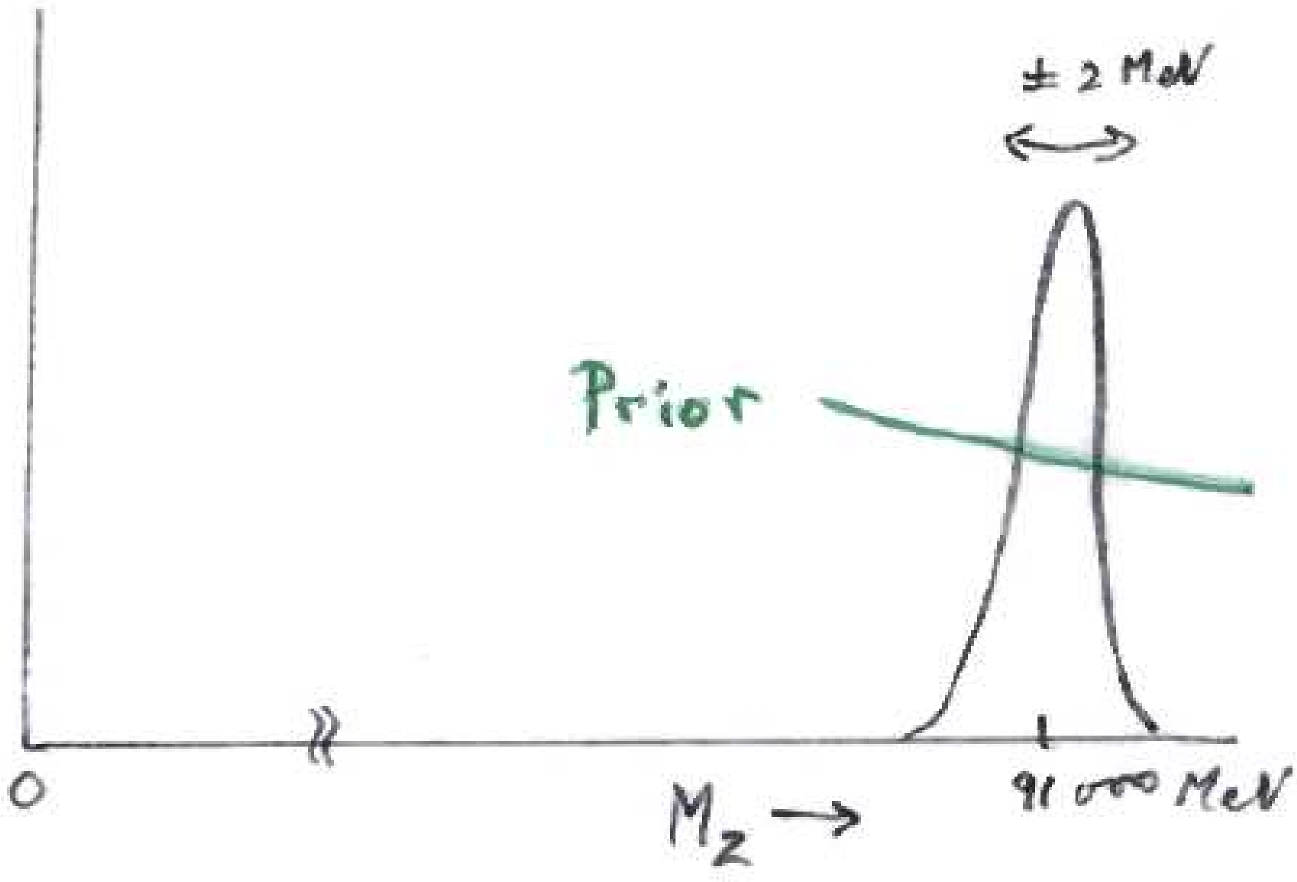
In which variable? e.g. $m, m^2, \ln m, ...?$

Even more problematic with more params

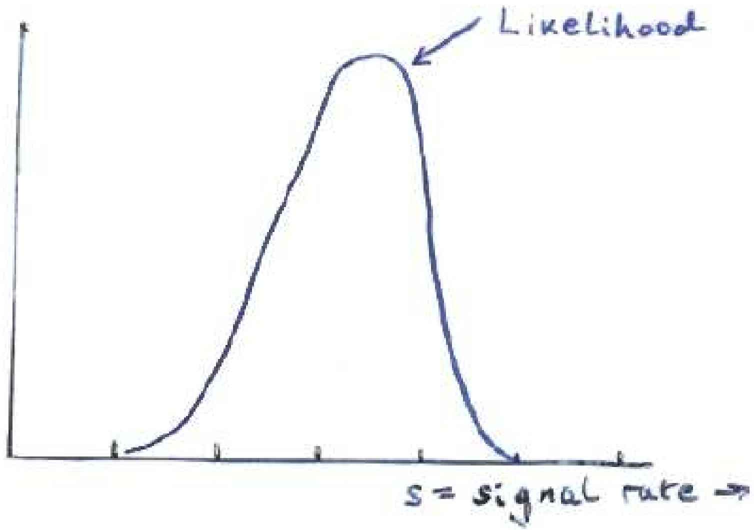
Unimportant if "data overshadows prior"

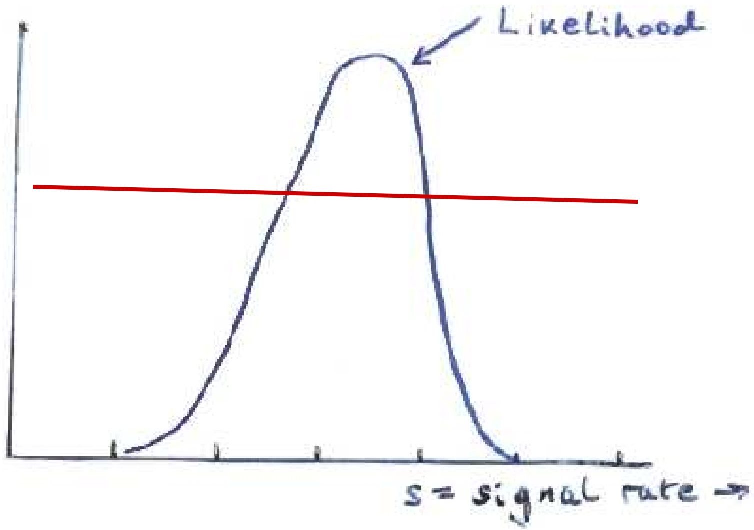
Important for limits

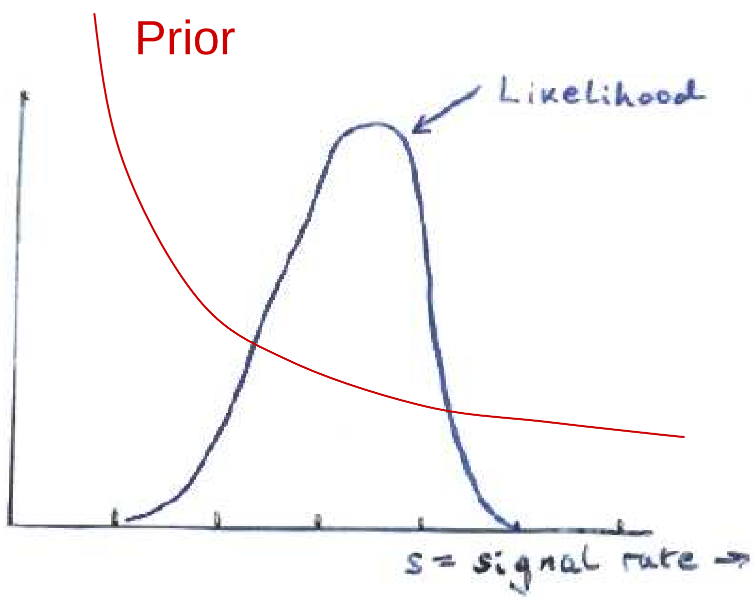
Subjective or Objective prior?

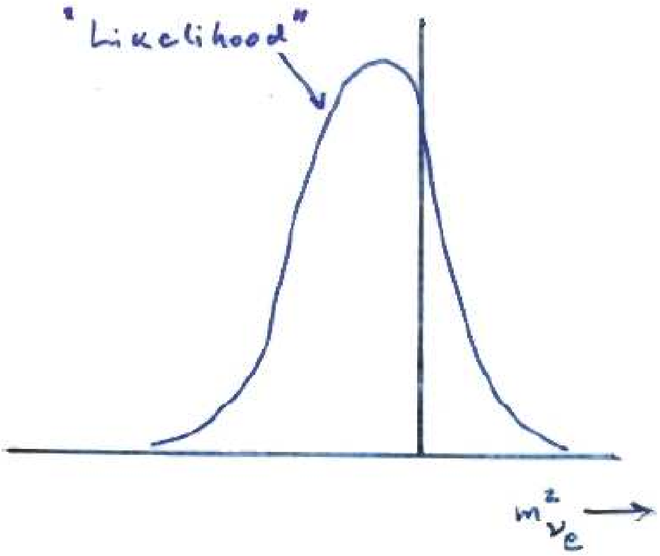


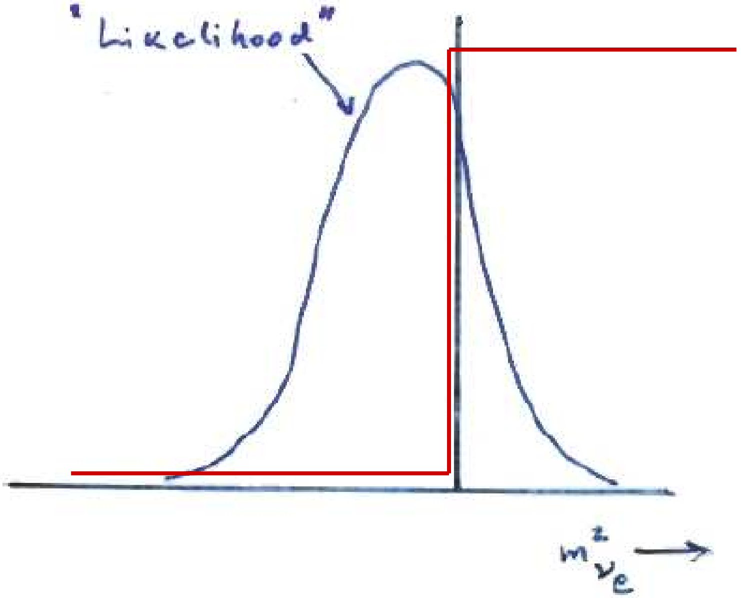
Data overshadows prior


Likelihood
s = signal rate →


Likelihood
s = signal rate →


Prior
Likelihood
s = signal rate →





Prior = zero in unphysical region

Bayes: Specific example

Particle decays exponentially: $dn/dt = (1/\tau) \exp(-t/\tau)$

Observe 1 decay at time t_1: $\mathcal{L}(\tau) = (1/\tau) \exp(-t_1/\tau)$

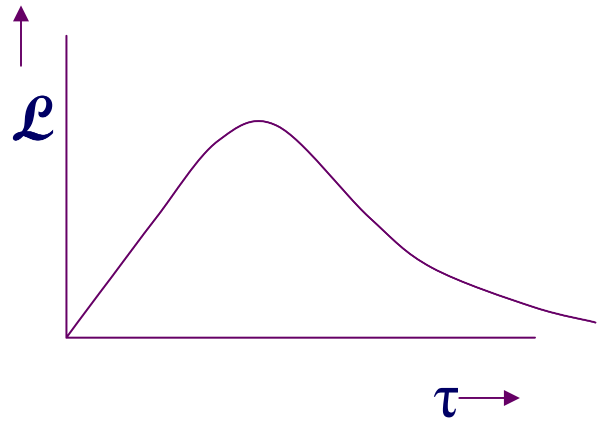
Choose prior $\pi(\tau)$ for τ

e.g. constant up to some large τ

Then posterior $p(\tau) = \mathcal{L}(\tau) * \pi(\tau)$

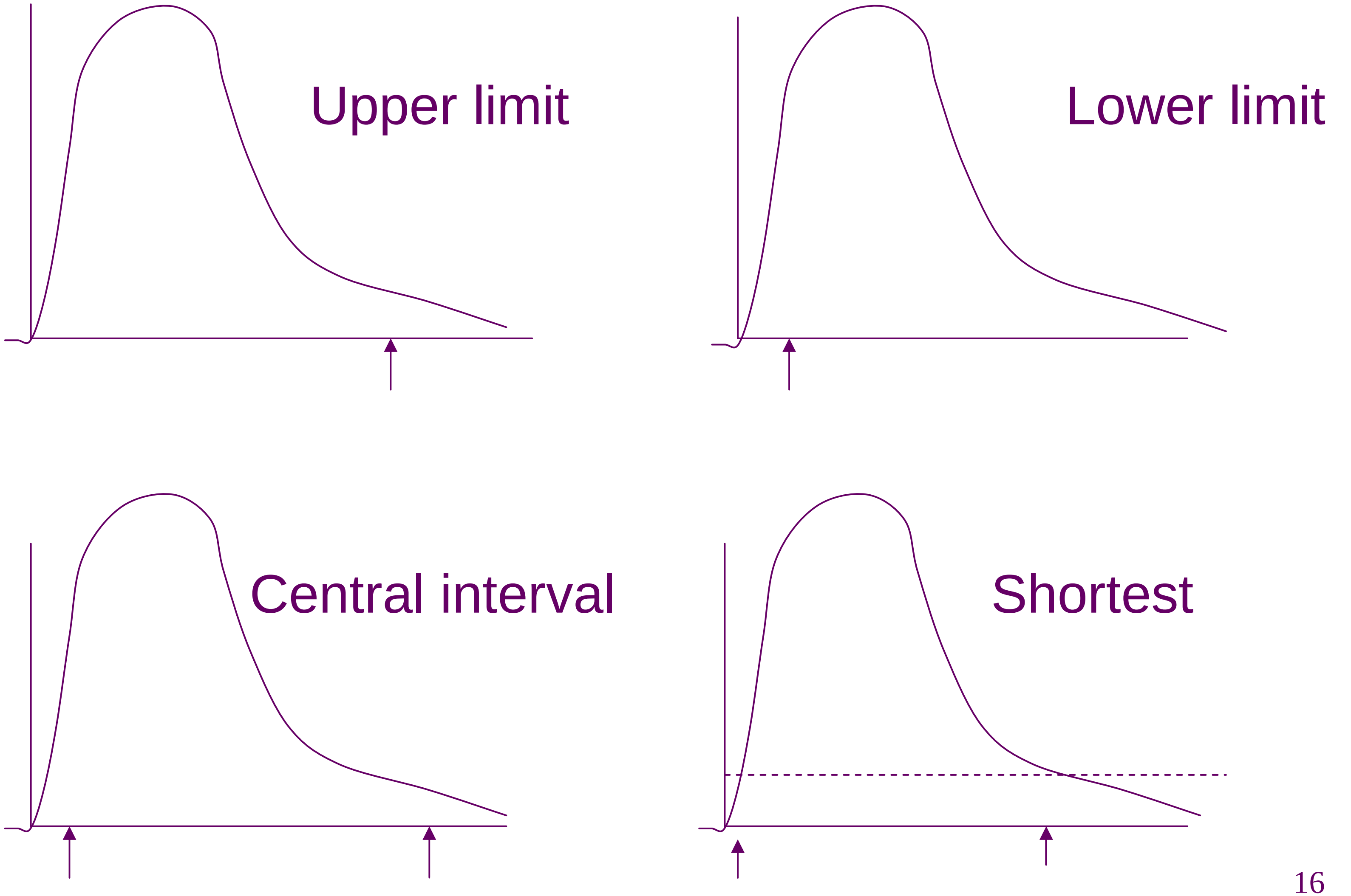
has almost same shape as $\mathcal{L}(\tau)$

Use $p(\tau)$ to choose interval for τ in usual way

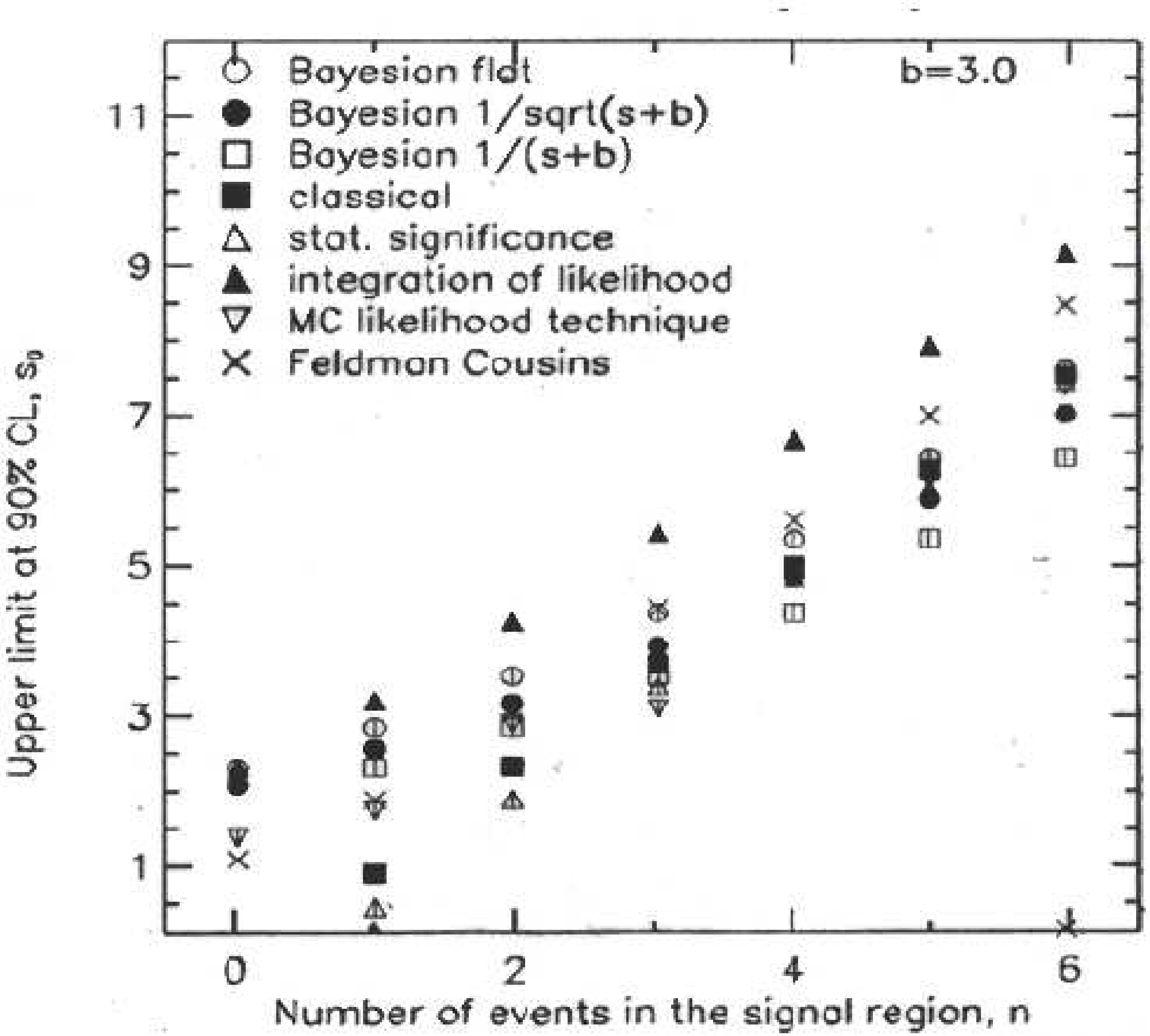


Contrast frequentist method for same situation later.

Bayesian posterior → intervals



Ilya Narsky, FNAL CLW 2000



P (Data;Theory) $\neq$ P (Theory;Data)

HIGGS SEARCH at CERN

Is data consistent with Standard Model?

or with Standard Model + Higgs?

End of Sept 2000: Data not very consistent with S.M.
Prob (Data ; S.M.) < 1% valid frequentist statement

Turned by the press into: Prob (S.M. ; Data) < 1%
and therefore Prob (Higgs ; Data) > 99%

i.e. "It is almost certain that the Higgs has been seen"

P (Data;Theory) $\neq$ P (Theory;Data)

Theory = male or female

Data = pregnant or not pregnant

P (pregnant ; female) ~ 3%

P (Data;Theory) $\neq$ P (Theory;Data)

Theory = male or female

Data = pregnant or not pregnant

P (pregnant ; female) ~ 3%

but

P (female ; pregnant) >>>3%

Example 1 :   Is coin fair ?

Toss coin: 5 consecutive tails

What is   P(unbiased; data) ? i.e. $p = ½$

Depends on Prior(p)

If village priest: prior $\sim \delta(p = 1/2)$

If stranger in pub: prior ~ 1 for $0 < p < 1$

(also needs cost function)

Example 2 : Particle Identification

Try to separate π's and protons

probability (p tag; real p) = 0.95

probability (π tag; real p) = 0.05

probability (p tag; real π) = 0.10

probability (π tag; real π) = 0.90

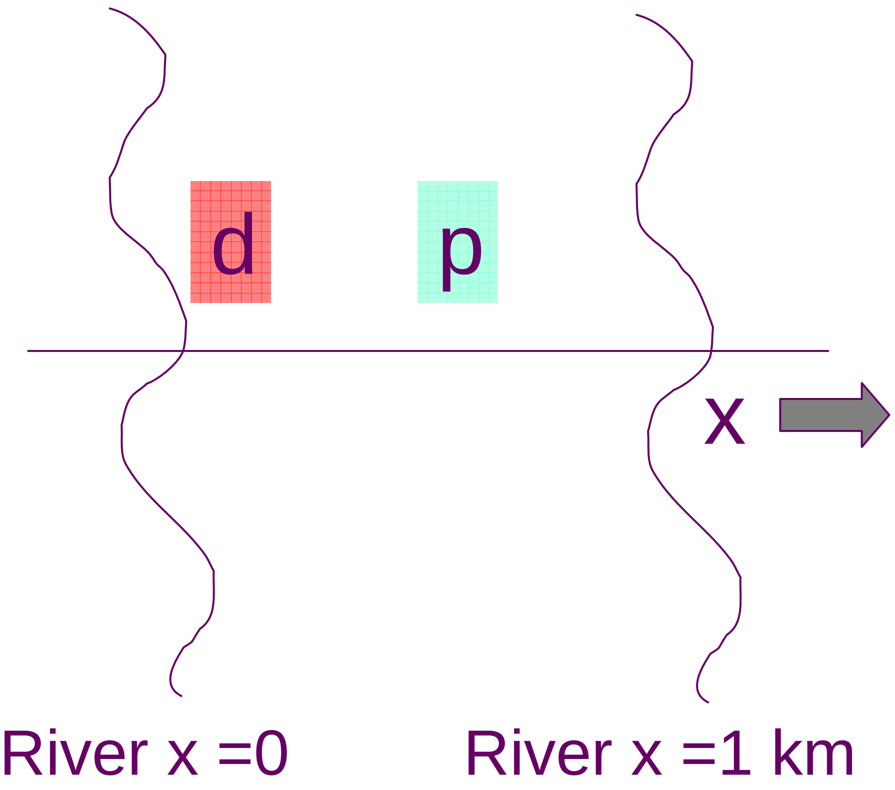
Particle gives proton tag. What is it?

Depends on prior = fraction of protons

If proton beam,	very likely
If general secondary particles,	more even
If pure π beam,	~ 0

Peasant and Dog

1) Dog d has 50% probability of being 100 m. of Peasant p
2) Peasant p has 50% probability of being within 100m of Dog d



Given that: a) Dog d has 50% probability of being 100 m. of Peasant,

is it true that: b) Peasant p has 50% probability of being within 100m of Dog d ?

Additional information

- Rivers at zero & 1 km. Peasant cannot cross them.

$$0 \leq h \leq 1\,\text{km}$$

- Dog can swim across river - Statement a) still true

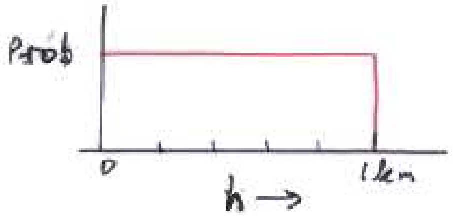
If dog at –101 m, Peasant cannot be within 100m of dog

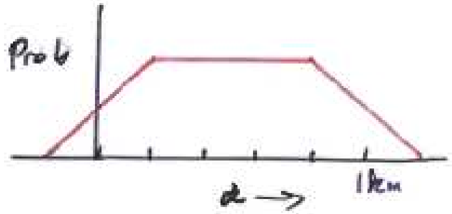
Statement b) untrue

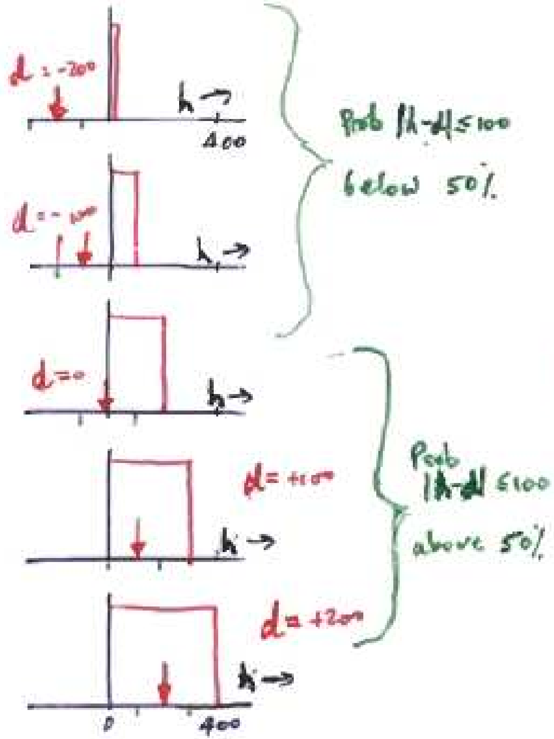
1) More specific on statement 1:

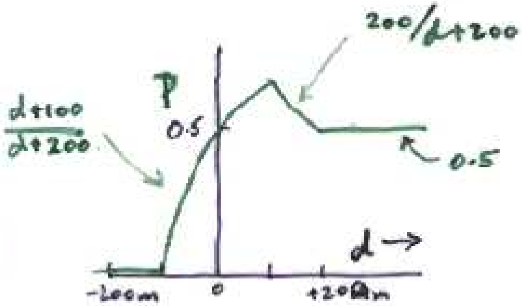
$$\text{Prob}(d-h) = \begin{cases} \text{const} & \text{for } |d-h| < 200\,\text{m} \\ 0 & \text{for } |d-h| > 200\,\text{m} \end{cases} \quad [\text{L'HOOD}]$$

2) Hunter h uniform in $0 \to 1$ km [PRIOR]









$P = \text{prob } |h-d| \le 100\text{ m}$

BF7 9''

Classical Approach

Neyman "confidence interval" avoids pdf for μ

Uses only $P(x;\mu)$

Confidence interval $\mu_1 \rightarrow \mu_2$:

$P(\mu_1 \rightarrow \mu_2$ contains $\mu) = \alpha$ True for any μ

Varying intervals from ensemble of experiments

fixed

Gives range of μ for which observed value x_0 was "likely" (α)

Contrast Bayes : Degree of belief = α that μ_t is in $\mu_1 \rightarrow \mu_2$

CLASSICAL (NEYMAN) CONFIDENCE INTERVALS

Uses only P (data | theory)

FIGURES

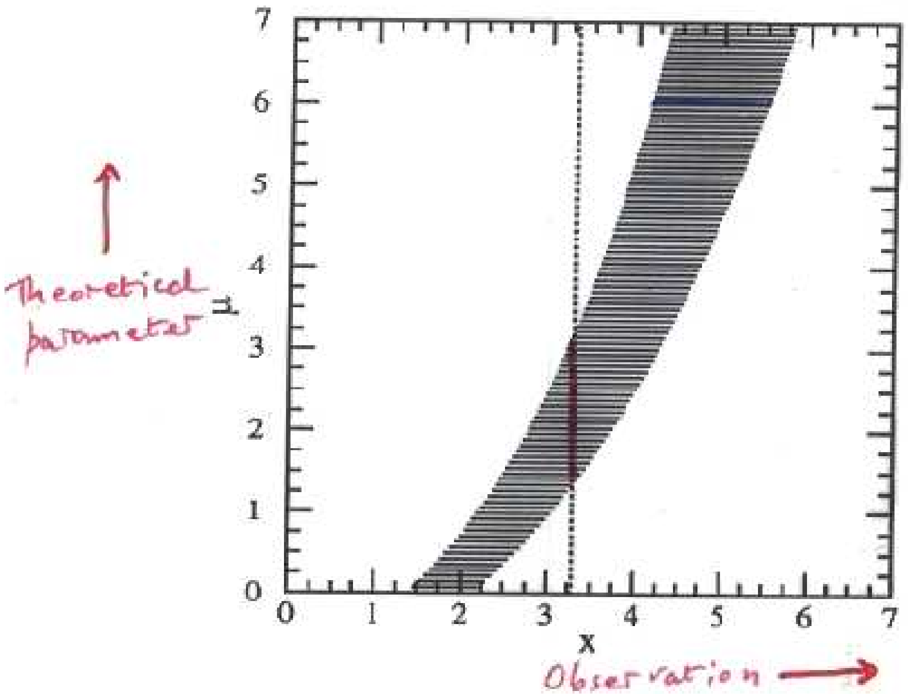


FIG. 1. A generic confidence belt construction and its use. For each value of μ, one draws a horizontal acceptance interval $[x_1, x_2]$ such that $P(x \in [x_1, x_2] | \mu) = \alpha$. Upon performing an experiment to measure x and obtaining the value x_0, one draws the dashed vertical line through x_0. The confidence interval $[\mu_1, \mu_2]$ is the union of all values of μ for which the corresponding acceptance interval is intercepted by the vertical line.

$\mu \geq 0$

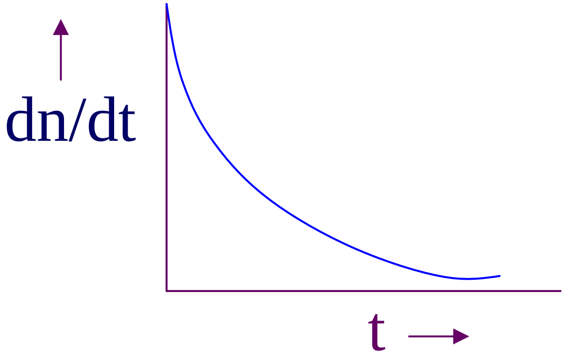
No prior for μ

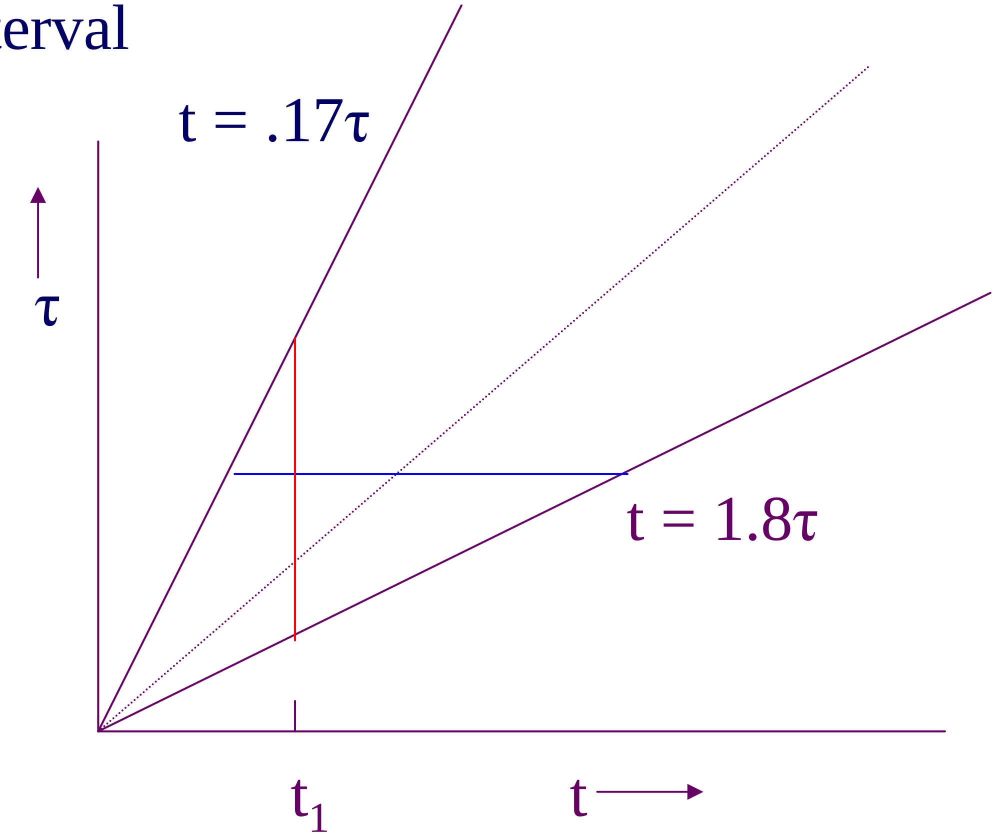
Frequentism: Specific example

Particle decays exponentially: $dn/dt = (1/\tau)\exp(-t/\tau)$

Observe 1 decay at time t_1: $\mathcal{L}(\tau) = (1/\tau)\exp(-t_1/\tau)$

Construct 68% central interval





90% Classical interval for Gaussian

$\sigma = 1 \qquad \mu \geq 0 \qquad$ e.g. $m^2(\nu_e)$

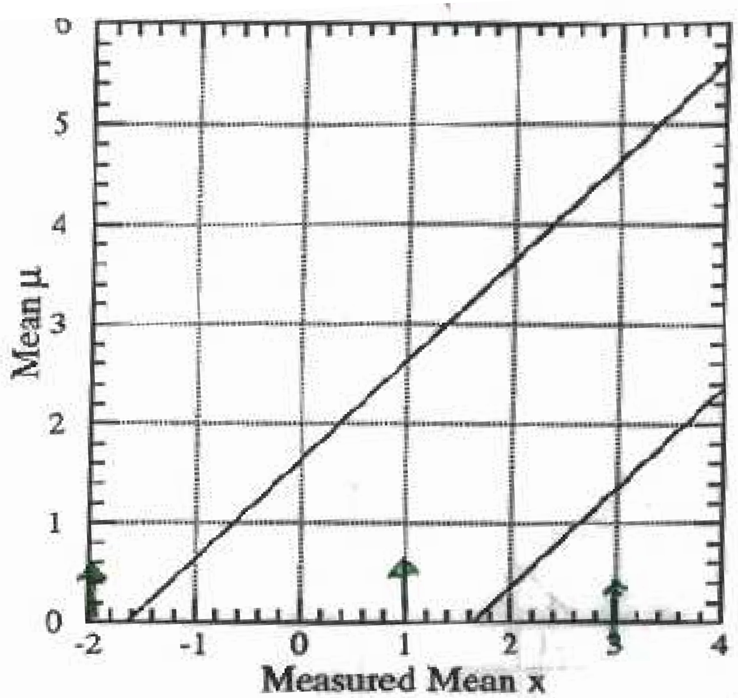


FIG. 3. Standard confidence belt for 90% C.L. central confidence intervals for the mean of a Gaussian, in units of the rms deviation.

$x_{obs} = 3$ Two sided limit

$x_{obs} = 1$ Upper limit

$x_{obs} = -2$ No region for μ

$\mu_l \le \mu \le \mu_u$ at 90% confidence

Frequentist

μ_l and μ_u known, but random

μ unknown, but fixed

Probability statement about μ_l and μ_u

Bayesian

μ_l and μ_u known, and fixed

μ unknown, and random

Probability/credible statement about μ

Coverage

Fraction of intervals containing true value

Property of method, not of result

Can vary with param

Frequentist concept. Built in to Neyman construction

Some Bayesians reject idea. Coverage not guaranteed

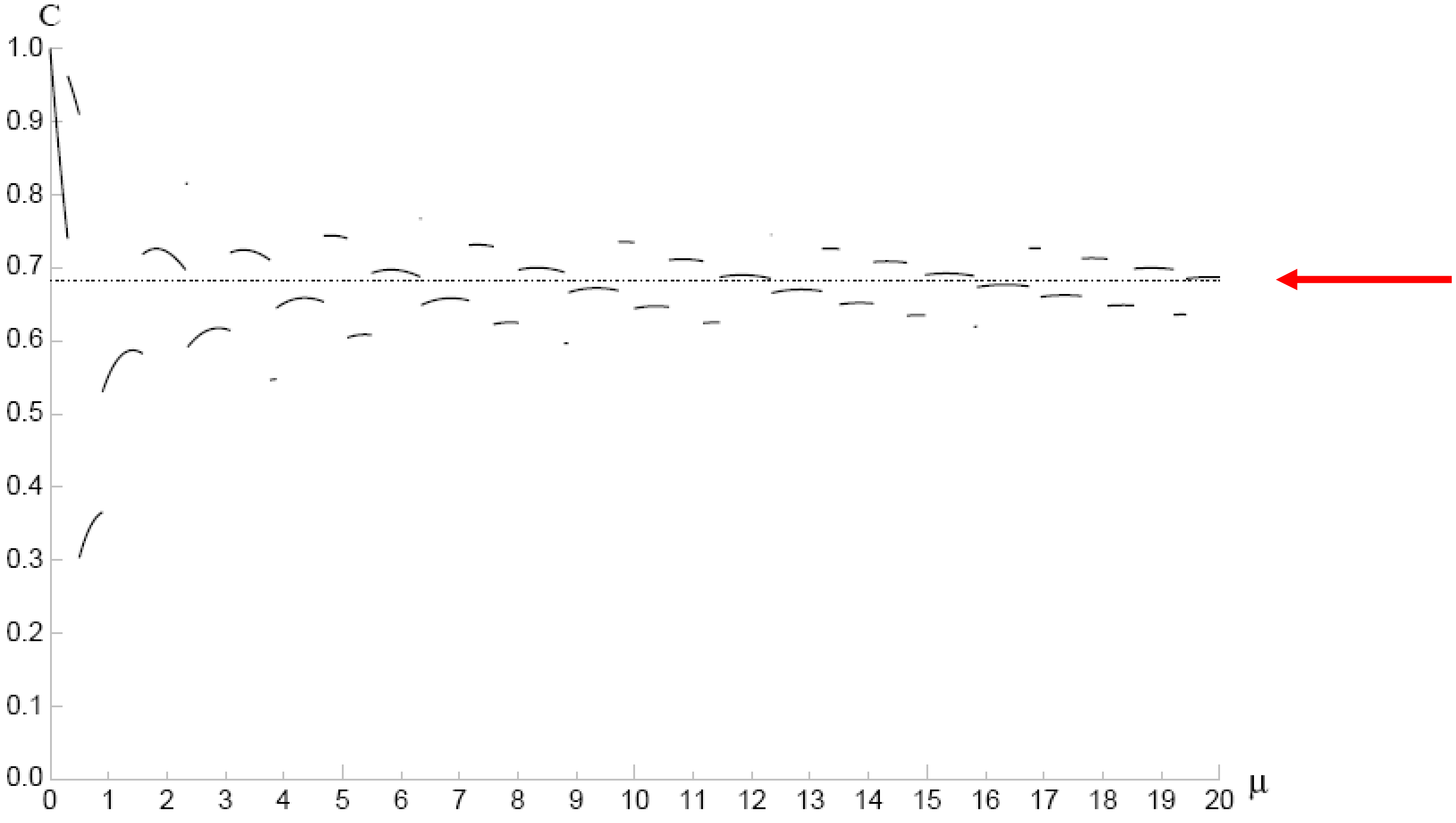
Integer data (Poisson) → discontinuities

Ideal coverage plot

Coverage : $\mathcal{L}$ approach (Not frequentist)

$P(n,\mu) = e^{-\mu}\mu^n/n!$ (Joel Heinrich CDF note 6438)

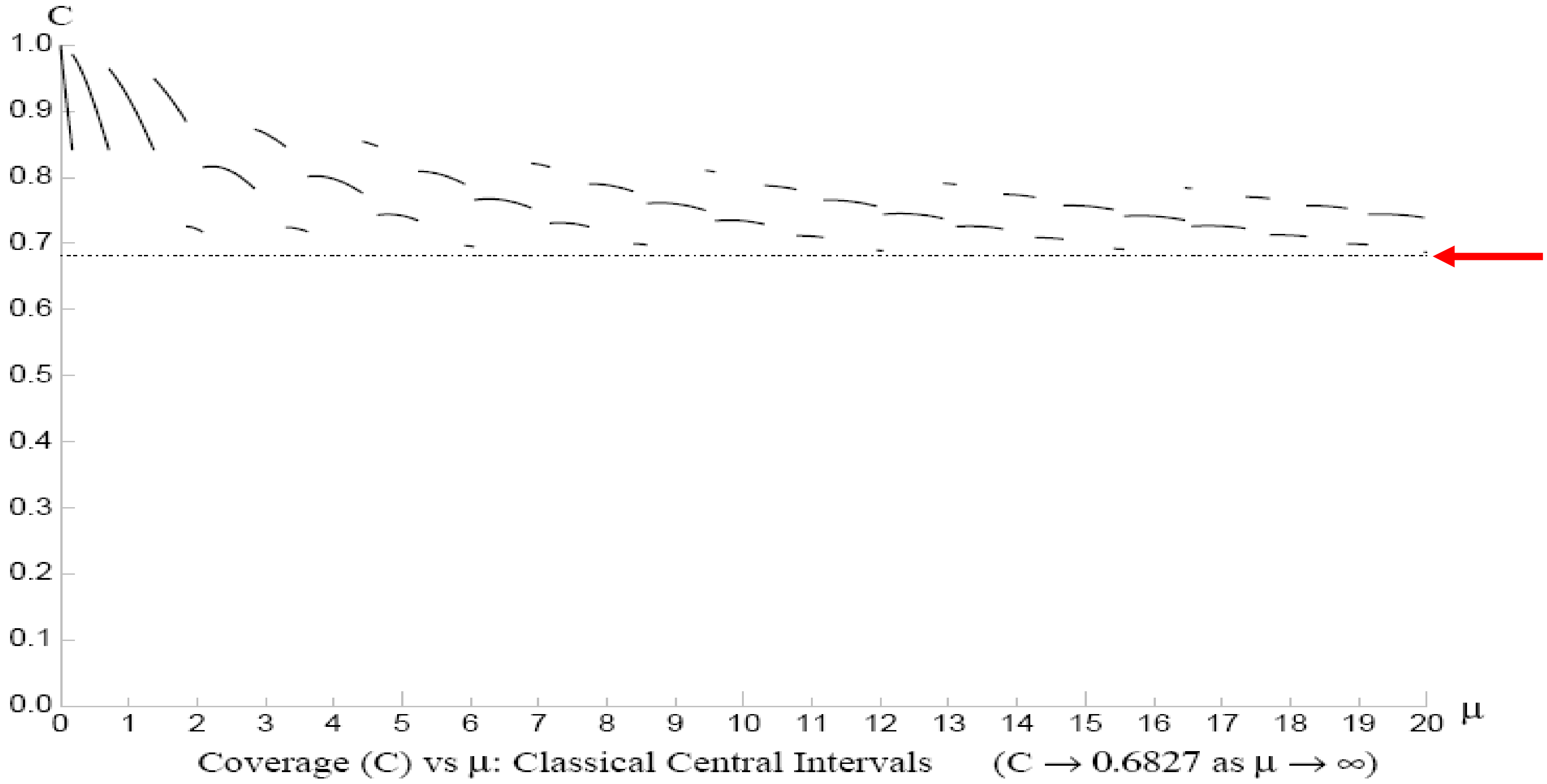
$-2\ \ln\lambda < 1$ $\lambda = P(n,\mu)/P(n,\mu_{best})$ UNDERCOVERS



Coverage (C) vs μ: $-2\ln\lambda < 1$ ($C \to 0.6827$ as $\mu \to \infty$)

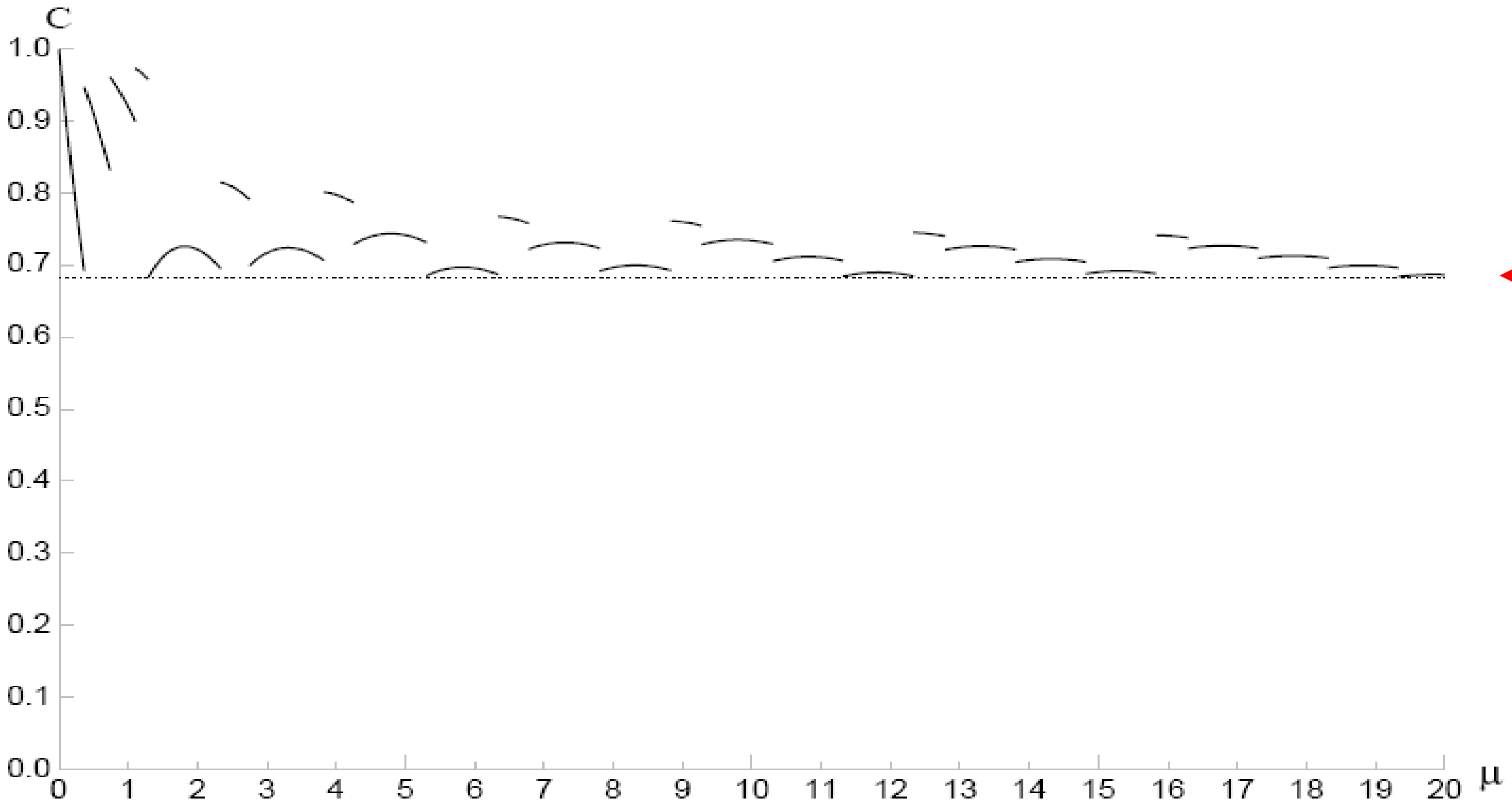
Frequentist central intervals, NEVER undercovers

(Conservative at both ends)



Feldman-Cousins Unified intervals

Frequentist, so NEVER undercovers



Coverage (C) vs μ: Unified Intervals (C → 0.6827 as μ → ∞)

Classical Intervals

- Problems

Hard to understand e.g. d'Agostini e-mail

Arbitrary choice of interval

Possibility of empty range

Nuisance parameters (systematic errors)

- Advantages

Widely applicable

Well defined coverage

FELDMAN - COUSINS

Wants to avoid empty classical intervals $\rightarrow$

Uses "$\mathcal{L}$-ratio ordering principle" to resolve ambiguity about "which 90% region?" $\rightarrow$

[Neyman + Pearson say $\mathcal{L}$-ratio is best for hypothesis testing]

No 'Flip-Flop' problem

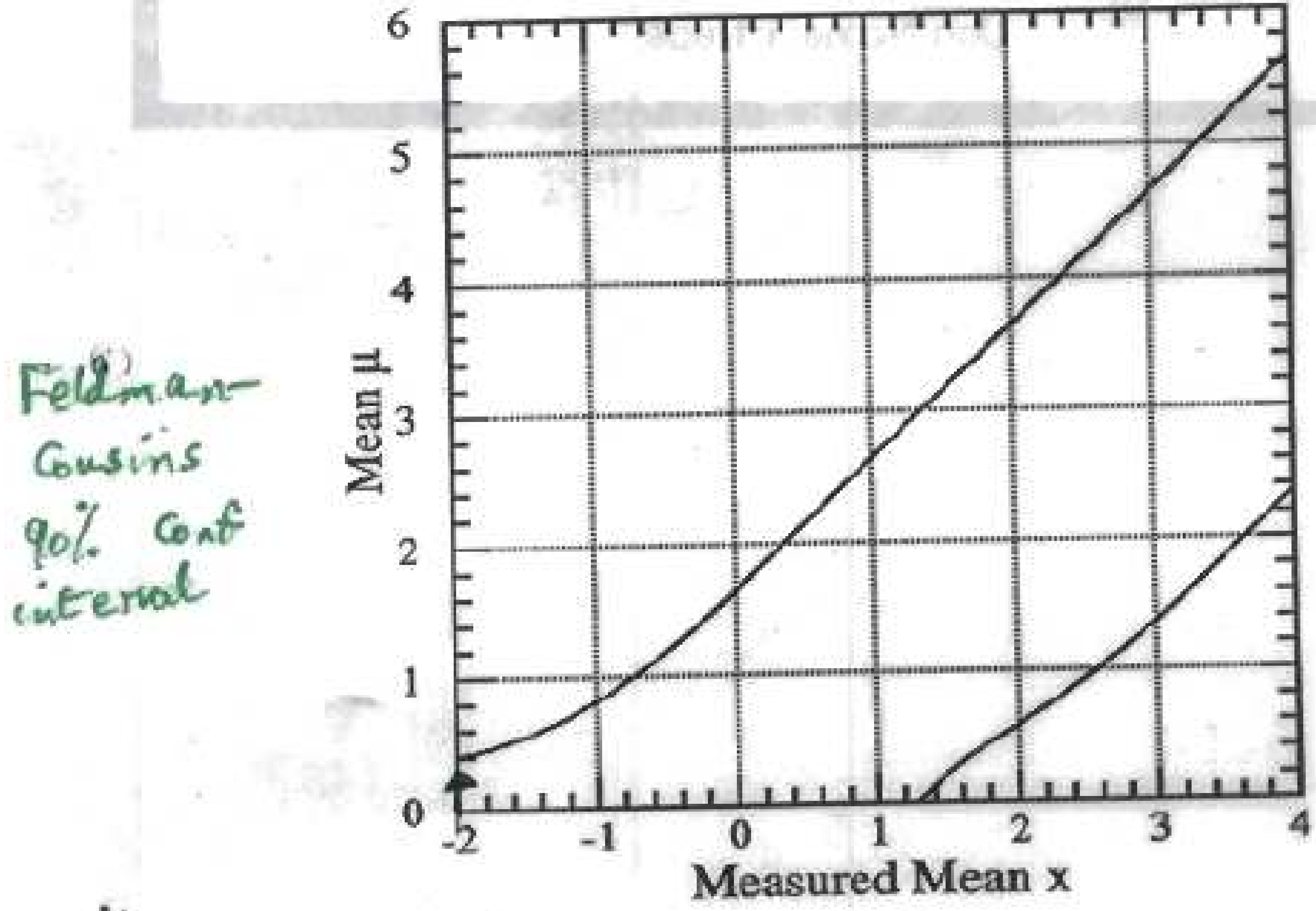
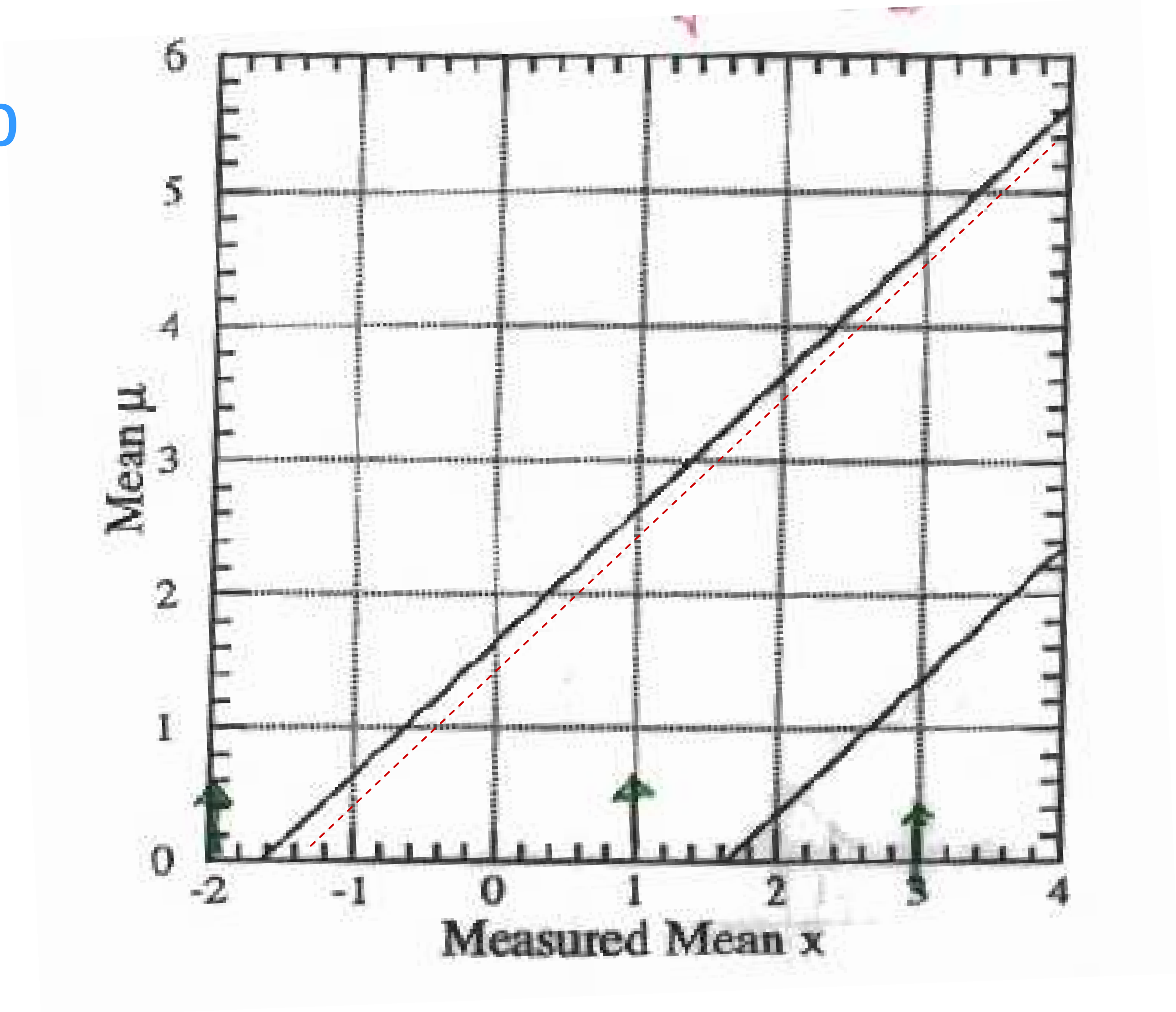


FIG. 10. Plot of our 90% confidence intervals for mean of a Gaussian, constrained to be non-negative, described in the text.

X_{obs} = -2 now gives upper limit

Flip-flop



Black lines Classical 90% central interval

Red dashed: Classical 90% upper limit

FLIP - FLOP

90% upper limit for $x_{obs} \le 3$

90% 2-sided interval for $x_{obs} > 3$

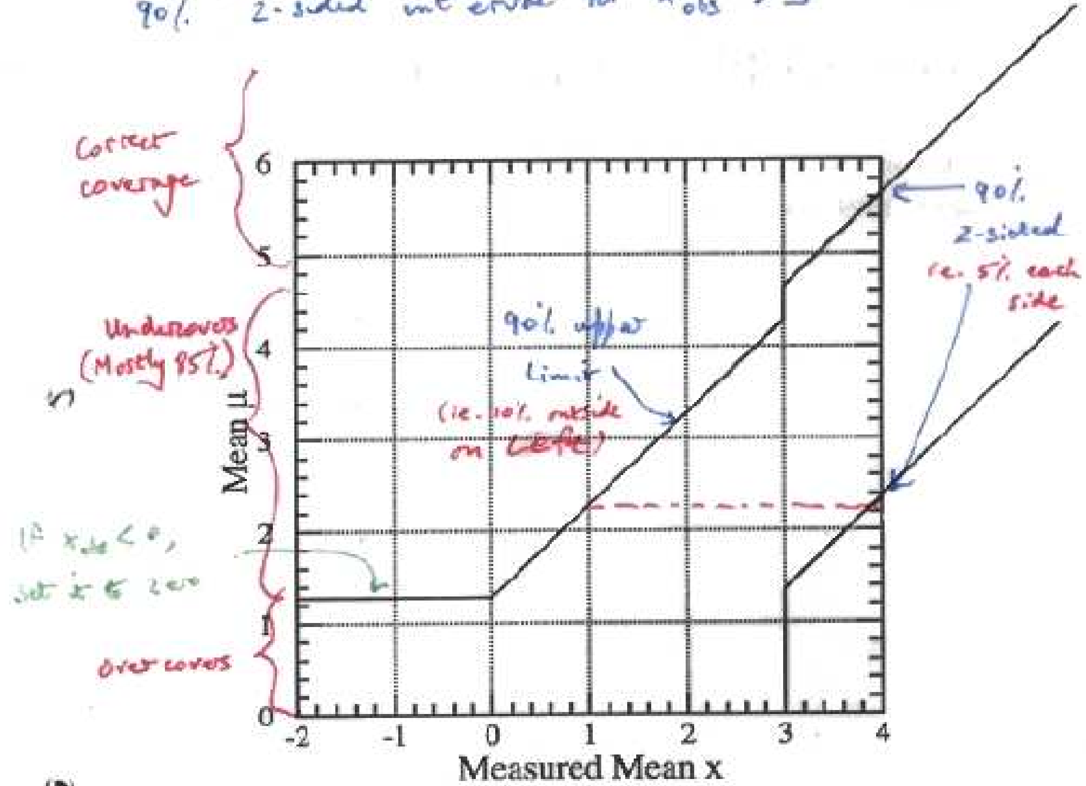


FIG. 4. Plot of confidence belts implicitly used for 90% C.L. confidence intervals (vertical intervals between the belts) quoted by flip-flopping Physicist X, described in the text. They are not valid confidence belts, since they can cover the true value at a frequency less than the stated confidence level. For $1.36 < \mu < 4.28$, the coverage (probability contained in the horizontal acceptance interval) is 85%.

Not good to let x_{obs} determine how result will be presented

F-C goes smoothly from 1-sided → 2-sided

Poisson confidence intervals. Background = 3

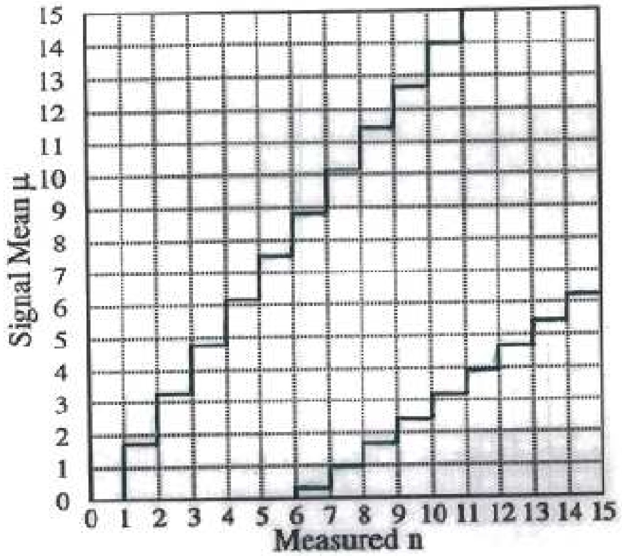


FIG. 6. Standard confidence belts for 90% C.L. central confidence intervals, for unknown Poisson signal mean μ in the presence of Poisson background with known mean $b = 3.0$.

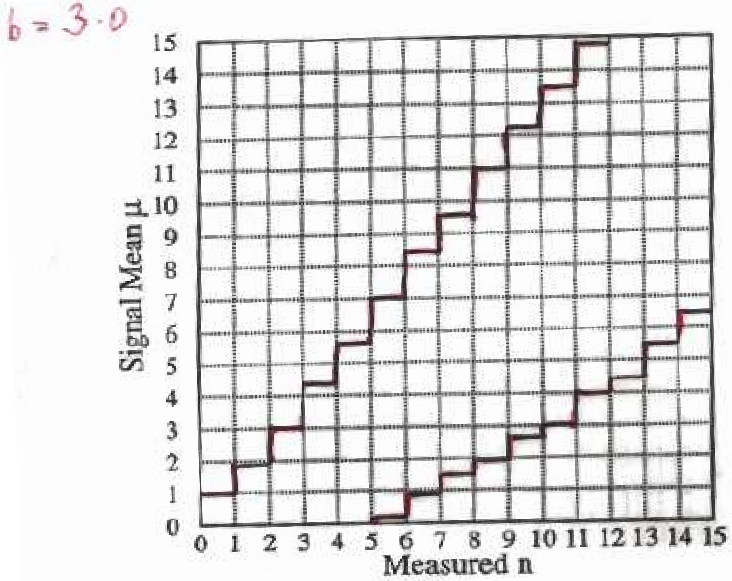


FIG. 7. Confidence belt based on our ordering principle, for 90% C.L. confidence intervals for unknown Poisson signal mean μ in the presence of Poisson background with known mean $b = 3.0$.

Standard Frequentist

Feldman - Cousins

FREQUENTIST POISSON C.B. CONSTRN.

< 10%

< 5%

Prob ordering

TABLES

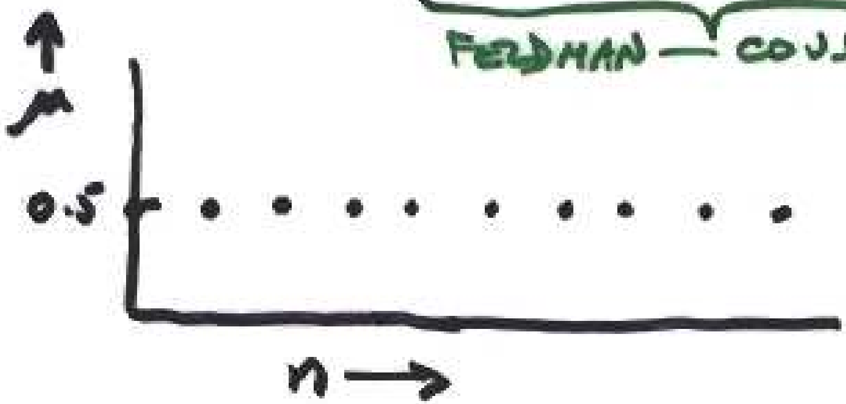
TABLE I. Illustrative calculations in the confidence belt construction for signal mean μ in the presence of known mean background $b = 3.0$. Here we find the acceptance interval for $\mu = 0.5$.

n	$P(n\|\mu)$	μ_{best}	$P(n\|\mu_{best})$	R	rank	U.L.	central
0	0.030	0.	0.050	0.607	6		
1	0.106	0.	0.149	0.708	5	✓	✓
2	0.185	0.	0.224	0.826	3	✓	✓
3	0.216	0.	0.224	0.963	2	✓	✓
4	0.189	1.	0.195	0.966	1	✓	✓
5	0.132	2.	0.175	0.753	4	✓	✓
6	0.077	3.	0.161	0.480	7	✓	✓
7	0.039	4.	0.149	0.259		✓	✓
8	0.017	5.	0.140	0.121		✓	
9	0.007	6.	0.132	0.050		✓	
10	0.002	7.	0.125	0.018		✓	
11	0.001	8.	0.119	0.006		✓	

5 3 1 2 4 6 7

FELDMAN — COUSINS

< 5%

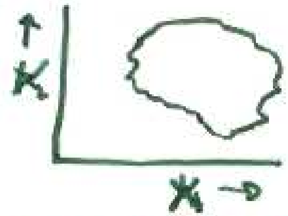


FEATURES OF F+C

REDUCES EMPTY INTERVALS

- UNIFIED 1-SIDED + 2-SIDED INTERVALS
- ELIMINATES FLIP-FLOP
- NO ARBITRARINESS OF INTERVAL

"READILY" EXTENDS TO SEVERAL DIMENSIONS



LESS OVERCOVERAGE THAN "5% AT ENDS"

MAX PROB DENSITY
5% AT ENDS ?

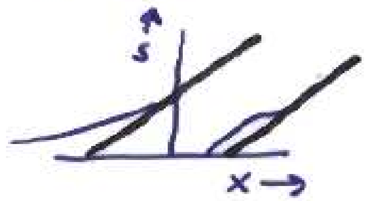
NEYMAN CONSTRUCTION ⇒ CPU-INTENSIVE
(ESP IN SEVERAL DIMENSIONS)

MINOR PATHOLOGIES : DISJOINT INTERVALS
WRONG BEHAVIOUR WRT BGD

TIGHT LIMITS FOR $b > n_{obs}$ e.g.

n_{obs}	bgd	90% Limit
0	3.0	1.08
0	0	2.44

UNIFIED ⇒ QUICKER EXCLUSION OF S=0



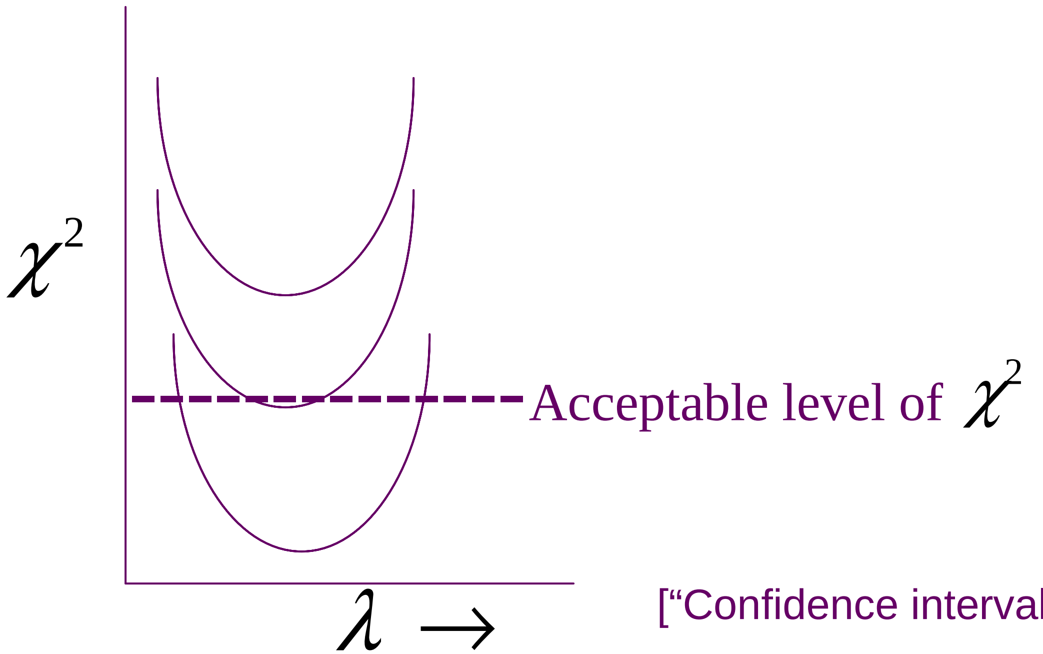
Importance of Ordering Rule

Neyman construction in 1 parameter μ

2 measurements $\mathbf{X}_1$ $\mathbf{X}_2$

$$p(x;\mu) = G(x-\mu,1)$$

An aside: Determination of single parameter p via χ^2



Range of parameters given by

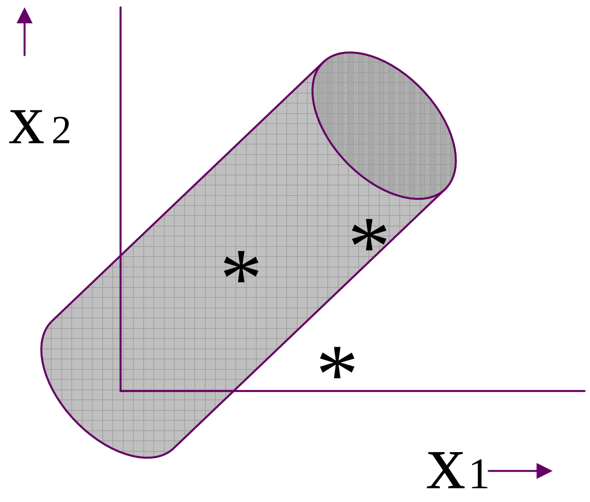
1) Values of λ for which data is likely i.e. p(χ^2) is acceptable or

2) $\chi^2(\lambda) < \chi^2_{\min}(\lambda) + 1$

2) is good

1) Range depends on $\chi^2_{\min}$

["Confidence interval coupled to goodness of fit"]



Neyman Construction

For given μ, acceptable (x_1 , x_2) satisfy

$$\chi^2 = (x_1-\mu)^2 + (x_2-\mu)^2 \leq Ccut$$

Defines cylinder in (μ, x_1, x_2) space

Experiment gives $(x_1, x_2) \rightarrow \mu$ interval

Range depends on $|x_1 - x_2|$

$$\mu = \frac{x_1+x_2}{2} \pm \sqrt{2-(x_1-x_2)^2}\ /\ 2$$

Range and goodness of fit are coupled

That was using Probability Ordering

Now change to Likelihood Ratio Ordering

For $x_1 \neq x_2$,no value of μ gives very good fit

For Neyman Construction at fixed μ , compare:

$$(x_1-\mu)^2+(x_2-\mu)^2 \quad \text{with} \quad (x_1-\mu_{best})^2+(x_2-\mu_{best})^2$$

where $\mu_{best} = (x_1+x_2)/2$

giving $$2\left[\mu^2-\mu(x_1+x_2)+\frac{1}{4}(x_1+x_2)^2\right] = 2\left[\mu-\frac{1}{2}(x_1+x_2)\right]^2$$

Cutting on Likelihood Ratio Ordering gives:

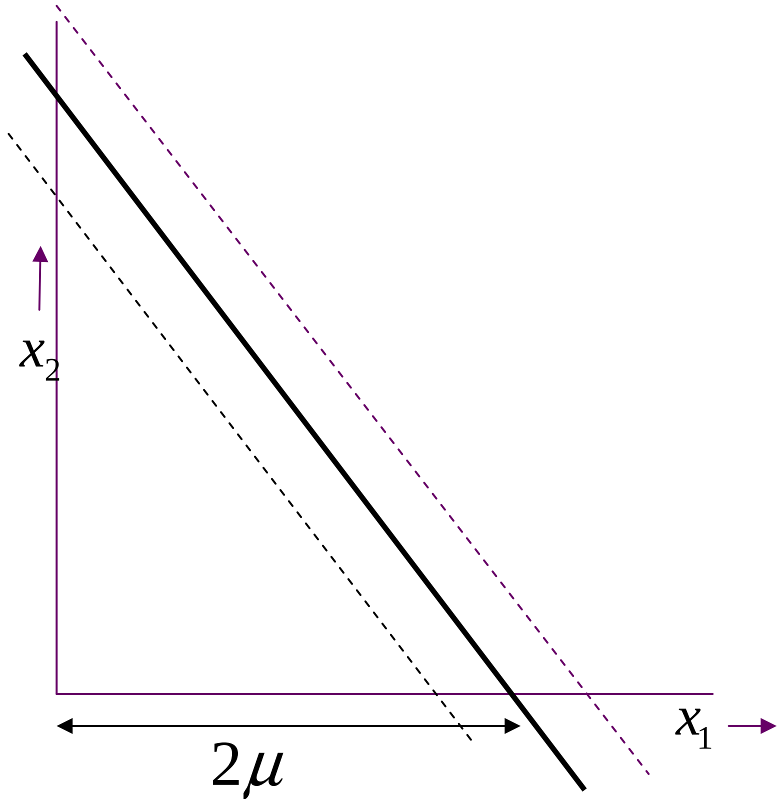
$$\mu = \frac{x_1+x_2}{2} \pm \sqrt{\frac{C}{2}}$$

$$\mu = \frac{x_1 + x_2}{2} \pm \sqrt{\frac{C}{2}}$$

Therefore, range of μ is

Constant Width

Independent of $x_1 - x_2$



Confidence Range and Goodness of Fit are completely decoupled

Standard Frequentist

Pros:

Coverage

Widely applicable

Cons:

Hard to understand

Small or empty intervals

Difficult in many variables (e.g. systematics)

Needs ensemble

Bayesian

Pros:

Easy to understand

Physical interval

Cons:

Needs prior

Coverage not guaranteed

Hard to combine

SYSTEMATICS

For example

$$N_{events} = \sigma LA + b$$

Observed

$$N \pm \sqrt{N}$$

for statistical errors

Physics parameter

we need to know these, probably from other measurements (and/or theory)

Uncertainties → error in σ

Some are arguably statistical errors

$$LA = LA_0 \pm \sigma_{LA}$$
$$b = b_0 \pm \sigma_b$$

Shift Central Value

Bayesian

Frequentist

Mixed

Shift Nuisance Parameters

$$N_{events} = \sigma \, \mathrm{LA} + \mathrm{b}$$

Simplest Method

Evaluate σ_0 using LA_0 and b_0

Move nuisance parameters (one at a time) by their errors → $\delta\sigma_{LA}$ & $\delta\sigma_{\mathrm{b}}$

If nuisance parameters are uncorrelated,

combine these contributions in quadrature

→ total systematic

Bayesian

Without systematics

$$p(\sigma; N) \propto p(N; \sigma)\Pi(\sigma)$$

$\uparrow$

prior

With systematics

$$p(\sigma, LA, b; N) \propto p(N; \sigma, LA, b)\Pi(\sigma, LA, b)$$

$\uparrow$

$$\sim \Pi_1(\sigma)\Pi_2(LA)\Pi_3(b)$$

Then integrate over LA and b

$$p(\sigma; N) = \iint p(\sigma, LA, b; N)\, dLA\, db$$

$$p(\sigma ; N) = \iint p(\sigma , LA , b ; N) dLA \ db$$

If $\Pi_1(\sigma)$ = constant and $\Pi_2(\mathrm{LA})$ = truncated Gaussian TROUBLE!

Upper limit on σ from $\int p(\sigma; N)\,\mathrm{d}\sigma$

Significance from likelihood ratio for $\sigma = 0$ and σ_{max}

Frequentist

Full Method

Imagine just 2 parameters σ and LA

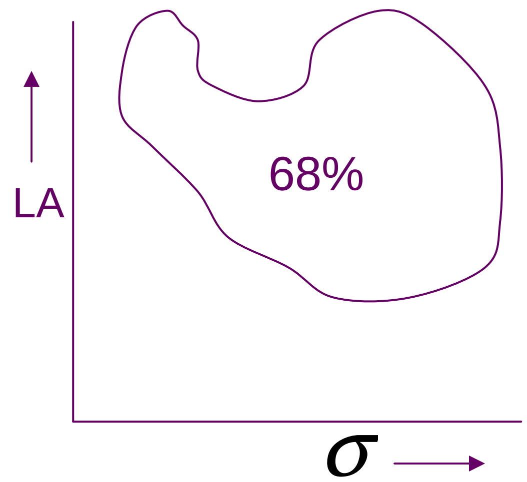
and 2 measurements N and M

$\uparrow$ Physics $\uparrow$ Nuisance

Do Neyman construction in 4-D

Use observed N and M, to give

Confidence Region for LA and σ



Then project onto σ axis

This results in OVERCOVERAGE

Aim to get better shaped region, by suitable choice of ordering rule

Example: Profile likelihood ordering

$$\frac{L(N_0 M_0; \sigma, LA_{best}(\sigma))}{L(N_0 M_0; \sigma_{best}, LA_{best}(\sigma))}$$

Full frequentist method hard to apply in several dimensions

Used in $\leq$ 3 parameters

For example: Neutrino oscillations (CHOOZ)

$$\sin^2 2\theta \, , \Delta m^2$$

Normalisation of data

Use approximate frequentist methods that reduce dimensions to just physics parameters

e.g. Profile pdf

i.e. $pdf_{profile}\left(N;\sigma\right) = pdf\left(N, M_0; \sigma, LA_{best}\right)$

Contrast Bayes marginalisation

Distinguish "profile ordering"

See Giovanni Punzi, PHYSTAT05 page 88

Talks at FNAL CONFIDENCE LIMITS WORKSHOP

(March 2000) by:

Gary Feldman

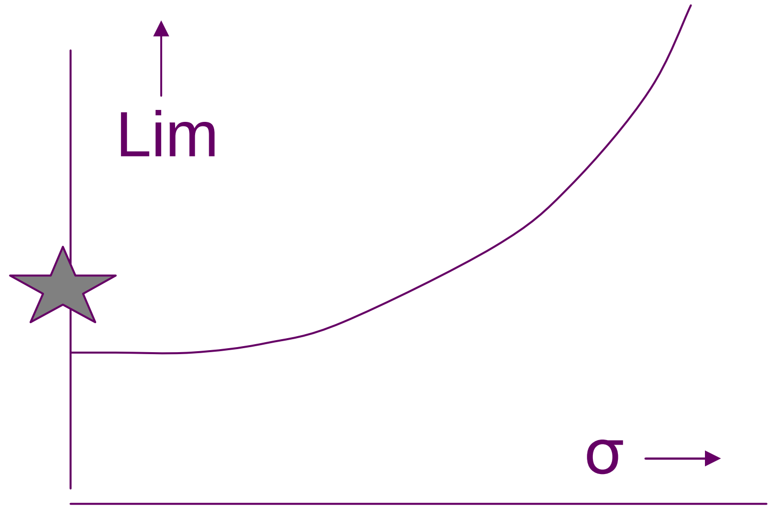
Wolfgang Rolke hep-ph/0005187 version 2

Acceptance uncertainty worse than Background uncertainty

Limit of C. Lim. as $\sigma \to 0$

$\neq$ C.L. for $\sigma = 0$

Need to check Coverage



Method: Mixed Frequentist - Bayesian

Bayesian for nuisance parameters and

Frequentist to extract range

Philosophical/aesthetic problems?

Highland and Cousins

(Motivation was paradoxical behaviour of Poisson limit when LA not known exactly)

Bayesian versus Frequentism

	Bayesian	Frequentist
Basis of method	Bayes Theorem → Posterior probability distribution	Uses pdf for data, for fixed parameters
Meaning of probability	Degree of belief	Frequentist definition
Prob of parameters?	Yes	Anathema
Needs prior?	Yes	No
Choice of interval?	Yes	Yes (except F+C)
Data considered	Only data you have	….+ other possible data
Likelihood principle?	Yes	No

Bayesian versus Frequentism

	Bayesian	Frequentist
Ensemble of experiment	No	Yes (but often not explicit)
Final statement	Posterior probability distribution	Parameter values → Data is likely
Unphysical/ empty ranges	Excluded by prior	Can occur
Systematics	Integrate over prior	Extend dimensionality of frequentist construction
Coverage	Unimportant	Built-in
Decision making	Yes (uses cost function)	Not useful

Bayesianism versus Frequentism

"Bayesians address the question everyone is interested in, by using assumptions no-one believes"

"Frequentists use impeccable logic to deal with an issue of no interest to anyone"